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On the Fractional Hermite-Hadamard Type Inequalities for (α, m)-Logarithmically Convex Functions

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Abstract. The purpose of this paper is to establish some refinements of Riemann-Liouville fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. By using our fractional integrals identities, we present some interesting and new left type fractional Hermite-Hadamard inequalities for once and twice differentiable (α, m)-logarithmically convex functions via powerful series.

1. Introduction

The subject of fractional calculus has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. For more recent development on fractional calculus, one can see the monographs [1–8]. Since the widely application of fractional integrals and importance of Hermite-Hadamard type inequalities, some authors extended to study fractional Hermite-Hadamard type inequalities according to the Hermite-Hadamard type inequalities for different classes functions. For more recent results, the readers can refer to [9–18] and the references therein.

Recently, Bai et al. [19] raised the new concept of (α, m)-logarithmically convex functions and established some interesting Hermite-Hadamard type inequalities of such functions. Next, Deng and Wang [17] generalized the results to right type fractional Hermite-Hadamard inequalities and improved the previous results [19] by replacing E_i, $i = 1, 2, 3$ by some series. However, the corresponding left type fractional Hermite-Hadamard inequalities have not been reported. Note that some interesting left type fractional Hermite-Hadamard identities including the first and second order derivative of the function have been reported in [10, 15]. Thus, we can expect to derive some new left type fractional Hermite-Hadamard inequalities by using or modifying our identities in [10, 15].

Motivated by [14, 17–19], we will go on studying left type Riemann-Liouville fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions. The purpose of this paper is to establish

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left type fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions via the powerful series.

To end this section, we collect left type fractional integral equalities which will be widely used in the sequel.

Definition 1.1. *(see [3]) Let $f \in L[a,b]$. The symbol ${}_{RL}J^{\alpha}_{a^+}f$ and ${}_{RL}J^{\alpha}_{b^-}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ are defined by*

$$({}_{RL}J^{\alpha}_{a^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ (0 \le a < x \le b),\ \ ({}_{RL}J^{\alpha}_{b^-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ (0 \le a \le x < b),$$

respectively. Here $\Gamma(\cdot)$ is the Gamma function.

Lemma 1.2. *(see [10]) Let $f : [a,b] \to R$ be a differentiable mapping on (a,b). If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a+}f(b) + {}_{RL}J^{\alpha}_{b-}f(a)] - f\left(\frac{a+b}{2}\right) \\ = & \frac{b-a}{2}\left[\int_0^1 h(t)f'(ta+(1-t)b)dt - \int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f'(ta+(1-t)b)dt\right] \\ = & \frac{b-a}{2}\int_0^1 [h(t) - (1-t)^{\alpha} + t^{\alpha}]f'(ta+(1-t)b)dt \end{aligned}$$

where

$$h(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1. \end{cases} \tag{1}$$

By Lemma 1.2 we have

Lemma 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < mb \le b$. If $f^{'} \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a+}f(mb) + {}_{RL}J^{\alpha}_{mb-}f(a)] - f\left(\frac{a+mb}{2}\right) = \frac{mb-a}{2}\int_0^1 [h(t) - (1-t)^{\alpha} + t^{\alpha}]f'(ta+m(1-t)b)dt,$$

where $h(\cdot)$ is defined by (1).

Lemma 1.4. *(see [15]) Let $f : [a,b] \to R$ be twice differentiable mapping on (a,b). If $f'' \in L[a,b]$, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a^+}f(b) + {}_{RL}J^{\alpha}_{b^-}f(a)] - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2}\int_0^1 g(t)f''(ta+(1-t)b)dt,$$

where

$$g(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [0, \frac{1}{2}), \\ 1 - t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [\frac{1}{2}, 1). \end{cases} \tag{2}$$

By Lemma 1.4 we have

Lemma 1.5. *Let $f : [a,b] \to R$ be twice differentiable mapping on (a,b) with $a < mb \le b$. If $f'' \in L[a,b]$, then*

$$\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a^+}f(mb) + {}_{RL}J^{\alpha}_{mb^-}f(a)] - f\left(\frac{a+mb}{2}\right) = \frac{(mb-a)^2}{2}\int_0^1 g(t)f''(ta+m(1-t)b)dt,$$

where $g(\cdot)$ is defined by (2).

Lemma 1.6. *(see [15]) Let $f : [a,b] \to R$ be twice differentiable mapping on (a,b) with $a < mb \le b$. If $f'' \in L^1[a,b]$, $r > 0$, then*

$$\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}[{}_{RL}J^\alpha_{a^+} f(mb) + {}_{RL}J^\alpha_{mb^-} f(a)] = (mb-a)^2 \int_0^1 u(t) f''(ta+m(1-t)b)dt,$$

where

$$u(t) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1}, & t \in [0, \frac{1}{2}), \\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1}, & t \in [\frac{1}{2}, 1). \end{cases}$$

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts.

Definition 2.1. *(see [19]) The function $f : [0,b] \to R^+$ is said to be (α,m)-logarithmically convex,if for every $x, y \in [0,b]$, $(\alpha,m) \in (0,1] \times (0,1]$,and $t \in [0,1]$ we have*

$$f(tx + m(1-t)y) \le (f(x))^{t^\alpha} (f(y))^{m(1-t^\alpha)}.$$

Lemma 2.2. *(see [18]) For $\alpha > 0$ and $k > 0$, we have*

$$I(\alpha,k) := \int_0^1 t^{\alpha-1} k^t dt = k \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} < +\infty,$$

where $(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1)$. Moreover, it holds

$$\left| I(\alpha,k) - k \sum_{i=1}^{m} \frac{(-\ln k)^{i-1}}{(\alpha)_i} \right| \le \frac{|\ln k|}{\alpha\sqrt{2\pi(m-1)}} \left(\frac{|\ln k| e}{m-1}\right)^{m-1}.$$

Lemma 2.3. *(see [14]) For $\alpha > 0$ and $k > 0, z > 0$, we have*

$$J(\alpha,k) := \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln\ k)^{i-1}}{(\alpha)_i} < \infty,$$

$$H(\alpha,k,z) := \int_0^z t^{\alpha-1} k^t dt = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z \ln\ k)^{i-1}}{(\alpha)_i} < \infty.$$

Lemma 2.4. *(see [17]) For $t \in [0,1]$, we have*

$$(1-t)^n \le 2^{1-n} - t^n \text{ for } n \in [0,1], \quad (1-t)^n \ge 2^{1-n} - t^n \text{ for } n \in [1,\infty).$$

3. The First Main Results

In this section, we apply Lemma 1.3 to derive some left Hermite-Hadamard type inequalities for differentiable (α,m)-logarithmically convex functions.

Theorem 3.1. *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f'|$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha \in (0,1]$, $0 \le a < mb \le b$, then the following inequality for fractional integrals holds*

$$\left| \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} [J^\alpha_{a+} f(mb) + J^\alpha_{mb-} f(a)] - f\left(\frac{a+mb}{2}\right) \right| \le I_k,$$

where

$$\begin{aligned} I_k &= (mb-a)|f'(b)|^m(1+2^{1-\alpha})\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}\alpha^{i-1}(k2^{i\alpha+1-\alpha}-k^{2^{-\alpha}})}{[\alpha;i-1]2^{i\alpha+2-\alpha}} \\ &\quad +(mb-a)|f'(b)|^m\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}\alpha^{i-1}(k^{2^{-\alpha}}-k2^{i\alpha})}{[\alpha;i]2^{i\alpha}},\ for\ k\neq 1, \\ I_k &= \frac{(mb-a)|f'(b)|^m(\alpha 2^{\alpha-1}+2^{\alpha-1}+1-2^{\alpha})}{(\alpha+1)2^{\alpha}},\ for\ k=1, \\ k &= \frac{|f'(a)|}{|f'(b)|^m}, \end{aligned}$$

and

$$[\alpha;0]:=1,\ [\alpha;i]:=(\alpha+1)(2\alpha+1)\cdots(i\alpha+1),\ i\in N.$$

Proof. (i) Case 1: $k\neq 1$. By Definition 2.1, Lemma 2.3, Lemma 2.4, and Lemma 1.3, we have

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J^{\alpha}_{a+}f(mb)+J^{\alpha}_{mb-}f(a)]-f\left(\frac{a+mb}{2}\right)\right| \\ \leq\ & \frac{mb-a}{2}\int_0^1|h(t)-(1-t)^{\alpha}+t^{\alpha}||f'(ta+m(1-t)b)|dt \\ \leq\ & \frac{mb-a}{2}\int_0^1|h(t)-(1-t)^{\alpha}+t^{\alpha}||f'(a)|^{t^{\alpha}}|f'(b)|^{m-mt^{\alpha}}dt \\ =\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^1|h(t)-(1-t)^{\alpha}+t^{\alpha}|k^{t^{\alpha}}dt \\ =\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}k^{t^{\alpha}}dt+\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}k^{(1-t)^{\alpha}}dt \\ & -\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}(1-t)^{\alpha}k^{t^{\alpha}}dt-\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}(1-t)^{\alpha}k^{(1-t)^{\alpha}}dt \\ & +\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}t^{\alpha}k^{t^{\alpha}}dt+\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}t^{\alpha}k^{(1-t)^{\alpha}}dt \\ =\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}k^{t^{\alpha}}dt+\frac{(mb-a)|f'(b)|^m}{2}\int_{\frac12}^1k^{t^{\alpha}}dt \\ & -\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}(1-t^{\alpha})k^{t^{\alpha}}dt-\frac{(mb-a)|f'(b)|^m}{2}\int_{\frac12}^1t^{\alpha}k^{t^{\alpha}}dt \\ & +\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac12}t^{\alpha}k^{t^{\alpha}}dt+\frac{(mb-a)|f'(b)|^m}{2}\int_{\frac12}^1(1-t)^{\alpha}k^{t^{\alpha}}dt \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} k^{t^\alpha}dt + \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 k^{t^\alpha}dt \\
& -\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}}(1-t^\alpha)k^{t^\alpha}dt - \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 t^\alpha k^{t^\alpha}dt \\
& +\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}dt + \frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 (2^{1-\alpha}-t^\alpha)k^{t^\alpha}dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2}\Big(1+2^{1-\alpha}\Big)\int_{\frac{1}{2}}^1 k^{t^\alpha}dt \\
& -(mb-a)|f'(b)|^m\int_{\frac{1}{2}}^1 t^\alpha k^{t^\alpha}dt + (mb-a)|f'(b)|^m\int_0^{\frac{1}{2}} t^\alpha k^{t^\alpha}dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2\alpha}\Big(1+2^{1-\alpha}\Big)\int_{\frac{1}{2^\alpha}}^1 t^{\frac{1}{\alpha}-1}k^t dt \\
& -(mb-a)|f'(b)|^m\frac{1}{\alpha}\int_{\frac{1}{2^\alpha}}^1 t^{(\frac{1}{\alpha}+1)-1}k^t dt + (mb-a)|f'(b)|^m\frac{1}{\alpha}\int_0^{\frac{1}{2^\alpha}} t^{(\frac{1}{\alpha}+1)-1}k^t dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2\alpha}\Big(1+2^{1-\alpha}\Big)\Big(\int_0^1 t^{\frac{1}{\alpha}-1}k^t dt - \int_0^{\frac{1}{2^\alpha}} t^{\frac{1}{\alpha}-1}k^t dt\Big) \\
& -(mb-a)|f'(b)|^m\frac{1}{\alpha}\int_0^1 t^{(\frac{1}{\alpha}+1)-1}k^t dt + 2(mb-a)|f'(b)|^m\frac{1}{\alpha}\int_0^{\frac{1}{2^\alpha}} t^{(\frac{1}{\alpha}+1)-1}k^t dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2\alpha}\Big(1+2^{1-\alpha}\Big)\Bigg(k\sum_{i=1}^\infty \frac{(-\ln\ k)^{i-1}}{(\frac{1}{\alpha})_i} - \Big(\frac{1}{2^\alpha}\Big)^{\frac{1}{\alpha}} k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty \frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac{1}{\alpha})_i}\Bigg) \\
& -(mb-a)|f'(b)|^m\frac{1}{\alpha}k\sum_{i=1}^\infty \frac{(-\ln\ k)^{i-1}}{(\frac{1}{\alpha}+1)_i} + 2(mb-a)|f'(b)|^m\frac{1}{\alpha}\Big(\frac{1}{2^\alpha}\Big)^{\frac{1}{\alpha}+1} k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty \frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac{1}{\alpha}+1)_i} \\
=\ & \frac{(mb-a)|f'(b)|^m}{2\alpha}\Big(1+2^{1-\alpha}\Big)\Bigg(k\sum_{i=1}^\infty \frac{(-\ln\ k)^{i-1}\alpha^i}{[\alpha;i-1]} - k^{2^{-\alpha}}\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i-1]2^{i\alpha+1-\alpha}}\Bigg) \\
& -(mb-a)|f'(b)|^m\frac{1}{\alpha}k\sum_{i=1}^\infty \frac{(-\ln\ k)^{i-1}\alpha^i}{[\alpha;i]} + 2(mb-a)|f'(b)|^m\frac{1}{\alpha}k^{2^{-\alpha}}\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^i}{[\alpha;i]2^{i\alpha+1}} \\
=\ & \frac{(mb-a)|f'(b)|^m}{2\alpha}\Big(1+2^{1-\alpha}\Big)\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^i(k2^{i\alpha+1-\alpha}-k^{2^{-\alpha}})}{[\alpha;i-1]2^{i\alpha+1-\alpha}} \\
& +(mb-a)|f'(b)|^m\frac{1}{\alpha}\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^i(k^{2^{-\alpha}}-k2^{i\alpha})}{[\alpha;i]2^{i\alpha}} \\
=\ & (mb-a)|f'(b)|^m(1+2^{1-\alpha})\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^{i-1}(k2^{i\alpha+1-\alpha}-k^{2^{-\alpha}})}{[\alpha;i-1]2^{i\alpha+2-\alpha}} \\
& +(mb-a)|f'(b)|^m\sum_{i=1}^\infty \frac{(-\ln k)^{i-1}\alpha^{i-1}(k^{2^{-\alpha}}-k2^{i\alpha})}{[\alpha;i]2^{i\alpha}}.
\end{aligned}
$$

(ii) Case 2: $k = 1$. By Definition 2.1 and Lemma 1.3, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}f(mb)+J^\alpha_{mb-}f(a)]-f\left(\frac{a+mb}{2}\right)\right| \\
\leq\ & \frac{mb-a}{2}\int_0^1 |h(t)-(1-t)^\alpha+t^\alpha||f^{'}(ta+m(1-t)b)|dt \\
\leq\ & \frac{mb-a}{2}\int_0^1 |h(t)-(1-t)^\alpha+t^\alpha||f'(a)|^{t^\alpha}|f'(b)|^{m-mt^\alpha}dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^1 |h(t)-(1-t)^\alpha+t^\alpha|dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} |1-(1-t)^\alpha+t^\alpha|dt+\frac{(mb-a)|f'(b)|^m}{2}\int_{\frac{1}{2}}^1 |-1-(1-t)^\alpha+t^\alpha|dt \\
=\ & \frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} |1-(1-t)^\alpha+t^\alpha|dt+\frac{(mb-a)|f'(b)|^m}{2}\int_0^{\frac{1}{2}} |-1-t^\alpha+(1-t)^\alpha|dt \\
=\ & (mb-a)|f'(b)|^m\int_0^{\frac{1}{2}} |1-(1-t)^\alpha+t^\alpha|dt \\
=\ & \frac{(mb-a)|f'(b)|^m(\alpha 2^{\alpha-1}+2^{\alpha-1}+1-2^\alpha)}{(\alpha+1)2^\alpha}.
\end{aligned}$$

The proof is completed. □

Theorem 3.2. *Let $f : [0,b] \to R$ be a differentiable mapping and $1 < q < \infty$. If $|f'|^q$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha \in (0,1]$, $0 \leq a < mb \leq b$, then the following inequality for fractional integrals holds*

$$\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}f(mb)+J^\alpha_{mb-}f(a)]-f\left(\frac{a+mb}{2}\right)\right| \leq I_k,$$

where

$$\begin{aligned}
I_k &= \frac{(mb-a)|f'(a)|}{(p\alpha+1)^{\frac{1}{p}}2^\alpha}\left[\sum_{i=1}^\infty \frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^{i-1}}{[\alpha;i-1]}\right]^{\frac{1}{q}}, \text{ for } k \neq 1, \\
I_k &= \frac{(mb-a)|f'(b)|^m}{(p\alpha+1)2^\alpha}, \text{ for } k = 1,
\end{aligned}$$

and $k = \frac{|f'(a)|^q}{|f'(b)|^{mq}}$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. (i) Case 1: $k \neq 1$. By Definition 2.1, Lemma 2.3, Lemma 1.3, and using Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}f(mb)+J^\alpha_{mb-}f(a)]-f\left(\frac{a+mb}{2}\right)\right| \\
\leq\ & \frac{mb-a}{2}\int_0^1 |h(t)-(1-t)^\alpha+t^\alpha||f^{'}(ta+m(1-t)b)|dt \\
\leq\ & \frac{mb-a}{2}\left[\int_0^1 |h(t)-(1-t)^\alpha+t^\alpha|^p dt\right]^{\frac{1}{p}}\left[\int_0^1 |f'(ta+m(1-t)b)|^q dt\right]^{\frac{1}{q}} \\
\leq\ & \frac{mb-a}{2}\left[2\int_0^{\frac{1}{2}} (1+t^\alpha-(1-t)^\alpha)^p dt\right]^{\frac{1}{p}}\left[\int_0^1 |f'(ta+m(1-t)b)|^q dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{mb-a}{2}\left[2\int_0^{\frac{1}{2}}(1+t^{\alpha}-(1-t^{\alpha}))^p dt\right]^{\frac{1}{p}}\left[\int_0^1 |f'(ta+m(1-t)b)|^q dt\right]^{\frac{1}{q}}\\
&\leq (mb-a)2^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t^{p\alpha}dt\right)^{\frac{1}{p}}\left[\int_0^1 |f'(ta+m(1-t)b)|^q dt\right]^{\frac{1}{q}}\\
&\leq \frac{mb-a}{(p\alpha+1)^{\frac{1}{p}}2^{\alpha}}\left[\int_0^1 |f'(ta+m(1-t)b)|^q dt\right]^{\frac{1}{q}}\\
&\leq \frac{mb-a}{(p\alpha+1)^{\frac{1}{p}}2^{\alpha}}\left[\int_0^1 |f'(a)|^{qt^{\alpha}}|f'(b)|^{mq-mqt^{\alpha}}dt\right]^{\frac{1}{q}}\\
&= \frac{(mb-a)|f'(b)|^m}{(p\alpha+1)^{\frac{1}{p}}2^{\alpha}}\left[\int_0^1 k^{t^{\alpha}}dt\right]^{\frac{1}{q}}\\
&= \frac{(mb-a)|f'(b)|^m}{(p\alpha+1)^{\frac{1}{p}}2^{\alpha}}\left[k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{\alpha(\frac{1}{\alpha})_i}\right]^{\frac{1}{q}}\\
&= \frac{(mb-a)|f'(a)|}{(p\alpha+1)^{\frac{1}{p}}2^{\alpha}}\left[\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{\alpha(\frac{1}{\alpha})_i}\right]^{\frac{1}{q}}\\
&= \frac{(mb-a)|f'(a)|}{(p\alpha+1)^{\frac{1}{p}}2^{\alpha}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^{i-1}}{[\alpha;i-1]}\right]^{\frac{1}{q}}.
\end{aligned}
$$

(ii) Case 2: $k=1$. By Definition 2.1, Lemma 2.3, Lemma1.3, and using Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J_{a+}^{\alpha}f(mb)+J_{mb-}^{\alpha}f(a)]-f\left(\frac{a+mb}{2}\right)\right|\\
&\leq \frac{mb-a}{(p\alpha+1)2^{\alpha+2p^{-1}}}\left[\int_0^1 |f'(a)|^{qt^{\alpha}}|f'(b)|^{mq-mqt^{\alpha}}dt\right]^{\frac{1}{q}}\\
&= \frac{(mb-a)|f'(b)|^m}{(p\alpha+1)2^{\alpha}}.
\end{aligned}
$$

The proof is done. □

4. The Second Main Results

In this section, we apply another fractional integrals identity to derive some new left Hermite-Hadamard type inequalities for twice differentiable (α,m)-logarithmically convex functions.

Now we are ready to present the main results in this section.

Theorem 4.1. *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f''|$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha\in(0,1]$, $0\leq a<mb\leq b$, then the following inequality for fractional integrals holds*

$$
\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)]-f\left(\frac{a+mb}{2}\right)\right| \quad\leq\quad I_k,
$$

where

$$\begin{aligned} I_k &= \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\sum_{i=1}^{\infty}\Big(\frac{k^{2-\alpha}}{2^{i\alpha+1-\alpha}}-k\Big)\Big(\frac{1}{(2\beta)_i}-\frac{1}{(\beta)_i}\Big)(-\ln\ k)^{i-1},\ for\ k\neq 1,\\ I_k &= \frac{(mb-a)^2|f''(b)|^m}{8},\ for\ k=1, \end{aligned}$$

and $k=\frac{|f''(a)|}{|f''(b)|^m}$.

Proof. (i) Case 1: $k\neq 1$. By Definition 2.1, Lemma 2.2, Lemma 2.4 and Lemma 1.5, we have

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)]-f\left(\frac{a+mb}{2}\right)\right|\\ \leq\ & \frac{(mb-a)^2}{2}\int_0^1|g(t)||f''(ta+m(1-t)b)|dt\\ \leq\ & \frac{(mb-a)^2}{2}\int_0^1|g(t)||f''(a)|^{t^\alpha}|f''(b)|^{m-mt^\alpha}dt\\ \leq\ & \frac{(mb-a)^2|f''(b)|^m}{2}\int_0^1|g(t)|k^{t^\alpha}dt\\ \leq\ & \frac{(mb-a)^2|f''(b)|^m}{2}\Big(\int_0^{\frac12}|g(t)|k^{t^\alpha}dt+\int_{\frac12}^1|g(t)|k^{t^\alpha}dt\Big)\\ =\ & \frac{(mb-a)^2|f''(b)|^m}{2}\Big(\int_0^{\frac12}\Big|t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\Big|k^{t^\alpha}dt+\int_{\frac12}^1\Big|1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\Big|k^{t^\alpha}dt\Big)\\ =\ & \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\Big(\int_0^{\frac12}\Big[t\alpha+t-1+(1-t)^{\alpha+1}+t^{\alpha+1}\Big]k^{t^\alpha}dt+\int_{\frac12}^1\Big|\alpha-t\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1}\Big|k^{t^\alpha}dt\Big)\\ \leq\ & \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\Big(\int_0^{\frac12}(t\alpha+t)k^{t^\alpha}dt+\int_{\frac12}^1(\alpha-t\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1})k^{t^\alpha}dt\Big)\\ \leq\ & \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\Big((\alpha+1)\int_0^{\frac12}tk^{t^\alpha}dt+(\alpha+1)\int_{\frac12}^1k^{t^\alpha}dt-(\alpha+1)\int_{\frac12}^1tk^{t^\alpha}dt\Big)\\ =\ & \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\Big(\int_0^{\frac{1}{2^\alpha}}t^{\frac{2}{\alpha}-1}k^tdt+\int_{\frac{1}{2^\alpha}}^1t^{\frac1\alpha-1}k^tdt-\int_{\frac{1}{2^\alpha}}^1t^{\frac2\alpha-1}k^tdt\Big)\\ =\ & \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\Big(2\int_0^{\frac{1}{2^\alpha}}t^{\frac2\alpha-1}k^tdt+\int_0^1t^{\frac1\alpha-1}k^tdt-\int_0^{\frac{1}{2^\alpha}}t^{\frac1\alpha-1}k^tdt-\int_0^1t^{\frac2\alpha-1}k^tdt\Big)\\ =\ & \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\Big[2\Big(\frac{1}{2^\alpha}\Big)^{\frac2\alpha}k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty\frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac2\alpha)_i}+k\sum_{i=1}^\infty\frac{(-\ln\ k)^{i-1}}{(\frac1\alpha)_i}\\ &-\Big(\frac{1}{2^\alpha}\Big)^{\frac1\alpha}k^{\frac{1}{2^\alpha}}\sum_{i=1}^\infty\frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac1\alpha)_i}-k\sum_{i=1}^\infty\frac{(-\ln\ k)^{i-1}}{(\frac2\alpha)_i}\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\Big[k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(2\beta)_i 2^{i\alpha+1-\alpha}} + k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(\beta)_i} \\
&\quad -k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(\beta)_i 2^{i\alpha+1-\alpha}} - k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(2\beta)_i}\Big] \\
&= \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\Big[k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{2^{i\alpha+1-\alpha}}\Big(\frac{1}{(2\beta)_i}-\frac{1}{(\beta)_i}\Big) \\
&\quad +k\sum_{i=1}^{\infty}(-\ln\ k)^{i-1}\Big(\frac{1}{(\beta)_i}-\frac{1}{(2\beta)_i}\Big)\Big] \\
&= \frac{(mb-a)^2|f''(b)|^m}{2\alpha}\sum_{i=1}^{\infty}\Big(\frac{k^{2^{-\alpha}}}{2^{i\alpha+1-\alpha}}-k\Big)\Big(\frac{1}{(2\beta)_i}-\frac{1}{(\beta)_i}\Big)(-\ln\ k)^{i-1}.
\end{aligned}
$$

(ii) Case 2: $k = 1$. By Definition 2.1, Lemma 2.2, Lemma 1.5, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[{}_{RL}J^{\alpha}_{a^+}f(mb) +_{RL} J^{\alpha}_{mb^-}f(a)] - f\left(\frac{a+mb}{2}\right)\right| \\
&\leq \frac{(mb-a)^2}{2}\int_0^1 |g(t)||f''(a)|^{t^\alpha}|f''(b)|^{m-mt^\alpha}dt \\
&\leq \frac{(mb-a)^2|f''(b)|^m}{2}\int_0^1 |g(t)|dt \\
&= \frac{(mb-a)^2|f''(b)|^m}{2}\Big(\int_0^{\frac{1}{2}}|g(t)|dt + \int_{\frac{1}{2}}^1 |g(t)|dt\Big) \\
&= \frac{(mb-a)^2|f''(b)|^m}{2}\Big(\int_0^{\frac{1}{2}}\Big|t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\Big|dt + \int_{\frac{1}{2}}^1\Big|1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\Big|dt\Big) \\
&= \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\Big(\int_0^{\frac{1}{2}}\Big[t\alpha+t-1+(1-t)^{\alpha+1}+t^{\alpha+1}\Big]dt + \int_{\frac{1}{2}}^1\Big|\alpha-t\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1}\Big|dt\Big) \\
&\leq \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\Big(\int_0^{\frac{1}{2}}(t\alpha+t)dt + \int_{\frac{1}{2}}^1(\alpha-t\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1})dt\Big) \\
&\leq \frac{(mb-a)^2|f''(b)|^m}{2}\Big(\int_0^{\frac{1}{2}}tdt + \int_{\frac{1}{2}}^1(1-t)dt\Big) \\
&= \frac{(mb-a)^2|f''(b)|^m}{8}.
\end{aligned}
$$

The proof is done. □

Theorem 4.2. *Let $f : [0,b] \to R$ be a differentiable mapping and $1 < q < \infty$. If $|f''|^q$ is measurable and (α,m)-logarithmically convex on $[a,b]$ for some fixed $\alpha \in (0,1]$, $0 \leq a < mb \leq b$, then the following inequality for fractional integrals holds*

$$
\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[{}_{RL}J^{\alpha}_{a^+}f(mb) +_{RL} J^{\alpha}_{mb^-}f(a)] - f\left(\frac{a+mb}{2}\right)\right| \leq I_k,
$$

where

$$I_k = \frac{(mb-a)^2|f''(a)|}{2(\alpha+1)(p+1)^{\frac{1}{p}}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^{i-1}}{\alpha[\alpha;i-1]}\right]^{\frac{1}{q}}, \text{ for } k\neq 1,$$
$$I_k = \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)(p+1)^{\frac{1}{p}}}, \text{ for } k=1,$$

and $k=\frac{|f''(a)|^q}{|f''(b)|^{mq}}$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. (i) Case 1: $k\neq 1$. By Definition 2.1, Lemma 2.2, Lemma 2.4, Lemma 1.5, and using Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)]-f\left(\frac{a+mb}{2}\right)\right|\\
\leq\ & \frac{(mb-a)^2}{2}\int_0^1|g(t)||f''(ta+m(1-t)b)|dt\\
\leq\ & \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1|g(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(ta+m(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{(mb-a)^2}{2(\alpha+1)}\left(\int_0^1|g(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f''(a)|^{qt^\alpha}|f''(b)|^{mq-mqt^\alpha}dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{(mb-a)^2|f''(b)|^m}{2(\alpha+1)}\left(\int_0^1|g(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 k^{t^\alpha}dt\right)^{\frac{1}{q}}\\
=\ & \frac{(mb-a)^2|f''(a)|}{2(\alpha+1)}\left(\frac{1}{(p+1)2^p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{\alpha(\frac{1}{\alpha})_i}\right)^{\frac{1}{q}}\\
=\ & \frac{(mb-a)^2|f''(a)|}{2(\alpha+1)(p+1)^{\frac{1}{p}}}\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^{i}}{\alpha[\alpha;i-1]}\right]^{\frac{1}{q}}.
\end{aligned}$$

where we use the following inequality

$$\begin{aligned}
\int_0^1|g(t)|^p dt &= \int_0^{\frac{1}{2}}|g(t)|^p dt+\int_{\frac{1}{2}}^1|g(t)|^p dt\\
&= \int_0^{\frac{1}{2}}\left|t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|^p dt+\int_{\frac{1}{2}}^1\left|1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|^p dt\\
&= \frac{1}{(\alpha+1)^p}\Big(\int_0^{\frac{1}{2}}\left[t\alpha+t-1+(1-t)^{\alpha+1}+t^{\alpha+1}\right]^p dt+\int_{\frac{1}{2}}^1\left|\alpha-t\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1}\right|^p dt\Big)\\
&\leq \frac{1}{(\alpha+1)^p}\Big(\int_0^{\frac{1}{2}}(t\alpha+t)^p dt+\int_{\frac{1}{2}}^1(\alpha-t\alpha-t+(1-t)^{\alpha+1}+t^{\alpha+1})^p dt\Big)\\
&\leq \int_0^{\frac{1}{2}}t^p dt+\int_{\frac{1}{2}}^1(1-t)^p dt\\
&= \frac{1}{(p+1)2^p}.
\end{aligned}$$

(ii) Case 2: $k = 1$. By Definition 2.1, Lemma 1.5, and using Hölder inequality, we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} [{}_{RL}J^\alpha_{a^+} f(mb) +_{RL} J^\alpha_{mb^-} f(a)] - f\left(\frac{a+mb}{2}\right) \right| \\ \leq & \frac{(mb-a)^2}{2(\alpha+1)} \left(\int_0^1 |g(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(a)|^{qt^\alpha} |f''(b)|^{mq-mqt^\alpha} dt \right)^{\frac{1}{q}} \\ = & \frac{(mb-a)^2 |f''(b)|^m}{2(\alpha+1)} \left(\int_0^1 |g(t)|^p dt \right)^{\frac{1}{p}} \\ = & \frac{(mb-a)^2 |f''(b)|^m}{(\alpha+1)} \left(\frac{1}{(p+1)2^p} \right)^{\frac{1}{p}} \\ = & \frac{(mb-a)^2 |f''(b)|^m}{2(\alpha+1)(p+1)^{\frac{1}{p}}}. \end{aligned}$$

The proof is done. □

5. The Third Main Results

In this section, we apply Lemma 1.6 to derive some new left Hermite-Hadamard type inequalities for twice differentiable (α, m)-logarithmically convex functions.

Now we are ready to present the main results in this section.

Theorem 5.1. *Let $f : [0, b] \to R$ be a differentiable mapping. If $|f''|$ is measurable and (α, m)-logarithmically convex on $[a, b]$ for some fixed $\alpha \in (0, 1]$, $0 \leq a < mb \leq b$, then the following inequality for fractional integrals holds*

$$\left| \frac{f(a) + f(mb)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha} [{}_{RL}J^\alpha_{a^+} f(mb) +_{RL} J^\alpha_{mb^-} f(a)] \right| \leq I_k,$$

where

$$\begin{aligned} I_k = & \frac{(mb-a)^2 |f''(b)|^m}{r(r+1)(\alpha+1)} \Big[k^{2^{-\alpha}} \sum_{i=1}^{\infty} \frac{(-\ln k)^{i-1}}{\alpha 2^{i\alpha-\alpha}} \Big(\frac{r+1}{(\beta)_i} + \frac{r\alpha + r}{4(2\beta)_i} \Big) \\ & + (2r + 2 + r\alpha + r) \sum_{i=1}^{\infty} \frac{(-\ln\ k)^{i-1}}{\alpha(\beta)_i} \Big(k - \frac{k^{2^{-\alpha}}}{2^{i\alpha+1-\alpha}} \Big) \\ & - r(\alpha+1) \sum_{i=1}^{\infty} \frac{(-\ln\ k)^{i-1}}{\alpha(2\beta)_i} \Big(k - \frac{k^{2^{-\alpha}}}{2^{i\alpha+2-\alpha}} \Big) \Big], \text{ for } k \neq 1, \\ I_k = & \frac{(mb-a)^2 |f''(b)|^m}{(\alpha+1)} \Big(\frac{2}{r} + \frac{\alpha+1}{4r+4} \Big), \text{ for } k = 1, \end{aligned}$$

and $k = \frac{|f''(a)|}{|f''(b)|^m}$.

Proof. (i) Case 1: $k \neq 1$. By Definition 2.1, Lemma 2.2, Lemma 2.4 and Lemma 1.6, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}[{}_{RL}J^{\alpha}_{a^+}f(mb)+{}_{RL}J^{\alpha}_{mb^-}f(a)]\right|\\
\le\;& (mb-a)^2\int_0^1 |u(t)||f''(ta+m(1-t)b)|dt\\
\le\;& (mb-a)^2\int_0^1 |u(t)||f''(a)|^{t^{\alpha}}|f''(b)|^{m-mt^{\alpha}}dt\\
=\;& (mb-a)^2|f''(b)|^m\int_0^1 |u(t)|k^{t^{\alpha}}dt\\
=\;& (mb-a)^2|f''(b)|^m\Big(\int_0^{\frac12}|u(t)|k^{t^{\alpha}}dt+\int_{\frac12}^1|u(t)|k^{t^{\alpha}}dt\Big)\\
=\;& (mb-a)^2|f''(b)|^m\Big(\int_0^{\frac12}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1}\right|k^{t^{\alpha}}dt\\
&+\int_{\frac12}^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{1-t}{r+1}\right|k^{t^{\alpha}}dt\Big)\\
=\;& \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(\int_0^{\frac12}\Big|(r+1)-(r+1)[(1-t)^{\alpha+1}+t^{\alpha+1}]-r(\alpha+1)t\Big|k^{t^{\alpha}}dt\\
&+\int_{\frac12}^1\Big|(r+1)-(r+1)[(1-t)^{\alpha+1}+t^{\alpha+1}]-r(\alpha+1)(1-t)\Big|k^{t^{\alpha}}dt\Big)\\
\le\;& \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(\int_0^{\frac12}\Big|(r+1)+(r+1)[(1-t)^{\alpha+1}+t^{\alpha+1}]+r(\alpha+1)t\Big|k^{t^{\alpha}}dt\\
&+\int_{\frac12}^1\Big|(r+1)+(r+1)[(1-t)^{\alpha+1}+t^{\alpha+1}]+r(\alpha+1)(1-t)\Big|k^{t^{\alpha}}dt\Big)\\
=\;& \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(\int_0^{\frac12}\Big|(r+1)+(r+1)+r(\alpha+1)t\Big|k^{t^{\alpha}}dt\\
&+\int_{\frac12}^1\Big|(r+1)+(r+1)+r(\alpha+1)(1-t)\Big|k^{t^{\alpha}}dt\Big)\\
=\;& \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(2(r+1)\int_0^{\frac12}k^{t^{\alpha}}dt+r(\alpha+1)\int_0^{\frac12}tk^{t^{\alpha}}dt\\
&+(2r+2+r\alpha+r)\int_{\frac12}^1 k^{t^{\alpha}}dt-r(\alpha+1)\int_{\frac12}^1 tk^{t^{\alpha}}dt\Big)\\
=\;& \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(2(r+1)\int_0^{\frac12}k^{t^{\alpha}}dt+r(\alpha+1)\int_0^{\frac12}tk^{t^{\alpha}}dt\\
&+(2r+2+r\alpha+r)\left[\int_0^1 k^{t^{\alpha}}dt-\int_0^{\frac12}k^{t^{\alpha}}dt\right]-r(\alpha+1)\left[\int_0^1 tk^{t^{\alpha}}dt-\int_0^{\frac12}tk^{t^{\alpha}}dt\right]\Big)\\
=\;& \frac{(mb-a)^2|f''(b)|^m}{r\alpha(r+1)(\alpha+1)}\Big(2(r+1)\int_0^{\frac{1}{2^{\alpha}}}t^{\frac{1}{\alpha}-1}k^t dt+r(\alpha+1)\int_0^{\frac{1}{2^{\alpha}}}t^{\frac{2}{\alpha}-1}k^t dt\\
&+(2r+2+r\alpha+r)\left[\int_0^1 t^{\frac{1}{\alpha}-1}k^t dt-\int_0^{\frac{1}{2^{\alpha}}}t^{\frac{1}{\alpha}-1}k^t dt\right]-r(\alpha+1)\left[\int_0^1 t^{\frac{2}{\alpha}-1}k^t dt-\int_0^{\frac{1}{2^{\alpha}}}t^{\frac{2}{\alpha}-1}k^t dt\right]\Big)
\end{aligned}$$

$$
\begin{aligned}
= & \frac{(mb-a)^2|f''(b)|^m}{r\alpha(r+1)(\alpha+1)}\Big[2(r+1)\Big(\frac{1}{2^\alpha}\Big)^{\frac{1}{\alpha}}k^{\frac{1}{2^\alpha}}\sum_{i=1}^{\infty}\frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac{1}{\alpha})_i}+r(\alpha+1)\Big(\frac{1}{2^\alpha}\Big)^{\frac{2}{\alpha}}k^{\frac{1}{2^\alpha}}\sum_{i=1}^{\infty}\frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac{2}{\alpha})_i}\\
& +(2r+2+r\alpha+r)\Big(k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(\frac{1}{\alpha})_i}-\Big(\frac{1}{2^\alpha}\Big)^{\frac{1}{\alpha}}k^{\frac{1}{2^\alpha}}\sum_{i=1}^{\infty}\frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac{1}{\alpha})_i}\Big)\\
& -r(\alpha+1)\Big(k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(\frac{2}{\alpha})_i}-\Big(\frac{1}{2^\alpha}\Big)^{\frac{2}{\alpha}}k^{\frac{1}{2^\alpha}}\sum_{i=1}^{\infty}\frac{\Big(-\frac{1}{2^\alpha}\ln k\Big)^{i-1}}{(\frac{2}{\alpha})_i}\Big)\Big]\\
= & \frac{(mb-a)^2|f''(b)|^m}{r\alpha(r+1)(\alpha+1)}\Big[(r+1)k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(\beta)_i2^{i\alpha-\alpha}}+r(\alpha+1)k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(2\beta)_i2^{i\alpha+2-\alpha}}\\
& +(2r+2+r\alpha+r)\Big(k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(\beta)_i}-k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(\beta)_i2^{i\alpha+1-\alpha}}\Big)\\
& -r(\alpha+1)\Big(k\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(2\beta)_i}-k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{(2\beta)_i2^{i\alpha+2-\alpha}}\Big)\Big]\\
= & \frac{(mb-a)^2|f''(b)|^m}{r\alpha(r+1)(\alpha+1)}\Big[k^{2^{-\alpha}}\sum_{i=1}^{\infty}\frac{(-\ln k)^{i-1}}{2^{i\alpha-\alpha}}\Big(\frac{r+1}{(\beta)_i}+\frac{r\alpha+r}{4(2\beta)_i}\Big)\\
& +(2r+2+r\alpha+r)\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(\beta)_i}\Big(k-\frac{k^{2^{-\alpha}}}{2^{i\alpha+1-\alpha}}\Big)-r(\alpha+1)\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{(2\beta)_i}\Big(k-\frac{k^{2^{-\alpha}}}{2^{i\alpha+2-\alpha}}\Big)\Big].
\end{aligned}
$$

(ii) Case 2: $k=1$. By Definition 2.1, Lemma 1.6, we have

$$
\begin{aligned}
& \left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)]\right|\\
\leq & (mb-a)^2\int_0^1|u(t)||f''(ta+m(1-t)b)|dt\\
\leq & (mb-a)^2\int_0^1|u(t)||f''(a)|^{t^\alpha}|f''(b)|^{m-mt^\alpha}dt\\
= & (mb-a)^2|f''(b)|^m\int_0^1|u(t)|dt\\
= & (mb-a)^2|f''(b)|^m\Big(\int_0^{\frac{1}{2}}|u(t)|k^{t^\alpha}dt+\int_{\frac{1}{2}}^1|u(t)|dt\Big)\\
= & (mb-a)^2|f''(b)|^m\Big(\int_0^{\frac{1}{2}}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1}\right|dt+\int_{\frac{1}{2}}^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{1-t}{r+1}\right|dt\Big)\\
= & \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(\int_0^{\frac{1}{2}}\Big|(r+1)+(r+1)[(1-t)^{\alpha+1}+t^{\alpha+1}]+r(\alpha+1)t\Big|dt\\
& +\int_{\frac{1}{2}}^1\Big|(r+1)+(r+1)[(1-t)^{\alpha+1}+t^{\alpha+1}]+r(\alpha+1)(1-t)\Big|dt\Big)
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(\int_0^{\frac{1}{2}}\Big|(r+1)+(r+1)+r(\alpha+1)t\Big|dt+\int_{\frac{1}{2}}^1\Big|(r+1)+(r+1)+r(\alpha+1)(1-t)\Big|dt\Big)\\
&= \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(\Big|(r+1)\frac{1}{2}+(r+1)\frac{1}{2}+r(\alpha+1)\frac{1}{8}\Big|+\Big|(r+1)\frac{1}{2}+(r+1)\frac{1}{2}+r(\alpha+1)\frac{1}{2}-r(\alpha+1)\frac{3}{8}\Big|\Big)\\
&= \frac{(mb-a)^2|f''(b)|^m}{r(r+1)(\alpha+1)}\Big(2(r+1)+\frac{r(\alpha+1)}{4}\Big)\\
&= \frac{(mb-a)^2|f''(b)|^m}{(\alpha+1)}\Big(\frac{2}{r}+\frac{\alpha+1}{4r+4}\Big).
\end{aligned}$$

The proof is done. □

Theorem 5.2. *Let $f : [0, b] \to R$ be a differentiable mapping and $1 < q < \infty$. If $|f''|^q$ is measurable and (α, m)-logarithmically convex on $[a, b]$ for some fixed $\alpha \in (0, 1]$, $0 \le a < mb \le b$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)]\right| \le I_k,$$

where

$$\begin{aligned}
I_k &= \frac{(mb-a)^2|f''(a)|}{(\alpha+1)}\left(\frac{p}{P+1}\right)^{\frac{1}{p}}\left(\frac{1-2^{-\alpha}}{r(\alpha+1)}+\frac{1}{2(r+1)}\right)\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^i}{\alpha[\alpha;i-1]}\right]^{\frac{1}{q}}, \textit{ for } k\neq 1,\\
I_k &= (mb-a)^2|f''(b)|^m\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left(\frac{1-2^{-\alpha}}{r(\alpha+1)}+\frac{1}{2(r+1)}\right), \textit{ for } k=1,
\end{aligned}$$

and $k = \frac{|f''(a)|^q}{|f''(b)|^{mq}}$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. (i) Case 1: $k \neq 1$. By Definition 2.1, Lemma 2.2, Lemma 2.4, Lemma 1.6, and using Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)]\right|\\
&\le (mb-a)^2\int_0^1|u(t)||f''(ta+m(1-t)b)|dt\\
&\le (mb-a)^2\left(\int_0^1|u(t)|^pdt\right)^{\frac{1}{p}}\left(\int_0^1|f''(ta+m(1-t)b)|^qdt\right)^{\frac{1}{q}}\\
&\le (mb-a)^2\left(\int_0^1|u(t)|^pdt\right)^{\frac{1}{p}}\left(\int_0^1|f''(a)|^{qt^\alpha}|f''(b)|^{mq-mqt^\alpha}dt\right)^{\frac{1}{q}}\\
&= (mb-a)^2|f''(b)|^m\left(\int_0^1|u(t)|^pdt\right)^{\frac{1}{p}}\left(\int_0^1k^{t^\alpha}dt\right)^{\frac{1}{q}}\\
&= (mb-a)^2|f''(a)|\left(\frac{p}{P+1}\right)^{\frac{1}{p}}\left(\frac{1-2^{-\alpha}}{r(\alpha+1)}+\frac{1}{2(r+1)}\right)\left(\sum_{i=1}^{\infty}\frac{(-\ln\ k)^{i-1}}{\alpha(\frac{1}{\alpha})_i}\right)^{\frac{1}{q}}\\
&= \frac{(mb-a)^2|f''(a)|}{(\alpha+1)}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left(\frac{1-2^{-\alpha}}{r(\alpha+1)}+\frac{1}{2(r+1)}\right)\left[\sum_{i=1}^{\infty}\frac{(mq\ln|f'(b)|-q\ln|f'(a)|)^{i-1}\alpha^i}{\alpha[\alpha;i-1]}\right]^{\frac{1}{q}},
\end{aligned}$$

where we use the following inequality

$$\begin{aligned}
\int_0^1 |u(t)|^p dt &= \int_0^{\frac{1}{2}} \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1} \right|^p dt \\
&\quad + \int_{\frac{1}{2}}^1 \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1} \right|^p dt \\
&\leq \int_0^{\frac{1}{2}} \left| \frac{1-2^{-\alpha}}{r(\alpha+1)} + \frac{t}{r+1} \right|^p dt + \int_{\frac{1}{2}}^1 \left| \frac{1-2^{-\alpha}}{r(\alpha+1)} + \frac{1-t}{r+1} \right|^p dt \\
&= \frac{p}{p+1} \left(\frac{1-2^{-\alpha}}{r(\alpha+1)} + \frac{1}{2(r+1)} \right)^p.
\end{aligned}$$

(ii) Case 2: $k = 1$. By Definition 2.1, Lemma 1.6, and using Hölder inequality, we have

$$\begin{aligned}
&\left| \frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha} [{}_{RL}J_{a^+}^{\alpha} f(mb) + {}_{RL}J_{mb^-}^{\alpha} f(a)] \right| \\
\leq\ & (mb-a)^2 \int_0^1 |u(t)||f''(ta+m(1-t)b)| dt \\
\leq\ & (mb-a)^2 \left(\int_0^1 |u(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta+m(1-t)b)|^q dt \right)^{\frac{1}{q}} \\
\leq\ & (mb-a)^2 \left(\int_0^1 |u(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(a)|^{qt^\alpha} |f''(b)|^{mq-mqt^\alpha} dt \right)^{\frac{1}{q}} \\
=\ & (mb-a)^2 |f''(b)|^m \left(\int_0^1 |u(t)|^p dt \right)^{\frac{1}{p}} \\
=\ & (mb-a)^2 |f''(b)|^m \left(\frac{p}{p+1} \right)^{\frac{1}{p}} \left(\frac{1-2^{-\alpha}}{r(\alpha+1)} + \frac{1}{2(r+1)} \right).
\end{aligned}$$

The proof is done. □

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References

[1] D. Baleanu, J. A. T. Machado, A. C. J. Luo, Fractional dynamics and control, Springer, 2012.
[2] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, 2010.
[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier Science B.V., 2006.
[4] V. Lakshmikantham, S. Leela, J. V. Devi, Theory of fractional dynamic systems, Cambridge Scientific Publishers, 2009.
[5] K. S. Miller, B. Ross, An introduction to the fractional calculus and differential equations, John Wiley, 1993.
[6] M. W. Michalski, Derivatives of noninteger order and their applications, Dissertationes Mathematicae, CCCXXVIII, Inst. Math., Polish Acad. Sci., 1993.
[7] I. Podlubny, Fractional differential equations, Academic Press, 1999.
[8] V. E. Tarasov, Fractional dynamics: Application of fractional calculus to dynamics of particles, fields and media, Springer, HEP, 2011.
[9] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., 57(2013), 2403-2407.

[10] C. Zhu, M. Fečkan, J. Wang, Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula, J. Appl. Math. Statistics Inform., 8(2012), 21-28.

[11] J. Wang, X. Li, C. Zhu, Refinements of Hermite-Hadamard type inequalities involving fractional integrals, Bull. Belg. Math. Soc. Simon Stevin, 20(2013), 655-666.

[12] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Comput. Math. Appl., 63(2012), 1147-1154.

[13] I. Işcan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, J. Inequal. Appl., 2013(2013):491, 1-15.

[14] J. Wang, J. Deng, M. Fečkan, Hermite-Hadamard type inequalities for r-convex functions via Riemann-Liouville fractional integrals, Ukrainian Math. J., 65(2013), 193-211.

[15] Y. Zhang, J. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, J. Inequal Appl., 2013(2013):220, 1-27.

[16] J. Wang, X. Li, M. Fečkan, Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, Appl. Anal., 92(2013), 2241-2253.

[17] J. Deng, J. Wang, Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions, J. Inequal. Appl., 2013(2013):364, 1-11.

[18] J. Wang, J. Deng, M. Fečkan, Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals, Math. Slovaca, 64 (2014), 1381–1396.

[19] R. Bai, F. Qi, B. Xi, Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions, Filomat, 27(2013), 1-7.