



On Strong Limit Theorems for Negatively Superadditive Dependent Random Variables

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Abstract. In this paper, we obtain strong convergence property for Jamison weighted sums of negatively superadditive dependent (NSD, in short) random variables, which extends the famous Jamison theorem. In addition, some sufficient conditions for complete convergence for weighed sums of NSD random variables are presented. These results generalize the corresponding results for independent identically distributed random variables to the case of NSD random variables without assumption of identical distribution.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space (Ω, \mathcal{F}, P) . The concept of negatively associated (NA) random variables was introduced by Joag-Dev and Proschan [6]. The concept of negatively superadditive dependent (NSD, in short) random variables was introduced by Hu [8], which was based on the class of superadditive functions. Superadditive structure functions have important reliability interpretations, which describe whether a system is more series-like or more parallel-like [1].

Definition 1.1. (Kemperman [7]) A function $\phi : R^n \rightarrow R$ is called superadditive if $\phi(x \vee y) + \phi(x \wedge y) \geq \phi(x) + \phi(y)$ for all $x, y \in R^n$, where \vee is for componentwise maximum and \wedge is for componentwise minimum.

Definition 1.2. (Hu [8]) A random vector $X = (X_1, X_2, \dots, X_n)$ is said to be negatively superadditive dependent (NSD, in short) if

$$E\phi(X_1, X_2, \dots, X_n) \leq E\phi(X_1^*, X_2^*, \dots, X_n^*), \quad (1)$$

where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1) exist.

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Hu [8] gave an example illustrating that NSD does not imply NA, and Hu posed an open problem whether NA implies NSD. Christofides and Vaggelatou [2] solved this open problem and indicated that NA implies NSD. Negatively superadditive dependent structure is an extension of negatively associated structure and sometimes more useful than negatively associated structure. For example, the structure function of a monotone coherent system can be superadditive [1], and inequalities derived from NSD can give one-side or two-side bounds of the system reliability. Eghbal *et al.* [3] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables under the assumption that $\{X_i, i \geq 1\}$ is a sequence of nonnegative NSD random variables with $EX_i^r < \infty$ for all $i \geq 1$ and some $r > 1$. Eghbal *et al.* [4] provided some Kolmogorov inequality for quadratic forms $T_n = \sum_{1 \leq i < j \leq n} X_i X_j$ and weighted quadratic forms $Q_n = \sum_{1 \leq i < j \leq n} a_{ij} X_i X_j$, where $\{X_i, i \geq 1\}$ is a sequence of nonnegative NSD uniformly bounded random variables. Shen *et al.* [9] obtained Khintchine-Kolmogorov convergence theorem and strong stability for NSD random variables. For more details about strong convergence properties for dependent sequence, one can refer to Yang *et al.* [14], Wang *et al.* [12,13], Zhou *et al.* [15], Zhou [16] and Shen *et al.* [10], and so forth.

In this paper, we study strong convergence property for Jamison weighted sums of NSD random variables, which extends the famous Jamison theorem in [5]. In addition, some sufficient conditions for complete convergence for weighed sums of NSD random variables are discussed.

The rest of the paper is organized as follows. In Section 2, some preliminary definition and lemmas for NSD random variables are presented. In Section 3, main results and their proofs are provided.

Throughout the paper, let $I(A)$ be the indicator function of the set A . C denotes a positive constant not depending on n , which may be different in various places. $a_n \ll b_n$ represents that there exists a constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all sufficiently large n .

2. Preliminaries

The following definition and lemmas will be needed in this paper.

Definition 2.1. A random variable sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant C , such that

$$P(|X_n| > x) \leq CP(|X| > x),$$

for all $x \geq 0$ and $n \geq 1$.

Lemma 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X . For any $\alpha > 0$ and $b > 0$, the following statement holds:

$$\begin{aligned} E|X_n|^\alpha I(|X_n| \leq b) &\leq C \{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\}, \\ E|X_n|^\alpha I(|X_n| > b) &\leq CE|X|^\alpha I(|X| > b), \end{aligned}$$

where C is a positive constant.

Lemma 2.3. (Shen *et al.* [9]) Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables. Assume that

$$\sum_{n=1}^{\infty} \text{Var}X_n < \infty,$$

then $\sum_{n=1}^{\infty} (X_n - EX_n)$ converges almost surely.

Lemma 2.4. (Sung [11]) Let $\phi(x)$ be a positive increasing function on $(0, +\infty)$ satisfying $\phi(x) \uparrow \infty$ as $x \rightarrow \infty$, and let $\psi(x)$ be the inverse function of $\phi(x)$. If $\psi(x)$ and $\phi(x)$ satisfy, respectively,

$$\psi(n) \sum_{i=1}^n \frac{1}{\psi(i)} = O(n), \quad E[\phi(|X|)] < \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{1}{\psi(n)} E|X|I(|X| > \psi(n)) < \infty.$$

Lemma 2.5. (Shen et al. [9]) Let X_1, X_2, \dots, X_n be NSD random variables with mean zero and finite second moments. Then

$$E \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^2 \right) \leq 2 \sum_{i=1}^n EX_i^2.$$

3. Main Results

In this section, we will provide strong convergence property for Jamison weighted sums of NSD random variables, which extends the result of Jamison *et al.* [5] for independent identically distributed random variables. And we will give some sufficient conditions for complete convergence for weighed sums of NSD random variables with different distributions.

Theorem 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables, which is stochastically dominated by random variable X . Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive numbers with $b_n \uparrow \infty$. Denote $c_n = b_n/a_n$ for $n \geq 1$. $1 \leq r < 2$. Assume that the following two conditions hold

$$EX = 0, E|X|^r < \infty, \tag{2}$$

$$N(n) = \text{Card}\{i : c_i \leq n\} \ll n^r, n \geq 1. \tag{3}$$

Then

$$\lim_{n \rightarrow \infty} b_n^{-1} \sum_{i=1}^n a_i X_i = 0, \text{ a.s.} \tag{4}$$

Proof. Let $N(0) = 0$. For every $n \geq 1$, denote $Y_i = -c_i I(X_i < -c_i) + X_i I(|X_i| \leq c_i) + c_i I(X_i > c_i)$. By (3), we can see that $c_n \rightarrow \infty$, as $n \rightarrow \infty$. Otherwise, there exist infinite subscripts i and some n_0 such that $c_i \leq n_0^r$. So, $N(n_0) = \infty$, which is contrary to $N(n_0) \ll n_0^r$ from (3). By (2) and (3), we have

$$\begin{aligned} \sum_{i=1}^{\infty} P(X_i \neq Y_i) &\ll \sum_{i=1}^{\infty} P(|X| > c_i) = \sum_{i=1}^{\infty} \sum_{c_i \leq j < c_{i+1}} P(|X| > c_i) \\ &\leq \sum_{j=1}^{\infty} \sum_{j-1 < c_i \leq j} P(|X| > j-1) \\ &= \sum_{j=1}^{\infty} (N(j) - N(j-1)) \sum_{k=j}^{\infty} P(k-1 < |X| \leq k) \\ &\ll \sum_{k=1}^{\infty} k^r P(k-1 < |X| \leq k) \\ &\ll E|X|^r < \infty, \end{aligned} \tag{5}$$

which implies $P(X_i \neq Y_i, \text{i.o.}) = 0$ from the Borel-Cantelli lemma. So in order to prove (4), we need only to prove

$$\lim_{n \rightarrow \infty} b_n^{-1} \sum_{i=1}^n a_i Y_i = 0, \text{ a.s.} \tag{6}$$

Combining (2), (3), Lemma 2.2 and the proof of (5), we have

$$\begin{aligned}
 \sum_{i=1}^{\infty} \text{Var}\left(\frac{a_i}{b_i} Y_i\right) &\ll \sum_{i=1}^{\infty} c_i^{-2} EX^2 I(|X| \leq c_i) + \sum_{i=1}^{\infty} P(|X| > c_i) \\
 &\ll \sum_{j=2}^{\infty} (N(j) - N(j-1))(j-1)^{-2} \sum_{k=1}^j EX^2 I(k-1 < |X| \leq k) + C \\
 &= \sum_{j=2}^{\infty} (N(j) - N(j-1))(j-1)^{-2} EX^2 I(0 < |X| \leq 1) \\
 &+ \sum_{k=2}^{\infty} EX^2 I(k-1 < |X| \leq k) \sum_{j=k}^{\infty} (N(j) - N(j-1))(j-1)^{-2} + C \\
 &\leq \sum_{j=2}^{\infty} N(j)((j-1)^{-2} - j^{-2}) EX^2 I(0 < |X| \leq 1) \\
 &+ \sum_{k=2}^{\infty} EX^2 I(k-1 < |X| \leq k) \sum_{j=k}^{\infty} N(j)((j-1)^{-2} - j^{-2}) + C \\
 &\ll E|X|^r + C < \infty.
 \end{aligned} \tag{7}$$

By the property of NSD random variables Lemma 2.3, (7) and Kronecker’s lemma, we can get

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{i=1}^n a_i (Y_i - EY_i) = 0, \text{ a.s.}$$

To complete the proof of (6), it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{i=1}^n a_i EY_i = 0. \tag{8}$$

By (2) and (3), we obtain

$$\begin{aligned}
 \sum_{i=1}^{\infty} \left| \frac{a_i}{b_i} EY_i \right| &\leq \sum_{i=1}^{\infty} c_i^{-1} [E|X_i| I(|X_i| \leq c_i) + E(c_i I(|X_i| > c_i))] \\
 &\ll \sum_{i=1}^{\infty} c_i^{-1} E|X| I(|X| \leq c_i) + \sum_{i=1}^{\infty} P(|X_i| > c_i).
 \end{aligned} \tag{9}$$

Notice that $EX = 0$, then we have

$$\begin{aligned}
 \sum_{i=1}^{\infty} c_i^{-1} E|X| I(|X| \leq c_i) &= \sum_{i=1}^{\infty} c_i^{-1} E|X| I(|X| > c_i) \\
 &\leq \sum_{j=1}^{\infty} \sum_{j-1 < c_i \leq j} c_i^{-1} E|X| I(|X| > j-1) \\
 &\leq C + \sum_{j=2}^{\infty} (N(j) - N(j-1))(j-1)^{-1} \sum_{i=j-1}^{\infty} E|X| I(i < |X| \leq i+1)
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{i=1}^{\infty} E|X|I(i < |X| \leq i + 1) \sum_{j=2}^i N(j)((j - 1)^{-1} - j^{-1}) \\
 &+ C \sum_{i=1}^{\infty} E|X|I(i < |X| \leq i + 1) \frac{N(i + 1)}{i} \\
 &\leq C \sum_{i=1}^{\infty} E|X|^r I(i < |X| \leq i + 1) \\
 &\leq CE|X|^r < \infty.
 \end{aligned} \tag{10}$$

It follows from (5), (9) and (10) that

$$\sum_{i=1}^{\infty} \left| \frac{a_i}{b_i} EY_i \right| < \infty.$$

Hence,

$$\sum_{i=1}^{\infty} \frac{a_i}{b_i} EY_i \text{ converges,}$$

which implies (8) from Kronecker’s lemma. \square

Remark 3.2. It is easy to check that we can take $a_n = n$ and $b_n = n^2$ for $n \geq 1$ such that they satisfy the conditions of Theorem 3.1. Sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are sequences of positive numbers with $b_n \uparrow \infty, c_n = b_n/a_n = n$, and (3) holds.

The following theorem gives some sufficient conditions for complete convergence for weighed sums of NSD random variables with different distributions.

Theorem 3.3. Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables, which is stochastically dominated by a random variable X , with $EX = 0, EX^2 < \infty$. Let the inverse function $\psi(x)$ of $\phi(x)$ satisfies Lemma 2.4. Assume that the following conditions hold for the array of positive constants $\{a_{ni}, n \geq 1, i \geq 1\}$

$$(i) \max_{1 \leq i \leq n} a_{ni} = O\left(\frac{1}{\psi(n)}\right), \quad (ii) \text{ for some } \delta > 0, \sum_{i=1}^n a_{ni}^2 = O(\log^{-1-\delta} n).$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon\right) < \infty. \tag{11}$$

Proof. For $n \geq 1$, let

$$X_i^{(n)} = -\psi(n)I(X_i < -\psi(n)) + X_i I(|X_i| \leq \psi(n)) + \psi(n)I(X_i > \psi(n)), 1 \leq i \leq n,$$

$$A = \bigcap_{i=1}^n (X_i = X_i^{(n)}) = \bigcap_{i=1}^n (|X_i| \leq \psi(n)), \quad \bar{A} = \bigcup_{i=1}^n (X_i \neq X_i^{(n)}) = \bigcup_{i=1}^n (|X_i| > \psi(n)),$$

$$E_n = \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon \right), \quad T_j^{(n)} = \sum_{i=1}^j (a_{ni} X_i^{(n)} - Ea_{ni} X_i^{(n)}), 1 \leq j \leq n.$$

$$\begin{aligned} \sum_{i=1}^j a_{ni} X_i &= \sum_{i=1}^j a_{ni} (X_i - X_i^{(n)}) + \sum_{i=1}^j (a_{ni} X_i^{(n)} - E a_{ni} X_i^{(n)}) + \sum_{i=1}^j E a_{ni} X_i^{(n)} \\ &= T_j^{(n)} + \sum_{i=1}^j E a_{ni} X_i^{(n)} + \sum_{i=1}^j a_{ni} X_i I(|X_i| > \psi(n)) \\ &\quad + \sum_{i=1}^j a_{ni} \psi(n) [I(X_i < -\psi(n)) - I(X_i > \psi(n))]. \end{aligned}$$

So it is easy to see that

$$\begin{aligned} E_n &= (E_n \cap A) \cup (E_n \cap \bar{A}) = \left(\max_{1 \leq j \leq n} |T_j^{(n)} + \sum_{i=1}^j E a_{ni} X_i^{(n)}| > \varepsilon \right) \cup (E_n \cap \bar{A}) \\ &\subset \left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \varepsilon - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| \right) \cup \bar{A}, \end{aligned}$$

hence

$$P(E_n) \leq P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \varepsilon - \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| \right) + \sum_{i=1}^n P(|X_i| > \psi(n)). \tag{12}$$

It follows from Lemma 2.4 and Kronecker’s lemma that

$$\lim_{n \rightarrow \infty} \frac{1}{\psi(n)} \sum_{i=1}^n E|X|I(|X| > \psi(i)) = 0. \tag{13}$$

By Condition (i), $EX = 0$, $\psi(n) \uparrow \infty$, Lemma 2.2 and (13), we can get

$$\begin{aligned} \max_{1 \leq j \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| &\leq \sum_{i=1}^n a_{ni} E|X_i|I(|X_i| \leq \psi(n)) + \sum_{i=1}^n a_{ni} \psi(n) E I(|X_i| > \psi(n)) \\ &\ll \sum_{i=1}^n a_{ni} E|X|I(|X| \leq \psi(n)) + \sum_{i=1}^n a_{ni} \psi(n) E I(|X| > \psi(n)) \\ &\ll \frac{1}{\psi(n)} \sum_{i=1}^n E|X|I(|X| > \psi(i)) \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

So, for all $\varepsilon > 0$ and enough large value n ,

$$\max_{1 \leq i \leq n} \left| \sum_{i=1}^j E a_{ni} X_i^{(n)} \right| < \frac{\varepsilon}{2}, \quad a.s. \tag{14}$$

Combining(12) with (14), we have for sufficiently large n

$$P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^j a_{ni} X_i \right| > \varepsilon \right) \leq P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2} \right) + \sum_{i=1}^n P(|X_i| > \psi(n)). \tag{15}$$

By $E[\phi(|X|)] < \infty$, it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P(|X_i| > \psi(n)) &\ll \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n P(|X| > \psi(n)) = \sum_{n=1}^{\infty} P(|X| > \psi(n)) \\ &= \sum_{n=1}^{\infty} P(\phi(X) > n) \ll E[\phi(|X|)] < \infty. \end{aligned} \tag{16}$$

By property of NSD sequence Lemma 2.5, Cr inequality, $EX^2 < \infty$ and condition (ii), we can get

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |T_j^{(n)}| > \frac{\varepsilon}{2}\right) &\ll \sum_{n=1}^{\infty} n^{-1} E\left(\max_{1 \leq j \leq n} |T_j^{(n)}|^2\right) \ll \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n E|a_{ni} X_i^{(n)}|^2 \\ &\ll \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n a_{ni}^2 EX^2 I(|X| \leq \psi(n)) + \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n a_{ni}^2 \psi^2(n) EI(|X| > \psi(n)) \\ &\leq C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^n a_{ni}^2 \leq C \sum_{n=1}^{\infty} n^{-1} \log^{-1-\alpha} n < \infty. \end{aligned} \quad (17)$$

Then (11) holds following from (15)-(17). \square

Remark 3.4. As we can see, the family of NSD sequences contains independent sequences and negatively associated sequences as special cases. Theorem 3.1 gives strong convergence property for Jamison weighted sums of NSD random variables. It extends the result of Jamison *et al.* [5] for independent identically distributed random variables. In addition, the result of Theorem 3.3 holds for independent sequences and negatively associated sequences. And we obtain these results without assumption of identical distribution.

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