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Neighborhood Structures of Graded Ditopological Texture Spaces

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Abstract. Graded ditopological texture spaces have been presented and discussed in categorical aspects by Alexander Šostak and Lawrence M. Brown in [18]. In this paper, the authors study the generalization of two sorts of neighborhood structure defined in [10] to the graded ditopological texture spaces.

1. Introduction

The concept of fuzzy topological space was defined in 1968 by C. Chang as ordinary subset of the family of all fuzzy subsets of a given set [8]. As a more suitable approach to the idea of fuzzyness, in 1985, Šostak and Kubiak independently redefined fuzzy topology where a fuzzy subset has a degree of openness rather than being open or not [13, 16] (for historical developments and basic ideas of the theory of fuzzy topology see [17]). Seven years later, this setting rediscovered in [9] under the name "graded fuzzy topology" and in [15] under the name "smooth topology".

M. Demirci introduced two sorts of neighborhood structure of graded fuzzy topological spaces and investigated their properties [10].

In classical topology the notion of open set is usually taken as primitive with that of closed set being auxiliary. However, since the closed sets are easily obtained as the complements of open sets they often play an important, sometimes dominating role in topological arguments. A similar situation holds for topologies on lattices where the role of set complement is played by an order reversing involution. It is the case, however, that there may be no order reversing involution available, or that the presence of such an involution is otherwise irrelevant to the topic under consideration. To deal with such cases it is natural to consider a topological structure consisting of *a priori* unrelated families of open sets and of closed sets. This was the approach adapted from the beginning for the topological structures on textures, originally introduced as a point-based representation for fuzzy sets [2, 3]. These topological structures were given the name dichotomous topology, or ditopology for short. They consist of a family τ of open sets and a generally unrelated family κ of closed sets. Hence, both the open and the closed sets are regarded as primitive concepts for a ditopology.

A ditopology (τ, κ) on the discrete texture $(X, \mathcal{P}(X))$ gives rise to a bitopological space (X, τ, κ^c). This link with bitopological spaces has had a powerful influence on the development of the theory of ditopological texture spaces, but it should be emphasized that a ditopology and a bitopology are conceptually different.

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Indeed, a bitopology consists of two separate topological structures (complete with their open and closed sets) whose interrelations we wish to study, whereas a ditopology represents a single topological structure.

Ditopological texture spaces were introduced by L. M. Brown as a natural extention of the work of second autor on the representation of lattice-valued topologies by bitopologies in [11]. The concept of ditopology is more general than general topology, bitopology and fuzzy topology in Chang's sense. An adequate introduction to the theory of texture spaces and ditopological texture spaces may be obtained from [2–7].

Recently, A. Šostak and L.M. Brown have presented the concept "graded ditopology" on textures as an extension of the concept of ditopology to the case where openness and closedness are given in terms of a priori unrelated grading functions [18]. The concept of graded ditopology is more general than ditopology and graded fuzzy topology.

The aim of this work is to generalize two sorts of neighborhood structure in graded fuzzy topological spaces defined by M. Demirci [10] to the graded ditopological texture spaces which is introduced by A. Šostak and L. M. Brown in [18]. The material in this work forms a part of first named author's Ph.D. Thesis, currently being written under the supervision of the second name author Dr. Rıza Ertürk.

2. Preliminaries

We recall various concepts and properties from [3, 5–7] under the following subtitle.

Ditopological Texture Spaces: Let S be a set. A texturing $\mathcal{S}$ on S is a subset of $\mathcal{P}(S)$ which is a point separating, complete, completely distributive lattice with respect to inclusion which contains S, $\emptyset$ and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S,\mathcal{S})$ is then called a texture or a texture space.

In general, a texturing of S need not be closed under set complementation, but there may exist a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a complementation on $(S,\mathcal{S})$ and $(S,\mathcal{S},\sigma)$ is said to be a complemented texture.

For a texture $(S,\mathcal{S})$, most properties are conveniently defined in terms of the $p-sets$

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the $q-sets$

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.$$

A texture $(S,\mathcal{S})$ is called a plain texture if it satisfies any of the following equivalent conditions:

1. $P_s \not\subseteq Q_s$ for all $s \in S$
2. $A = \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ for all $A_i \in \mathcal{S}, i \in I$

Recall that $M \in \mathcal{S}$ is called a molecule if $M \neq \emptyset$ and $M \subseteq A \cup B$, $A, B \in \mathcal{S}$ implies $M \subseteq A$ or $M \subseteq B$. The sets $P_s, s \in \mathcal{S}$ are molecules, and the texture $(S,\mathcal{S})$ is called "simple" if these are the only molecules in $\mathcal{S}$. For a set $A \in \mathcal{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Theorem 2.1. *[5] In any texture space $(S,\mathcal{S})$, the following statements hold:*

1. $s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ *for all* $s \in S$, $A \in \mathcal{S}$.
2. $A^\flat = \{s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
3. *For* $A_j \in \mathcal{S}$, $j \in J$ *we have* $(\bigvee_{j\in J} A_j)^\flat = \bigcup_{j\in J} A_j^\flat$.
4. *A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all $A \in \mathcal{S}$.*
5. *For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*
6. $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.

7. *$A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*

Let $\mathbb{L}$ be a fuzzy lattice, i.e. a completely distributive lattice with order reversing involution $'$ and L denote the set of molecules in $\mathbb{L}$ and $\mathcal{L} = \{\varphi(a) | a \in \mathbb{L}\}$ where $\varphi(a) = \{m \in L | m \leq a\}$ for $a \in \mathbb{L}$. Then:

Theorem 2.2. *[3] For the above notations, $(L, \mathcal{L})$ is a simple texture with complement $\lambda(\varphi(a)) = \varphi(a')$, $a \in \mathbb{L}$ and $\varphi : \mathbb{L} \to \mathcal{L}$ is a lattice isomorphism which preserves complementation.*

Conversely, every complemented simple texture may be obtained in this way from a suitable fuzzy lattice.

Example 2.3. (1) If $\mathcal{P}(X)$ is the powerset of a set X, then $(X, \mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X, \mathcal{P}(X))$.
(2) Setting $\mathbb{I} = [0,1]$, $\mathcal{J} = \{[0,r), [0,r] \,|r \in \mathbb{I}\}$ gives the unit interval texture $(\mathbb{I}, \mathcal{J})$. For $r \in \mathbb{I}$, $P_r = [0,r]$ and $Q_r = [0,r)$. And the mapping $\iota : \mathcal{J} \to \mathcal{J}$, $\iota[0,r] = [0, 1-r)$, $\iota[0,r) = [0,1-r]$ is a complementation on this texture.
(3) The texture $(L, \mathcal{L}, \lambda)$ is defined by $L = (0,1]$, $\mathcal{L} = \{(0,r] \mid r \in [0,1]\}$, $\lambda((0,r]) = (0, 1-r]$. For $r \in L$, $P_r = (0,r] = Q_r$. This texture corresponds to fuzzy lattice $(\mathbb{I} = [0,1],')$ in the sense of Theorem 2.2.
(4) Let $X \neq \emptyset$, W be the set of "fuzzy points" of $\mathbb{I}^X$, i.e. the functions

$$x_m(z) = \begin{cases} m, & z = x \\ 0, & \text{otherwise} \end{cases}$$

for $x \in X$ and $m \in L = (0,1]$, where as before L is the set of molecules of $\mathbb{I}$. Representing x_m by the pair (x,m), it can be written that $W = X \times L$. Then $(W, \mathcal{W}, \omega)$ is the texture corresponds to fuzzy lattice $\mathbb{I}^X$ in the sense of Theorem 2.2. where $\mathcal{W} = \{\varphi(f) \mid f \in \mathbb{I}^X\}$, $\varphi(f) = \{(x,m) \in W \mid x_m \leq f\} = \{(x,m) \in W \mid m \leq f(x)\}$ and $\omega(\varphi(f)) = \varphi(f')$.
(5) $\mathcal{S} = \{\emptyset, \{a,b\}, \{b\}, \{b,c\}, S\}$ is a simple texturing of $S = \{a,b,c\}$. $P_a = \{a,b\}$, $P_b = \{b\}$, $P_c = \{b,c\}$. It is not possible to define a complementation on $(S, \mathcal{S})$.
(6) If $(S, \mathcal{S})$, $(V, \mathcal{V})$ are textures, the product texturing $\mathcal{S} \otimes \mathcal{V}$ of $S \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{V}$, and $(S \times V, \mathcal{S} \otimes \mathcal{V})$ is called the product of $(S, \mathcal{S})$ and $(V, \mathcal{V})$. For $s \in S, v \in V$, $P_{(s,v)} = P_s \times P_v$ and $Q_{(s,v)} = (Q_s \times V) \cup (S \times Q_v)$.

A dichotomous topology, or ditopology for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of open sets τ satisfies

(T_1) $S, \emptyset \in \tau$
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no priori relation between the open and closed sets.

Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

1. If $s \in S^\flat$, a neighborhood of s is a set $N \in \mathcal{S}$ for which there exist $G \in \tau$ satisfying $P_s \subseteq G \subseteq N \not\subseteq Q_s$.
2. If $s \in S$, a coneighborhood of s is a set $M \in \mathcal{S}$ for which there exist $K \in \kappa$ satisfying $P_s \not\subseteq M \subseteq K \subseteq Q_s$.

If the set of nhds (conhds) of s is denoted by $\eta(s)$ $(\mu(s))$ respectively, then (η, μ) is called dinhd system of (τ, κ).

Theorem 2.4. *[7] For a ditopology (τ, κ) on $(S, \mathcal{S})$ let the families $\eta(s)$, $s \in S^\flat$ and $\mu(s)$, $s \in S$ be defined as above.*
(1) For each $s \in S^\flat$ we have $\eta(s) \neq \emptyset$ and these families satisfy the following conditions:

(i) $N \in \eta(s) \Rightarrow N \not\subseteq Q_s$
(ii) $N \in \eta(s), N \subseteq N' \in \mathcal{S} \Rightarrow N' \in \eta(s)$
(iii) $N_1, N_2 \in \eta(s), N_1 \cap N_2 \not\subseteq Q_s \Rightarrow N_1 \cap N_2 \in \eta(s)$
(iv) *(a)* $N \in \eta(s) \Rightarrow \exists N^\star \in \mathcal{S}, P_s \subseteq N^\star \subseteq N$, *so that* $N^\star \not\subseteq Q_t \Rightarrow N^\star \in \eta(t), \forall t \in S^\flat$
(b) For $N \in \mathcal{S}$ *and* $N \not\subseteq Q_s$*, if there exists* $N^\star \in \mathcal{S}, P_s \subseteq N^\star \subseteq N$*, which satisfies* $N^\star \not\subseteq Q_t \Rightarrow N^\star \in \eta(t), \forall t \in S^\flat$*, then* $N \in \eta(s)$*.*

Moreover, the sets G in τ *are characterized by the condition that* $G \in \eta(s)$ *for all s with* $G \not\subseteq Q_s$*.*
(2) For each $s \in S$ *we have* $\mu(s) \neq \emptyset$ *and these families satisfy the following conditions:*

(i) $M \in \mu(s) \Rightarrow P_s \not\subseteq M$
(ii) $M \in \mu(s), M \supseteq M' \in \mathcal{S} \Rightarrow M' \in \mu(s)$
(iii) $M_1, M_2 \in \mu(s) \Rightarrow M_1 \cup M_2 \in \mu(s)$
(iv) *(a)* $M \in \mu(s) \Rightarrow \exists M^\star \in \mathcal{S}, M \subseteq M^\star \subseteq Q_s$*, so that* $P_t \not\subseteq M^\star \Rightarrow M^\star \in \mu(t), \forall t \in S$
(b) For $M \in \mathcal{S}$ *and* $P_s \not\subseteq M$*, if there exists* $M^\star \in \mathcal{S}$*,* $M \subseteq M^\star \subseteq Q_s$*, which satisfies* $P_t \not\subseteq M^\star \Rightarrow M^\star \in \mu(t), \forall t \in S$*, then* $M \in \mu(s)$*.*

Moreover, the sets K in κ *are characterized by the condition that* $K \in \mu(s)$ *for all s with* $P_s \not\subseteq K$*.*

Conversely, if $\eta(s), s \in S^\flat$ *and* $\mu(s), s \in S$ *are non-empty families of sets in* $\mathcal{S}$ *which satisfy conditions (1) and (2) above, respectively, then there exists a ditopology* (τ, κ) *on* $(S, \mathcal{S})$ *for which* $\eta(s)$ $(\mu(s))$ *are the families of nhds (resp., conhds).*

A complementation σ on a texture $(S, \mathcal{S})$ is called "grounded" [14] if there is an involution $s \mapsto s'$ on S such that $\sigma(P_s) = Q_{s'}$ and $\sigma(Q_s) = P_{s'}$ (s' will be denoted by $\sigma(s)$) for all $s \in S$ and in this case the complemented texture space $(S, \mathcal{S}, \sigma)$ is called "complemented grounded texture space". It is obtained that a complemented plain texture is grounded in [19].

It is well known that, in the classical set theory, "$A \cap B = \emptyset$ if and only if $A \subset X - B$ for any subsets, A and B, of X" and in the fuzzy set theory, " $A \cap B = 0$ implies $A \subset 1 - B$ for any fuzzy subsets, A and B, of X". So it can be defined an alternative binary implication in the fuzzy set theory such as; " A is quasi-coincident with B (denoted by AqB) if and only if there exists an $x \in X$ such that $A(x) + B(x) > 1$ for any fuzzy sets, A and B, of X", and "A is not-quasi-coincident with B (denoted by $A\bar{q}B$) if and only if $A(x) + B(x) \leq 1$ for all $x \in X$". These notions are generalized to the complemented texture spaces in [15] as follows:

Let $(S, \mathcal{S}, \sigma)$ be a complemented texture space and $A, B \in \mathcal{S}$. It is called that "A is quasi-coincident with B" (denoted by AqB) if $A \not\subseteq \sigma(B)$ and "A is not quasi-coincident with B" (denoted by $A\bar{q}B$) if $A \subseteq \sigma(B)$ [12].

Graded Ditopological Texture Spaces: In this subsection, we recall various concepts and properties from [18] that will be needed later on this paper.

Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and consider $\mathcal{T}, \mathcal{K} : \mathcal{S} \to \mathcal{V}$ satisfying

(GT_1) $\mathcal{T}(S) = \mathcal{T}(\emptyset) = V$
(GT_2) $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)$ $\forall A_1, A_2 \in \mathcal{S}$
(GT_3) $\bigcap_{j \in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$

and

(GCT_1) $\mathcal{K}(S) = \mathcal{K}(\emptyset) = V$
(GCT_2) $\mathcal{K}(A_1) \cap \mathcal{K}(A_2) \subseteq \mathcal{K}(A_1 \cup A_2)$ $\forall A_1, A_2 \in \mathcal{S}$
(GCT_3) $\bigcap_{j \in J} \mathcal{K}(A_j) \subseteq \mathcal{K}(\bigcap_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$

Then $\mathcal{T}$ is called a $(V, \mathcal{V})$-graded topology, $\mathcal{K}$ a $(V, \mathcal{V})$-graded cotopology and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. The tuple $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is called graded ditopological texture space. For $v \in V$ it is defined that

$$\mathcal{T}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{T}(A)\}, \ \mathcal{K}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{K}(A)\}.$$

Then $(\mathcal{T}^v, \mathcal{K}^v)$ is a ditopology on $(S, \mathcal{S})$ for each $v \in V$. That is, if $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is any graded ditopological texture space, then there exists a ditopology $(\mathcal{T}^v, \mathcal{K}^v)$ on the texture space $(S, \mathcal{S})$ for each $v \in V$

If $(S, \mathcal{S}, \sigma)$ is a complemented texture and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$, then $(\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$ is also a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. $(\mathcal{T}, \mathcal{K})$ is called complemented if $(\mathcal{T}, \mathcal{K}) = (\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$.

Example 2.5. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathcal{V})$ the discrete texture on a singleton. Take $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and define $\tau^g : \mathcal{S} \to \mathcal{P}(1)$ by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$. Then τ^g is a $(V, \mathcal{V})$-graded topology on $(S, \mathcal{S})$. Likewise, κ^g defined by $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ is a $(V, \mathcal{V})$-graded cotopology on $(S, \mathcal{S})$. (τ^g, κ^g) is called a graded ditopology on $(S, \mathcal{S})$ corresponding to ditopology (τ, κ).

Therefore graded ditopological texture spaces are more general than ditopological texture spaces.

For a mapping $K : \mathcal{S} \to \mathcal{V}$ and $v \in V$, we define the family ${}^vK = \{A \in \mathcal{S} : K(A) \not\subseteq Q_v\}$ and thus for each $v \in V$, we have ${}^vK \subseteq K^v$.

3. Graded Dineighborhood Systems

Definition 3.1. Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$ and $N : S^\flat \to \mathcal{V}^\mathcal{S}$, $M : S \to \mathcal{V}^\mathcal{S}$ mappings where $N(s) = N_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S^\flat$ and $M(s) = M_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S$. Then the mapping N_s is called a "graded neighborhood system of s" if

$$ {}^vN_s = \{A \in \mathcal{S} : \mathcal{T}(B) \not\subseteq Q_v \textit{ and } P_s \subseteq B \subseteq A \not\subseteq Q_s \textit{ for some } B \in \mathcal{S}\} \tag{1} $$

for each $v \in V^\flat$ and the mapping M_s is called a "graded coneighborhood system of s" if

$$ {}^vM_s = \{A \in \mathcal{S} : \mathcal{K}(B) \not\subseteq Q_v \textit{ and } P_s \not\subseteq A \subseteq B \subseteq Q_s \textit{ for some } B \in \mathcal{S}\} \tag{2} $$

for each $v \in V^\flat$. The mapping N (M) is called a "graded neighborhood system" ("graded coneighborhood system") of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ if N_s (M_s) is a graded neighborhood system for each $s \in S^\flat$ (graded coneighborhood system for each $s \in S$) respectively. (N, M) is called "graded dineighborhood system" of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ if N is a graded neighborhood system and M is a graded coneighborhood system of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

Proposition 3.2. *For the above notations, (N, M) is a graded dinhd system of a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ iff*

$$ N_s(A) = \begin{cases} sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}, & A \not\subseteq Q_s \\ \emptyset, & A \subseteq Q_s \end{cases} \tag{3} $$

for each $s \in S^\flat$, $A \in \mathcal{S}$ and

$$ M_s(A) = \begin{cases} sup\{\mathcal{K}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}, & P_s \not\subseteq A \\ \emptyset, & P_s \subseteq A \end{cases} \tag{4} $$

for each $s \in S$, $A \in \mathcal{S}$.

Proof. $(\Longrightarrow)$: Let $s \in S^\flat$, $A \in \mathcal{S}$ and N_s be a graded nhd system of s. If $A \subseteq Q_s$ then by the equation (1), $A \notin {}^vN_s$ for each $v \in V^\flat$ so $N_s(A) \subseteq Q_v$ for each $v \in V^\flat$ since ${}^vN_s = \{A \in \mathcal{S} : N_s(A) \not\subseteq Q_v\}$. If we assume that $N_s(A) \neq \emptyset$, then we get $N_s(A)^\flat \neq \emptyset$ and therefore there exists a $t \in V^\flat$ such that $t \in N_s(A)^\flat$. That implies the contradiction $N_s(A) \not\subseteq Q_t$. This means that if $A \subseteq Q_s$, then we have $N_s(A) = \emptyset$.

Now let $A \not\subseteq Q_s$, and for any $s \in S^\flat$ suppose that $N_s(A) \not\subseteq sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s\}$. Then we have $N_s(A) \not\subseteq Q_v$ and $P_v \not\subseteq sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$ for some $v \in V$. Moreover $v \in V^\flat$ since $V \neq Q_v$, so we have $A \in {}^vN_s$. Thus, considering the equation (1), there exists a $B \in \mathcal{S}$ such that $\mathcal{T}(B) \not\subseteq Q_v$ and $P_s \subseteq B \subseteq A \not\subseteq Q_s$. It follows that $sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\} \not\subseteq Q_v$ which is in contradiction with $P_v \not\subseteq sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$, i.e. $N_s(A) \subseteq sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$. Similarly it can be obtained that $sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\} \subseteq N_s(A)$. So we have $N_s(A) = sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$.

$(\Longleftarrow)$: Let (3) holds for each $s \in S^\flat$, $A \in \mathcal{S}$. Then we obtain

$\mathcal{T}(B) \not\subseteq Q_v$ *and* $P_s \subseteq B \subseteq A \not\subseteq Q_s$ *for some* $B \in \mathcal{S} \Leftrightarrow N_s(A) = sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\} \not\subseteq Q_v \Leftrightarrow A \in {}^vN_s$

for each $s \in S^\flat$, $v \in V^\flat$. So N is a graded nhd system.

The proof of the conhd part can be shown by using the above method. □

Theorem 3.3. *Let $(\mathcal{T},\mathcal{K})$ be a $(V,\mathcal{V})$-graded ditopology on texture $(S,\mathcal{S})$. If (N,M) is the graded dinhd system of graded ditopological texture space $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$, then the following properties hold for all $A,A_1,A_2\in\mathcal{S}$:*
(**1**) *For each $s\in S^\flat$;*
(**N1**) $N_s(A)\neq\emptyset\Rightarrow A\not\subseteq Q_s$
(**N2**) $N_s(\emptyset)=\emptyset$ *and* $N_s(S)=V$
(**N3**) $A_1\subseteq A_2\Rightarrow N_s(A_1)\subseteq N_s(A_2)$
(**N4**) $A_1\cap A_2\not\subseteq Q_s\Rightarrow N_s(A_1)\wedge N_s(A_2)\subseteq N_s(A_1\cap A_2)$
(**N5**) $N_s(A)\subseteq sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}$

(**2**) *For each $s\in S$;*
(**M1**) $M_s(A)\neq\emptyset\Rightarrow P_s\not\subseteq A$
(**M2**) $M_s(S)=\emptyset$ *and* $M_s(\emptyset)=V$
(**M3**) $A_1\subseteq A_2\Rightarrow M_s(A_2)\subseteq M_s(A_1)$
(**M4**) $M_s(A_1)\wedge M_s(A_2)\subseteq M_s(A_1\cup A_2)$
(**M5**) $M_s(A)\subseteq sup\{\bigwedge_{s'\in(S\setminus B)}M_{s'}(B):P_s\not\subseteq A\subseteq B\subseteq Q_s, B\in\mathcal{S}\}$

Proof. (1) N1-N3: Straightforward from equation (3).
N4: Let $A_1\cap A_2\not\subseteq Q_s$.

$$N_s(A_1)\wedge N_s(A_2)=\bigvee_{P_s\subseteq B\subseteq A_1\not\subseteq Q_s}\mathcal{T}(B)\wedge\bigvee_{P_s\subseteq C\subseteq A_2\not\subseteq Q_s}\mathcal{T}(C)=\bigvee_{P_s\subseteq B\subseteq A_1\not\subseteq Q_s,\ P_s\subseteq C\subseteq A_2\not\subseteq Q_s}(\mathcal{T}(B)\wedge\mathcal{T}(C))$$
$$\subseteq\bigvee_{P_s\subseteq B\subseteq A_1\not\subseteq Q_s,\ P_s\subseteq C\subseteq A_2\not\subseteq Q_s}\mathcal{T}(B\cap C)\subseteq\bigvee_{P_s\subseteq D\subseteq(A_1\cap A_2)\not\subseteq Q_s}\mathcal{T}(D)=N_s(A_1\cap A_2)$$

N5: If $A\subseteq Q_s$, then $N_s(A)=\emptyset$, i.e. the property holds. Let $A\not\subseteq Q_s$. From equation (3), we have "$\forall s'\in B^\flat$ $\mathcal{T}(B)\subseteq N_{s'}(B)$" for all $B\in\mathcal{S}$. Thus it follows that
$N_s(A)=sup\{\mathcal{T}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}\subseteq sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}$.
(2) It can be shown similar as (1). □

Corollary 3.4. *If the texture $(V,\mathcal{V})$ or the texture $(S,\mathcal{S})$ is plain then the statements (N5) and (M5) in Theorem 3.3. can be strengthened as "$N_s(A)=sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}$" and "$M_s(A)=sup\{\bigwedge_{s'\in(S\setminus B)}M_{s'}(B):P_s\not\subseteq A\subseteq B\subseteq Q_s, B\in\mathcal{S}\}$" respectively.*

Proof. Let $(V,\mathcal{V})$ be a plain texture. Suppose that $\bigwedge_{s'\in B^\flat}N_{s'}(B)\not\subseteq\mathcal{T}(B)$ for a set $B\in\mathcal{S}$. Then we have $\bigwedge_{s'\in B^\flat}\bigvee_{P_{s'}\subseteq C\subseteq B\not\subseteq Q_{s'}}\mathcal{T}(C)\not\subseteq\mathcal{T}(B)$ and so $\bigwedge_{s'\in B^\flat}\bigvee_{P_{s'}\subseteq C\subseteq B\not\subseteq Q_{s'}}\mathcal{T}(C)\not\subseteq Q_v$ and $P_v\not\subseteq\mathcal{T}(B)$ for some $v\in V^\flat$. Since $(V,\mathcal{V})$ is plain, join coincides with union in $\mathcal{V}$. Therefore, we have

$$\bigwedge_{s'\in B^\flat}\bigvee_{P_{s'}\subseteq C\subseteq B\not\subseteq Q_{s'}}\mathcal{T}(C)\not\subseteq Q_v\Rightarrow P_v\subseteq\bigwedge_{s'\in B^\flat}\bigvee_{P_{s'}\subseteq C\subseteq B\not\subseteq Q_{s'}}\mathcal{T}(C)\Rightarrow\forall s'\in B^\flat,\ P_v\subseteq\bigvee_{P_{s'}\subseteq C\subseteq B\not\subseteq Q_{s'}}\mathcal{T}(C)=\bigcup_{P_{s'}\subseteq C\subseteq B\not\subseteq Q_{s'}}\mathcal{T}(C)$$
$$\Rightarrow\forall s'\in B^\flat,\ \exists C_{s'}\in\mathcal{S}\ :\ P_{s'}\subseteq C_{s'}\subseteq B\not\subseteq Q_{s'}\text{ and }P_v\subseteq\mathcal{T}(C_{s'})$$

and so we have $P_v\subseteq\bigcap_{s'\in B^\flat}\mathcal{T}(C_{s'})$. Thus, using (GT_3) we obtain

$$\mathcal{T}(B)=\mathcal{T}(\bigvee_{s'\in B^\flat}C_{s'})\supseteq\bigcap_{s'\in B^\flat}\mathcal{T}(C_{s'})\supseteq P_v$$

which is in contradiction with $P_v\not\subseteq\mathcal{T}(B)$. So, by means of the conclusion "$\bigwedge_{s'\in B^\flat}N_{s'}(B)\subseteq\mathcal{T}(B)$ for all $B\in\mathcal{S}$", we get $sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}\subseteq sup\{\mathcal{T}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}=N_s(A)$. Thus by N5 of Theorem 3.3, they are equal.

Similarly it can be obtained that $sup\{\bigwedge_{s'\in(S\setminus B)}M_{s'}(B):P_s\not\subseteq A\subseteq B\subseteq Q_s, B\in\mathcal{S}\}\subseteq M_s(A)$.

Let $(S,\mathcal{S})$ be a plain texture. Suppose that $sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}\not\subseteq N_s(A)$. Then $sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}\not\subseteq Q_v$ and $P_v\not\subseteq N_s(A)$ for some $v\in V^\flat$.

$$sup\{\bigwedge_{s'\in B^\flat}N_{s'}(B):P_s\subseteq B\subseteq A\not\subseteq Q_s, B\in\mathcal{S}\}\not\subseteq Q_v\Rightarrow\exists B\in\mathcal{S}:P_s\subseteq B\subseteq A\not\subseteq Q_s\text{ and }"\forall s'\in B^\flat\ N_{s'}(B)\not\subseteq Q_v"$$

Since $(S,\mathcal{S})$ is plain, $P_s \not\subseteq Q_s$ i.e. $B \not\subseteq Q_s$ or $s \in B^\flat$. Then it follows that $N_s(B) \not\subseteq Q_v$ and considering (N3), $N_s(A) \not\subseteq Q_v$ which is in contradiction with $P_v \not\subseteq N_s(A)$.

Similarly it can be obtained that $sup\{\bigwedge_{s'\in(S\setminus B)} M_{s'}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\} \subseteq M_s(A)$. $\square$

Proposition 3.5. *If the mappings* $N : S^\flat \to \mathcal{V}^S$, $M : S \to \mathcal{V}^S$ *satisfy the conditions* $N1 - N5$ *and* $M1 - M5$ *in Theorem 3.3. then the mappings* $\mathcal{T}_N, \mathcal{K}_M : \mathcal{S} \to \mathcal{V}$, *defined by*

$$\mathcal{T}_N(A) = \bigcap_{s\in A^\flat} N_s(A) \tag{5}$$

$$\mathcal{K}_M(A) = \bigcap_{s\in S\setminus A} M_s(A) \tag{6}$$

where $A \in \mathcal{S}$, *form a* $(V,\mathcal{V})$*-graded ditopology on texture* $(S,\mathcal{S})$. *Moreover, if* $(N^{\mathcal{T}_N}, M^{\mathcal{K}_M})$ *is the graded dinhd system of* $(S,\mathcal{S},\mathcal{T}_N,\mathcal{K}_M,V,\mathcal{V})$ *then* $(N,M) \subseteq (N^{\mathcal{T}_N}, M^{\mathcal{K}_M})$ *(i.e.* $N_s(A) \subseteq N_s^{\mathcal{T}_N}(A)$ *for all* $s \in S^\flat, A \in \mathcal{S}$ *and* $M_s(A) \subseteq M_s^{\mathcal{K}_M}(A)$ *for all* $s \in S, A \in \mathcal{S}$*).*

Proof. $(GT_1) : \mathcal{T}_N(\emptyset) = \bigcap_{s\in\emptyset} N_s(\emptyset) = V$ and $\mathcal{T}_N(S) = \bigcap_{s\in S^\flat} N_s(S) = \bigcap_{s\in S^\flat} V = V$
(GT_2) : Let $A_1, A_2 \in \mathcal{S}$. If $(A_1 \cap A_2)^\flat = \emptyset$ then $\mathcal{T}_N(A_1 \cap A_2) = V$ i.e. (GT_2) holds. Let $(A_1 \cap A_2)^\flat \neq \emptyset$. If $s \in (A_1 \cap A_2)^\flat$ then $A_1 \cap A_2 \not\subseteq Q_s$, so $N_s(A_1) \wedge N_s(A_2) \subseteq N_s(A_1 \cap A_2)$ from (N4). Furthermore, we have $A_1 \cap A_2 \subseteq A_1 \Rightarrow (A_1 \cap A_2)^\flat \subseteq A_1^\flat$ and $A_1 \cap A_2 \subseteq A_2 \Rightarrow (A_1 \cap A_2)^\flat \subseteq A_2^\flat$. Therefore, we get;

$$\begin{aligned}\mathcal{T}_N(A_1 \cap A_2) = \bigcap_{s\in(A_1\cap A_2)^\flat} N_s(A_1 \cap A_2) \supseteq \bigcap_{s\in(A_1\cap A_2)^\flat} (N_s(A_1) \wedge N_s(A_2)) \supseteq \bigcap_{s\in(A_1\cap A_2)^\flat} N_s(A_1) \wedge \bigcap_{s\in(A_1\cap A_2)^\flat} N_s(A_2) \\ \supseteq \bigcap_{s\in A_1^\flat} N_s(A_1) \cap \bigcap_{s\in A_2^\flat} N_s(A_2) = \mathcal{T}_N(A_1) \cap \mathcal{T}_N(A_2)\end{aligned}$$

(GT_3) : Let $A_j \in \mathcal{S}$ for all $j \in J$ where J is a nonempty index set. If $(\bigvee_{j\in J} A_j)^\flat = \emptyset$ then $\mathcal{T}_N(\bigvee_{j\in J} A_j) = V$ i.e. (GT_3) holds. Let $(\bigvee_{j\in J} A_j)^\flat \neq \emptyset$. If $s \in (\bigvee_{j\in J} A_j)^\flat = \bigcup_{j\in J} A_j^\flat$, (by Therem 2.1), then we have $\bigvee_{j\in J} A_j \not\subseteq Q_s$, furthermore, since $A_j \subseteq \bigvee_{i\in J} A_i$ for all $j \in J$ we get $N_s(A_j) \subseteq N_s(\bigvee_{i\in J} A_i)$ for all $j \in J$ from (N3). Thus, we get;

$$\mathcal{T}_N(\bigvee_{j\in J} A_j) = \bigcap_{s\in(\bigvee_{j\in J} A_j)^\flat} N_s(\bigvee_{i\in J} A_i) = \bigcap_{s\in\bigcup_{j\in J} A_j^\flat} N_s(\bigvee_{i\in J} A_i) = \bigcap_{j\in J}\bigcap_{s\in A_j^\flat} N_s(\bigvee_{i\in J} A_i) \supseteq \bigcap_{j\in J}\bigcap_{s\in A_j^\flat} N_s(A_j) = \bigcap_{j\in J} \mathcal{T}_N(A_j)$$

Hence, $\mathcal{T}_N$ is a $(V,\mathcal{V})$-graded topology on $(S,\mathcal{S})$ and similarly it follows that $\mathcal{K}_M$ is a $(V,\mathcal{V})$-graded cotopology on $(S,\mathcal{S})$.

Let $s \in S^\flat, A \in \mathcal{S}$. If $A \subseteq Q_s$ then $N_s(A) = N_s^{\mathcal{T}_N}(A) = \emptyset$. If $A \not\subseteq Q_s$, using (N5) and the equations (3), (5) we get $N_s(A) \subseteq sup\{\bigwedge_{s'\in B^\flat} N_{s'}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\} = sup\{\mathcal{T}_N(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\} = N_s^{\mathcal{T}_N}(A)$. Hence $N_s(A) \subseteq N_s^{\mathcal{T}_N}(A)$ for all $s \in S^\flat, A \in \mathcal{S}$. Similarly, using (M5) and the equations (4), (6) we have $M_s(A) \subseteq M_s^{\mathcal{K}_M}(A)$ for all $s \in S, A \in \mathcal{S}$. $\square$

Corollary 3.6. *If* (N,M) *is the graded dinhd system of a graded ditopological texture space* $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ *then* $(\mathcal{T},\mathcal{K}) = (\mathcal{T}_N,\mathcal{K}_M)$.

Proof. Let (N,M) be a graded dinhd system of graded ditopological texture space $(S,\mathcal{S},\mathcal{T},\mathcal{K},V,\mathcal{V})$ and $A \in \mathcal{S}$. Suppose that $\mathcal{T}_N(A) \not\subseteq \mathcal{T}(A)$. Then, $\mathcal{T}_N(A) \not\subseteq Q_v$ and $P_v \not\subseteq \mathcal{T}(A)$ for some $v \in V$. Hence,

$$\begin{aligned}\mathcal{T}_N(A) \not\subseteq Q_v \Rightarrow \bigcap_{s\in A^\flat} N_s(A) \not\subseteq Q_v \Rightarrow \forall s \in A^\flat,\ N_s(A) \not\subseteq Q_v \Rightarrow \forall s \in A^\flat,\ sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s\} \not\subseteq Q_v \\ \Rightarrow \forall s \in A^\flat,\ \exists B_s \in \mathcal{S} : \mathcal{T}(B_s) \not\subseteq Q_v,\ P_s \subseteq B_s \subseteq A \not\subseteq Q_s\end{aligned}$$

Thus, it follows that

$$A = \bigvee_{s\in A^\flat} P_s \subseteq \bigvee_{s\in A^\flat} B_s \subseteq A \Rightarrow A = \bigvee_{s\in A^\flat} B_s$$

Using (GT_3), this follows that $\mathcal{T}(A) = \mathcal{T}(\bigvee_{s\in A^\flat} B_s) \supseteq \bigcap_{s\in A^\flat} \mathcal{T}(B_s) \supseteq P_v$, which is a contradiction.

On the other hand, since $P_s \subseteq A \subseteq A \not\subseteq Q_s$ for all $s \in A^\flat$, we get $\mathcal{T}(A) \subseteq sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s\} = N_s(A)$ for all $s \in A^\flat$. It follows that $\mathcal{T}_N(A) = \bigcap_{s\in A^\flat} N_s(A) \supseteq \mathcal{T}(A)$. This means $\mathcal{T}_N = \mathcal{T}$.

Using similar method, it can be obtained that $\mathcal{K}_M = \mathcal{K}$. □

Corollary 3.7. *If the texture $(V, \mathcal{V})$ or the texture $(S, \mathcal{S})$ is plain then for the graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}_N, \mathcal{K}_M, V, \mathcal{V})$ we have $(N, M) = (N^{\mathcal{T}_N}, M^{\mathcal{K}_M})$, i.e. (N, M) is the graded dinhd system of $(S, \mathcal{S}, \mathcal{T}_N, \mathcal{K}_M, V, \mathcal{V})$.*

Proof. It is clear from Corollary 3.4. □

Proposition 3.8. *Let $(S, \mathcal{S}, \sigma)$ be a complemented grounded texture space and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S}, \sigma)$. If (N, M) is the graded dinhd system of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then the mappings $N^* : S^\flat \to \mathcal{V}^{\mathcal{S}}, M^* : S \to \mathcal{V}^{\mathcal{S}}$ defined by*

$$N^*_s(A) = M_{\sigma(s)}(\sigma(A)),\ s \in S^\flat\ ;\ \ M^*_s(A) = N_{\sigma(s)}(\sigma(A)),\ s \in S$$

for all $A \in \mathcal{S}$, form a graded dinhd system (N^*, M^*) for the graded ditopological texture space $(S, \mathcal{S}, \mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma, V, \mathcal{V})$. Moreover, $(N, M) = (N^*, M^*)$ if $(\mathcal{T}, \mathcal{K})$ is complemented.

Proof. Let $s \in S^\flat, v \in V^\flat, A \in \mathcal{S}$. Using Definition 3.1. we get

$$\begin{aligned} A \in {}^vN^*_s &\Leftrightarrow N^*_s(A) \not\subseteq Q_v \Leftrightarrow M_{\sigma(s)}(\sigma(A)) \not\subseteq Q_v \Leftrightarrow \sigma(A) \in {}^vM_{\sigma(s)} \Leftrightarrow \exists B \in \mathcal{S} : P_{\sigma(s)} \not\subseteq \sigma(A) \subseteq B \subseteq Q_{\sigma(s)},\ \mathcal{K}(B) \not\subseteq Q_v \\ &\Leftrightarrow \exists B \in \mathcal{S} : P_s \subseteq \sigma(B) \subseteq A \not\subseteq Q_s,\ \mathcal{K}(B) \not\subseteq Q_v \Leftrightarrow \exists B'(=\sigma(B)) \in \mathcal{S} : P_s \subseteq B' \subseteq A \not\subseteq Q_s,\ \mathcal{K}(\sigma(B')) \not\subseteq Q_v \end{aligned}$$

Thus, we have ${}^vN^*_s = \{A \in \mathcal{S} \mid (\mathcal{K} \circ \sigma)(B) \not\subseteq Q_v \textit{ and } P_s \subseteq B \subseteq A \not\subseteq Q_s \textit{ for some } B \in \mathcal{S}\}$ and so N^* is a graded nhd system of $(S, \mathcal{S}, \mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma, V, \mathcal{V})$. Similarly, it can be obtained that M^* is a graded conhd system of $(S, \mathcal{S}, \mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma, V, \mathcal{V})$.

Now, let $(\mathcal{T}, \mathcal{K})$ be complemented, $s \in S^\flat$ and $A \in \mathcal{S}$. If $A \subseteq Q_s$ then $N_s(A) = \emptyset$, $P_{\sigma(s)} \subseteq \sigma(A)$ and so $N^*_s(A) = M_{\sigma(s)}(\sigma(A)) = \emptyset$. Hence, $N_s = N^*_s$. If $A \not\subseteq Q_s$ then $P_{\sigma(s)} \not\subseteq \sigma(A)$ and so using Proposition 3.2. we get

$$\begin{aligned} N^*_s(A) = M_{\sigma(s)}(\sigma(A)) &= sup\{\mathcal{K}(B) \mid P_{\sigma(s)} \not\subseteq \sigma(A) \subseteq B \subseteq Q_{\sigma(s)},\ B \in \mathcal{S}\} \\ &= sup\{\mathcal{K}(\sigma(B')) \mid P_{\sigma(s)} \not\subseteq \sigma(A) \subseteq \sigma(B') \subseteq Q_{\sigma(s)},\ B' \in \mathcal{S}\} \\ &= sup\{\mathcal{T}(B') \mid P_s \subseteq B' \subseteq A \not\subseteq Q_s,\ B' \in \mathcal{S}\} = N_s(A). \end{aligned}$$

Therefore we have $N = N^*$. Using the above method it can be shown that $M = M^*$. □

Example 3.9. (1) Let (η, μ) be the dinhd system of a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$. Then the mappings $N : S^\flat \to \mathcal{P}(1)^{\mathcal{S}}, M : S \to \mathcal{P}(1)^{\mathcal{S}}$ defined by

$$N_s(A) = \begin{cases} 1, & A \in \eta(s) \\ \emptyset, & A \notin \eta(s) \end{cases}$$

and

$$M_s(A) = \begin{cases} 1, & A \in \mu(s) \\ \emptyset, & A \notin \mu(s) \end{cases}$$

for all $A \in \mathcal{S}$, form a graded dinhd system for the graded ditopological texture space $(S, \mathcal{S}, \tau^g, \kappa^g, 1, \mathcal{P}(1))$ corresponding to $(S, \mathcal{S}, \tau, \kappa)$.

On the other hand, if (N, M) is a graded dinhd system for a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, 1, \mathcal{P}(1))$ then the families $\eta(s), s \in S^\flat$ and $\mu(s), s \in S$ defined by

$$\eta(s) = \{A \in \mathcal{S} : N_s(A) = 1\}, \mu(s) = \{A \in \mathcal{S} : M_s(A) = 1\}$$

form a dinhd system (η, μ) for the ditopological texture space $(S, \mathcal{S}, \mathcal{T}^0, \mathcal{K}^0)$ corresponding to $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, 1, \mathcal{P}(1))$.
(2) Let (N, M) be the graded dinhd system of a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ and (η^v, μ^v) be the dinhd system of $(\mathcal{T}^v, \mathcal{K}^v)$ for each $v \in V$. Then for the families (η^v_N, μ^v_M) defined by

$$s \in S^\flat, \eta^v_N(s) = \{A \in \mathcal{S} \mid P_v \subseteq N_s(A)\}$$
$$s \in S, \mu^v_M(s) = \{A \in \mathcal{S} \mid P_v \subseteq M_s(A)\}$$

we have $(\eta^v, \mu^v) \subseteq (\eta^v_N, \mu^v_M)$ for each $v \in V$. If the texture $(V, \mathcal{V})$ is plain then we get $(\eta^v, \mu^v) = (\eta^v_N, \mu^v_M)$ for each $v \in V$.

4. Graded Q-dineighborhood Systems

Throughout this section we assume that the texture $(S, \mathcal{S})$ has a complementation σ.

Definition 4.1. Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$ and $\widetilde{N}, \widetilde{M} : S \to \mathcal{V}^{\mathcal{S}}$ mappings where $\widetilde{N}(s) = \widetilde{N}_s : \mathcal{S} \to \mathcal{V}$, $\widetilde{M}(s) = \widetilde{M}_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S$. Then the mapping $\widetilde{N}_s$ is called a "graded Q-nhd system of s" if

$${}^v\widetilde{N}_s = \{A \in \mathcal{S} : \mathcal{T}(B) \not\subseteq Q_v \text{ and } P_s q B \subseteq A \text{ for some } B \in \mathcal{S}\} \tag{7}$$

for each $v \in V^\flat$ and the mapping $\widetilde{M}_s$ is called a "graded Q-conhd system of s" if

$${}^v\widetilde{M}_s = \{A \in \mathcal{S} : \mathcal{K}(B) \not\subseteq Q_v \text{ and } P_s \not\subseteq B, A \subseteq B \text{ for some } B \in \mathcal{S}\} \tag{8}$$

for each $v \in V^\flat$. The mapping $\widetilde{N}$ ($\widetilde{M}$) is called a "graded Q-nhd system" ("graded Q-conhd system") of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ if $\widetilde{N}_s$ ($\widetilde{M}_s$) is a graded Q-nhd system (graded Q-conhd system) for each $s \in S$ respectively. $(\widetilde{N}, \widetilde{M})$ is called a "graded Q-dinhd system" of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ if $\widetilde{N}$ is a graded Q-nhd system and $\widetilde{M}$ is a graded Q-conhd system of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

Proposition 4.2. *$(\widetilde{N}, \widetilde{M})$ is a graded Q-dinhd system of a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ iff*

$$\widetilde{N}_s(A) = \begin{cases} sup\{\mathcal{T}(B) : P_s q B \subseteq A, B \in \mathcal{S}\}, & P_s q A \\ \emptyset, & P_s \bar{q} A \end{cases} \tag{9}$$

$$\widetilde{M}_s(A) = \begin{cases} sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B, B \in \mathcal{S}\}, & P_s \not\subseteq A \\ \emptyset, & P_s \subseteq A \end{cases} \tag{10}$$

for each $s \in S$ and $A \in \mathcal{S}$.

Proof. We prove only the Q-conhd part, the Q-nhd part can be proved in a similar way.
$(\Longrightarrow)$: Let $s \in S$, $A \in \mathcal{S}$ and $\widetilde{M}_s$ be a graded Q-conhd system of s. If $P_s \subseteq A$ then by (8), $A \notin {}^v\widetilde{M}_s$ for each $v \in V^\flat$ so $\widetilde{M}_s(A) \subseteq Q_v$ for each $v \in V^\flat$. If we assume that $\widetilde{M}_s(A) \neq \emptyset$, then we get $\widetilde{M}_s(A)^\flat \neq \emptyset$ and therefore there exists $t \in V^\flat$ such that $t \in \widetilde{M}_s(A)^\flat$. So we have $\widetilde{M}_s(A) \not\subseteq Q_t$. This means contradiction with the statement "$\widetilde{M}_s(A) \subseteq Q_v$ for each $v \in V^\flat$" and hence we obtain that $\widetilde{M}_s(A) = \emptyset$.

Now let $P_s \not\subseteq A$. Suppose that $\widetilde{M}_s(A) \not\subseteq sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\}$. Then we have $\widetilde{M}_s(A) \not\subseteq Q_v$ and $P_v \not\subseteq sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\}$ for some $v \in V$. Moreover $v \in V^\flat$ since $V \neq Q_v$, so we have $A \in {}^v\widetilde{M}_s$. Thus, from (8) it follows that there exists $B \in \mathcal{S}$ such that $\mathcal{K}(B) \not\subseteq Q_v$ and $P_s \not\subseteq B, A \subseteq B$. It follows that $sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\} \not\subseteq Q_v$ which is in contradiction with $P_v \not\subseteq sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\}$, i.e. $\widetilde{M}_s(A) \subseteq sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\}$. Similarly it can be obtained that $sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\} \subseteq \widetilde{M}_s(A)$. So we have $\widetilde{M}_s(A) = sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\}$.
$(\Longleftarrow)$: Let (10) hold for each $s \in S$, $A \in \mathcal{S}$. Then we obtain
$\mathcal{K}(B) \not\subseteq Q_v$ *and* $P_s \subseteq B, A \subseteq B$ *for some* $B \in \mathcal{S} \Leftrightarrow \widetilde{M}_s(A) = sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\} \not\subseteq Q_v \Leftrightarrow A \in {}^v\widetilde{M}_s$
for each $s \in S$, $v \in V^\flat$. So $\widetilde{M}$ is a graded Q-conhd system. □

Theorem 4.3. *Let* $(\mathcal{T}, \mathcal{K})$ *be a* $(V, \mathcal{V})$*-graded ditopology on texture* $(S, \mathcal{S})$*. If* $(\widetilde{N}, \widetilde{M})$ *is the graded Q-dinhd system of graded ditopological texture space* $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$*, then the following properties hold for each* $s \in S$*,* $A, A_1, A_2 \in \mathcal{S}$*:*
(1) $(\widetilde{\mathbf{N}}\mathbf{1})$ $\widetilde{N}_s(A) \neq \emptyset \Rightarrow P_s q A$
$(\widetilde{\mathbf{N}}\mathbf{2})$ $\widetilde{N}_s(\emptyset) = \emptyset$ *and* $\widetilde{N}_s(S) = V$
$(\widetilde{\mathbf{N}}\mathbf{3})$ $A_1 \subseteq A_2 \Rightarrow \widetilde{N}_s(A_1) \subseteq \widetilde{N}_s(A_2)$
$(\widetilde{\mathbf{N}}\mathbf{4})$ $\widetilde{N}_s(A_1) \wedge \widetilde{N}_s(A_2) \subseteq \widetilde{N}_s(A_1 \cap A_2)$
$(\widetilde{\mathbf{N}}\mathbf{5})$ $\widetilde{N}_s(A) = sup\{\bigwedge_{P_{s'} q B} \widetilde{N}_{s'}(B) : P_s q B \subseteq A\}$

(2) $(\widetilde{\mathbf{M}}\mathbf{1})$ $\widetilde{M}_s(A) \neq \emptyset \Rightarrow P_s \not\subseteq A$
$(\widetilde{\mathbf{M}}\mathbf{2})$ $\widetilde{M}_s(S) = \emptyset$ *and* $\widetilde{M}_s(\emptyset) = V$
$(\widetilde{\mathbf{M}}\mathbf{3})$ $A_1 \subseteq A_2 \Rightarrow \widetilde{M}_s(A_2) \subseteq \widetilde{M}_s(A_1)$
$(\widetilde{\mathbf{M}}\mathbf{4})$ $\widetilde{M}_s(A_1) \wedge \widetilde{M}_s(A_2) \subseteq \widetilde{M}_s(A_1 \cup A_2)$
$(\widetilde{\mathbf{M}}\mathbf{5})$ $\widetilde{M}_s(A) = sup\{\bigwedge_{P_{s'} \not\subseteq B} \widetilde{M}_{s'}(B) : P_s \not\subseteq B, A \subseteq B\}$

Proof. (2) $\widetilde{M}1 - \widetilde{M}3$: Straightforward from equation (10).
$\widetilde{M}4$: If $P_s \subseteq A_1$ or $P_s \subseteq A_2$ then the property holds. Let $P_s \not\subseteq A_1$ and $P_s \not\subseteq A_2$. Since $P_s \not\subseteq B$, $A_1 \subseteq B$, $P_s \not\subseteq C$, $A_2 \subseteq C \Rightarrow P_s \not\subseteq (B \cup C), (A_1 \cup A_2) \subseteq (B \cup C)$,

$$\begin{aligned}\widetilde{M}_s(A_1) \wedge \widetilde{M}_s(A_2) &= \bigvee_{P_s \not\subseteq B, A_1 \subseteq B} \mathcal{K}(B) \wedge \bigvee_{P_s \not\subseteq C, A_2 \subseteq C} \mathcal{K}(C) = \bigvee_{P_s \not\subseteq B,\ A_1 \subseteq B,\ P_s \not\subseteq C,\ A_2 \subseteq C} (\mathcal{K}(B) \cap \mathcal{K}(C)) \\ &\subseteq \bigvee_{P_s \not\subseteq B,\ A_1 \subseteq B,\ P_s \not\subseteq C,\ A_2 \subseteq C} \mathcal{K}(B \cup C) \subseteq \bigvee_{P_s \not\subseteq D, (A_1 \cup A_2) \subseteq D} \mathcal{K}(D) = \widetilde{M}_s(A_1 \cup A_2)\end{aligned}$$

$\widetilde{M}5$: From equation (10), we have "$P_{s'} \not\subseteq B \Rightarrow \mathcal{K}(B) \subseteq \widetilde{M}_{s'}(B)$" for all $B \in \mathcal{S}$. Thus it follows that $\widetilde{M}_s(A) = sup\{\mathcal{K}(B) : P_s \not\subseteq B, A \subseteq B\} \subseteq sup\{\bigwedge_{P_{s'} \not\subseteq B} \widetilde{M}_{s'}(B) : P_s \not\subseteq B, A \subseteq B\}$. Now, suppose that $sup\{\bigwedge_{P_{s'} \not\subseteq B} \widetilde{M}_{s'}(B) : P_s \not\subseteq B, A \subseteq B\} \not\subseteq \widetilde{M}_s(A)$. Then, $sup\{\bigwedge_{P_{s'} \not\subseteq B} \widetilde{M}_{s'}(B) : P_s \not\subseteq B, A \subseteq B\} \not\subseteq Q_v$ and $P_v \not\subseteq \widetilde{M}_s(A)$ for some $v \in V^\flat$.

$$sup\{\bigwedge_{P_{s'} \not\subseteq B} \widetilde{M}_{s'}(B) : P_s \not\subseteq B, A \subseteq B\} \not\subseteq Q_v \Rightarrow \exists B \in \mathcal{S} : P_s \not\subseteq B, A \subseteq B \text{ and } "P_{s'} \not\subseteq B \Rightarrow \widetilde{M}_{s'}(B) \not\subseteq Q_v"$$

Since $P_s \not\subseteq B$, it follows that $\widetilde{M}_s(B) \not\subseteq Q_v$. Using $A \subseteq B$ and $(\widetilde{M}3)$, we have $\widetilde{M}_s(A) \not\subseteq Q_v$ which is in contradiction with $P_v \not\subseteq \widetilde{M}_s(A)$.
(1) can be shown similar to (2). □

Proposition 4.4. *If the mappings* $\widetilde{N}, \widetilde{M} : S \to \mathcal{V}^S$ *satisfy the conditions* $\widetilde{N}1 - \widetilde{N}5$ *and* $\widetilde{M}1 - \widetilde{M}5$ *in Theorem 4.3. then the mappings* $\mathcal{T}_{\widetilde{N}}, \mathcal{K}_{\widetilde{M}} : \mathcal{S} \to \mathcal{V}$*, defined by*

$$\mathcal{T}_{\widetilde{N}}(A) = \bigcap_{P_s q A} \widetilde{N}_s(A) \tag{11}$$

$$\mathcal{K}_{\widetilde{M}}(A) = \bigcap_{P_s \not\subseteq A} \widetilde{M}_s(A) \tag{12}$$

where $A \in \mathcal{S}$, form a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$. Moreover, $(\widetilde{N}, \widetilde{M})$ is the graded Q-dinhd system of $(S, \mathcal{S}, \mathcal{T}_{\widetilde{N}}, \mathcal{K}_{\widetilde{M}}, V, \mathcal{V})$.

Proof. (GCT_1) $\mathcal{K}_{\widetilde{M}}(\emptyset) = \bigcap_{P_s \not\subseteq \emptyset} \widetilde{M}_s(\emptyset) = \bigcap_{s \in S} V = V$ and $\mathcal{K}_{\widetilde{M}}(S) = \bigcap_{P_s \not\subseteq S} \widetilde{M}_s(S) = \bigcap_{s \in \emptyset} \widetilde{M}_s(S) = V$
(GCT_2) Let $A_1, A_2 \in \mathcal{S}$. If $A_1 \cup A_2 = S$ then $\mathcal{K}_{\widetilde{M}}(A_1 \cup A_2) = V$ and (GCT_2) holds. Let $A_1 \cup A_2 \neq S$. Using $(\widetilde{M}4)$, we get;

$$\mathcal{K}_{\widetilde{M}}(A_1 \cup A_2) = \bigcap_{P_s \not\subseteq (A_1 \cup A_2)} \widetilde{M}_s(A_1 \cup A_2) \supseteq \bigcap_{P_s \not\subseteq (A_1 \cup A_2)} (\widetilde{M}_s(A_1) \wedge \widetilde{M}_s(A_2)) = \bigcap_{P_s \not\subseteq (A_1 \cup A_2)} \widetilde{M}_s(A_1) \cap \bigcap_{P_s \not\subseteq (A_1 \cup A_2)} \widetilde{M}_s(A_2)$$
$$\supseteq \bigcap_{P_s \not\subseteq A_1} \widetilde{M}_s(A_1) \cap \bigcap_{P_s \not\subseteq A_2} \widetilde{M}_s(A_2) = \mathcal{K}_{\widetilde{M}}(A_1) \cap \mathcal{K}_{\widetilde{M}}(A_2)$$

(GCT_3) Let $A_j \in \mathcal{S}$ for all $j \in J$ where J is a nonempty index set. If $(\bigcap_{j \in J} A_j) = S$ then $\mathcal{K}_{\widetilde{M}}(\bigcap_{j \in J} A_j) = V$ i.e. (GCT_3) holds. Let $(\bigcap_{j \in J} A_j) \neq S$. We have $P_s \not\subseteq (\bigcap_{j \in J} A_j) \Leftrightarrow s \in S \setminus (\bigcap_{j \in J} A_j) = \bigcup_{j \in J} (S \setminus A_j)$. Thus, using $(\widetilde{M}3)$ we get;

$$\mathcal{K}_{\widetilde{M}}(\bigcap_{j \in J} A_j) = \bigcap_{P_s \not\subseteq (\bigcap_{j \in J} A_j)} \widetilde{M}_s(\bigcap_{i \in J} A_i) \supseteq \bigcap_{j \in J} (\bigcap_{P_s \not\subseteq A_j} \widetilde{M}_s(\bigcap_{i \in J} A_i)) \supseteq \bigcap_{j \in J} (\bigcap_{P_s \not\subseteq A_j} \widetilde{M}_s(A_j)) = \bigcap_{j \in J} \mathcal{K}_{\widetilde{M}}(A_j)$$

Hence, $\mathcal{K}_{\widetilde{M}}$ is a $(V, \mathcal{V})$-graded cotopology on $(S, \mathcal{S})$ and similarly it follows that $\mathcal{T}_{\widetilde{N}}$ is a $(V, \mathcal{V})$-graded topology on $(S, \mathcal{S})$. Furthermore, using $(\widetilde{N}5)$, $(\widetilde{M}5)$ we get that $(\widetilde{N}, \widetilde{M})$ is the graded Q-dinhd system of $(S, \mathcal{S}, \mathcal{T}_{\widetilde{N}}, \mathcal{K}_{\widetilde{M}}, V, \mathcal{V})$. □

Corollary 4.5. *In Proposition 4.4., if σ is grounded then $\mathcal{T}_{\widetilde{N}}$ and $\mathcal{K}_{\widetilde{M}}$ can be also characterized as follows:*

$$\mathcal{T}_{\widetilde{N}}(A) = \bigcap_{s \in A^\flat} \widetilde{N}_{\sigma(s)}(A)\ ,\ \mathcal{K}_{\widetilde{M}}(A) = \bigcap_{s \in \sigma(A)^\flat} \widetilde{M}_{\sigma(s)}(A)\ ; \qquad A \in \mathcal{S} \tag{13}$$

Proof. Considering $P_s q A \Leftrightarrow P_s \not\subseteq \sigma(A) \Leftrightarrow A = \sigma(\sigma(A)) \not\subseteq \sigma(P_s) = Q_{\sigma(s)} \Leftrightarrow \sigma(s) \in A^\flat$ with (11) and $P_s \not\subseteq A \Leftrightarrow \sigma(A) \not\subseteq \sigma(P_s) = Q_{\sigma(s)} \Leftrightarrow \sigma(s) \in \sigma(A)^\flat$ with (12), it can be obtained. □

Corollary 4.6. *If $(\widetilde{N}, \widetilde{M})$ is the graded Q-dinhd system of a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then $(\mathcal{T}, \mathcal{K}) = (\mathcal{T}_{\widetilde{N}}, \mathcal{K}_{\widetilde{M}})$.*

Proposition 4.7. *Let $(S, \mathcal{S}, \sigma)$ be a complemented texture space and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S}, \sigma)$. If $(\widetilde{N}, \widetilde{M})$ is the graded Q-dinhd system of $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then the mappings $\widetilde{N}^*, \widetilde{M}^* : S \to \mathcal{V}^{\mathcal{S}}$ defined by*

$$\widetilde{N}^*_s(A) = \widetilde{M}_s(\sigma(A)),\ \ \widetilde{M}^*_s(A) = \widetilde{N}_s(\sigma(A)),\ s \in S$$

for all $A \in \mathcal{S}$, form a graded Q-dinhd system (N^*, M^*) for the graded ditopological texture space $(S, \mathcal{S}, \mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma, V, \mathcal{V})$. Moreover, $(\widetilde{N}, \widetilde{M}) = (\widetilde{N}^*, \widetilde{M}^*)$ iff $(\mathcal{T}, \mathcal{K})$ is complemented.

Proof. It is clear from Proposition 4.2. □

Example 4.8. First, we recall the concept of "smooth Q-neighborhood system" [10]. Let (X, τ) be a graded fuzzy (smooth) topological space, $x_m \in \mathbb{I}^X$ a fixed fuzzy point and $Q_{x_m} : \mathbb{I}^X \to \mathbb{I}$ "smooth Q-neighborhood system of x_m", i.e.,

$$Q_{x_m}(\mu) = \begin{cases} sup\{\tau(\eta) : \eta \in \mathbb{I}^X,\ x_m q \eta, \eta \leq \mu\}, & x_m q \mu \\ 0, & x_m \bar{q} \mu \end{cases} \tag{14}$$

for each $\mu \in \mathbb{I}^X$ [10]. Then, remembering that the texture $(L, \mathcal{L}, \lambda)$ corresponds to $\mathbb{I}$ and the texture $(W, \mathcal{W}, \omega)$ corresponds to $\mathbb{I}^X$ from Example 2.3., the mappings $\mathcal{T}, \mathcal{K} : \mathcal{W} \to \mathcal{L}$ defined by

$$\mathcal{T}(\varphi(\mu)) = (0, \tau(\mu)],\ \mathcal{K}(\varphi(\mu)) = \mathcal{T}(\omega(\varphi(\mu))) = \mathcal{T}(\varphi(\mu'))\ ;\ \mu \in \mathbb{I}^X \tag{15}$$

form a complemented $(L, \mathcal{L})$-graded ditopology on $(W, \mathcal{W}, \omega)$. Moreover, the mappings $\widetilde{N}, \widetilde{M} : W \to \mathcal{L}^{\mathcal{W}}$ defined by

$$\widetilde{N}_{x_m}(\varphi(\mu)) = \begin{cases} sup\{\mathcal{T}(\varphi(\eta)) :\ x_m q \varphi(\eta) \subseteq \varphi(\mu)\}, & x_m q \varphi(\mu) \\ \emptyset, & x_m \bar{q} \varphi(\mu) \end{cases} \tag{16}$$

$$\widetilde{M}_{x_m}(\varphi(\mu)) = \widetilde{N}_{x_m}(\omega(\varphi(\mu))) \tag{17}$$

for each $\varphi(\mu) \in \mathcal{W}$, $x_m \in W$ form a graded Q-nhd system $(\widetilde{N}, \widetilde{M})$ of $(W, \mathcal{W}, \omega, \mathcal{T}, \mathcal{K}, L, \mathcal{L})$.

5. Categorical Aspects

Graded Q-dinhd systems give a complete characterization of graded ditopologies on complemented texture spaces but this structure is not a generalization of dinhd systems on ditopological texture spaces. On the other hand, the structure of graded dinhd systems is a generalization of dinhd systems on ditopological texture spaces and it characterizes graded ditopologies on plain textures. In this section we categorically investigate the relations between graded dinhd systems and graded ditopolological texture spaces. Our reference for category theory is [1]. We recall some basic concepts and properties from [5, 6, 18] that will be needed in this section.

Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures, the p-sets and q-sets for the product texture $(S \times V, P(S) \otimes \mathcal{V})$ be denoted by $\overline{P}_{(s,v)}, \overline{Q}_{(s,v)}$ respectively.

Difunctions: Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures.

1. $r \in P(S) \otimes \mathcal{V}$ is called a relation from $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies
   R1 $r \not\subseteq \overline{Q}_{(s,v)}, P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}_{(s',v)}$
   R2 $r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',v)}$.
2. $R \in P(S) \otimes \mathcal{V}$ is called a corelation from $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies
   CR1 $\overline{P}_{(s,v)} \not\subseteq R, P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}_{(s',v)} \not\subseteq R$
   CR2 $\overline{P}_{(s,v)} \not\subseteq R \Rightarrow \exists s' \in S$ such that $P'_s \not\subseteq Q_s$ and $\overline{P}_{(s',v)} \not\subseteq R$.
3. A pair (r, R), where r is a relation and R is a corelation from $(S, \mathcal{S})$ to $(V, \mathcal{V})$, is called a direlation from $(S, \mathcal{S})$ to $(V, \mathcal{V})$.
4. Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then (f, F) is called a difunction from $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies the following two conditions:
   DF1 For $s, s' \in S$, $P_s \not\subseteq Q'_s \Rightarrow \exists v \in V$ with $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F$
   DF2 For $v, v' \in V$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P'_v \not\subseteq Q_v$.
5. Let $(f, F) : (S, \mathcal{S}) \to (V, \mathcal{V})$ be a difunction. For $B \in \mathcal{V}$, the inverse image $f^{\leftarrow}B$ and inverse co-image $F^{\leftarrow}B$ are defined by

$$f^{\leftarrow}B = \bigvee\{P_s | \forall v,\ f \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$$

$$F^{\leftarrow}B = \bigcap\{Q_s | \forall v,\ \overline{P}_{(s,v)} \not\subseteq F \Rightarrow B \subseteq Q_v\}.$$

   For a difunction, the inverse image and the inverse co-image are equal.

6. A difunction $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is called continuous if $B \in \tau_2 \Rightarrow F^{\leftarrow}B \in \tau_1$, cocontinuous if $B \in \kappa_2 \Rightarrow f^{\leftarrow}B \in \kappa_1$, and bicontinuous if it is both continuous and cocontinuous.

The Category dfGDitop: Let $(S_k, \mathcal{S}_k, \mathcal{T}_k, \mathcal{K}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded ditopological texture spaces, $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. For the pair $((f, F), (h, H))$, (f, F) is called continuous with respect to (h, H) if $H^{\leftarrow}\mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow}A)$ $\forall A \in \mathcal{S}_2$, and cocontinuous with respect to (h, H) if $h^{\leftarrow}\mathcal{K}_2(A) \subseteq \mathcal{K}_1(f^{\leftarrow}A)$ $\forall A \in \mathcal{S}_2$. The difunction (f, F) is called bicontinuous with respect to (h, H) if it is both continuous and cocontinuous with respect to (h, H).

The class of graded ditopological texture spaces and relatively bicontinuous difunction pairs between them form a category denoted by **dfGDitop**.

Definition 5.1. Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be texture spaces. A pair of mappings $N : S^{\flat} \to \mathcal{V}^S$, $M : S \to \mathcal{V}^S$ which satisfies $(N1) - (N5)$ and $(M1) - (M5)$ in Theorem 3.3. is called a $(V, \mathcal{V})$-graded dinhd function on $(S, \mathcal{S})$. In this case, $(S, \mathcal{S}, N, M, V, \mathcal{V})$ is called a graded dinhd texture space.

Definition 5.2. Let $(S_k, \mathcal{S}_k, N_k, M_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded dinhd texture spaces, $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. (f, F) is called n-continuous with respect to (h, H) if

$$H^{\leftarrow}(\bigcap_{s \in A^{\flat}} N_{2s}(A)) \subseteq \bigcap_{s \in (F^{\leftarrow}A)^{\flat}} N_{1s}(F^{\leftarrow}A) \qquad \forall A \in \mathcal{S}_2 \tag{18}$$

and n-cocontinuous with respect to (h, H) if

$$h^{\leftarrow}(\bigcap_{s \in S_2 \setminus A} M_{2s}(A)) \subseteq \bigcap_{s \in (S_1 \setminus f^{\leftarrow}A)} M_{1s}(f^{\leftarrow}A) \qquad \forall A \in \mathcal{S}_2. \tag{19}$$

The difunction (f, F) is called n-bicontinuous with respect to (h, H) if it is both n-continuous and n-cocontinuous with respect to (h, H).

For a graded dinhd texture space $(S, \mathcal{S}, N, M, V, \mathcal{V})$ and identity difunctions (i_S, I_S), (i_V, I_V) we have $I_V^{\leftarrow}(\bigcap_{s \in A^{\flat}} N_s(A)) = \bigcap_{s \in A^{\flat}} N_s(A) = \bigcap_{s \in (I_S^{\leftarrow}A)^{\flat}} N_s(I_S^{\leftarrow}A)$, $i_V^{\leftarrow}(\bigcap_{s \in S \setminus A} M_s(A)) = \bigcap_{s \in S \setminus A} M_s(A) = \bigcap_{s \in (S \setminus i_S^{\leftarrow}A)} M_s(i_S^{\leftarrow}A)$ for all $A \in \mathcal{S}$. So (i_S, I_S) is n-bicontinuous with respect to (i_V, I_V).

Proposition 5.3. *Relatively n-bicontinuity is preserved by composition of difunctions.*

Proof. Let $(S_k, \mathcal{S}_k, N_k, M_k, V_k, \mathcal{V}_k)$, $k = 1, 2, 3$ be graded dinhd texture spaces, $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$, $(g, G) : (S_2, \mathcal{S}_2) \to (S_3, \mathcal{S}_3)$, $(k, K) : (V_2, \mathcal{V}_2) \to (V_3, \mathcal{V}_3)$ difunctions where (f, F) is n-bicontinouos with respect to (h, H) and (g, G) is n-bicontinouos with respect to (k, K). For each $A \in \mathcal{S}_3$ we have

$$\begin{aligned}(K \circ H)^{\leftarrow}(\bigcap_{s \in A^{\flat}} N_{3s}(A)) &= H^{\leftarrow}(K^{\leftarrow}(\bigcap_{s \in A^{\flat}} N_{3s}(A))) \subseteq H^{\leftarrow}(\bigcap_{s \in (G^{\leftarrow}A)^{\flat}} N_{2s}(G^{\leftarrow}A)) \\ &\subseteq \bigcap_{s \in (F^{\leftarrow}(G^{\leftarrow}A))^{\flat}} N_{1s}(F^{\leftarrow}(G^{\leftarrow}A)) = \bigcap_{s \in ((G \circ F)^{\leftarrow}A)^{\flat}} N_{1s}((G \circ F)^{\leftarrow}A).\end{aligned}$$

So, $(g \circ f, G \circ F)$ is n-continuous with respect to $(k \circ h, K \circ H)$. Similarly $(g \circ f, G \circ F)$ is n-cocontinuous, and hence n-bicontinuous with respect to $(k \circ h, K \circ H)$. □

Graded dinhd texture spaces and relatively n-bicontionuous difunction pairs between them form a category that we will denote by **dfGDinhd**. From Proposition 3.5., for each $(S, \mathcal{S}, N, M, V, \mathcal{V}) \in Ob$**dfGDinhd** there is a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}_N, \mathcal{K}_M, V, \mathcal{V}) \in Ob$**dfGDitop**.

Corollary 5.4. *For the above notations, if (f, F) is $((N_1, M_1) - (N_2, M_2))$ n-bicontinuous with respect to (h, H) then it is $((\mathcal{T}_{N_1}, \mathcal{K}_{M_1}) - (\mathcal{T}_{N_2}, \mathcal{K}_{M_2}))$ bicontinuous with respect to (h, H). So, $\mathfrak{F}$: **dfGDinhd** $\to$ **dfGDitop** defined by*

$$\mathfrak{F}(((f, F), (h, H)) : (S_1, \mathcal{S}_1, N_1, M_1, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, N_2, M_2, V_2, \mathcal{V}_2))$$
$$= ((f, F), (h, H)) : (S_1, \mathcal{S}_1, \mathcal{T}_{N_1}, \mathcal{K}_{M_1}, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, \mathcal{T}_{N_2}, \mathcal{K}_{M_2}, V_2, \mathcal{V}_2).$$

is a faithful and full functor.

Proof. It is clear from Proposition 3.5. and from the definition of $\mathfrak{F}$. □

From Proposition 3.2., for each $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V}) \in Ob$**dfGDitop** there is a graded dineigborhood texture space $(S, \mathcal{S}, N^{\mathcal{T}}, M^{\mathcal{K}}, V, \mathcal{V}) \in Ob$**dfGDinhd** and from Corollary 3.6. $(\mathcal{T}, \mathcal{K}) = (\mathcal{T}_{N^{\mathcal{T}}}, \mathcal{K}_{M^{\mathcal{K}}})$.

Corollary 5.5. *For the above notations, if (f, F) is $((\mathcal{T}_1, \mathcal{K}_1) - (\mathcal{T}_2, \mathcal{K}_2))$ bicontinuous with respect to (h, H) then it is $((N^{\mathcal{T}_1}, M^{\mathcal{K}_1}) - (N^{\mathcal{T}_2}, M^{\mathcal{K}_2}))$ n-bicontinuous with respect to (h, H). So, the funktor $\mathfrak{G}$: **dfGDitop** $\to$ **dfGDinhd** defined by*

$$\mathfrak{G}(((f, F), (h, H)) : (S_1, \mathcal{S}_1, \mathcal{T}_1, \mathcal{K}_1, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, \mathcal{T}_2, \mathcal{K}_2, V_2, \mathcal{V}_2))$$
$$= ((f, F), (h, H)) : (S_1, \mathcal{S}_1, N^{\mathcal{T}_1}, M^{\mathcal{K}_1}, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, N^{\mathcal{T}_2}, M^{\mathcal{K}_2}, V_2, \mathcal{V}_2).$$

is a full embedding. Moreover $\mathfrak{F} \circ \mathfrak{G} = id_{\mathbf{dfGDinhd}}$.

Proof. From Proposition 3.2., $\mathfrak{G}$ is injective on objects and an embedding. □

Problem 5.6. *It is an open problem that whether $\mathfrak{F}$ is injective on objects (i.e. $(N, M) = (N^{\mathcal{T}_N}, M^{\mathcal{K}_M})$ in Proposition 3.5. so $\mathfrak{G} \circ \mathfrak{F} = id_{\mathbf{dfGDitop}}$) or not.*

6. Conclusion

Local devices have a crucial role in dealing with mathematical structures. For instance, in topological spaces neighborhood systems are convenient to characterize open sets (i.e. topology) by means of nhds of its points. In this work, two sorts of neighborhood structures which allows local investigation and another perspective for the theory of graded ditopological texture spaces have presented for graded ditopological texture spaces.

As expected, graded dinhd systems are more general than dinhd systems. Each graded ditopology $(\mathcal{T}, \mathcal{K})$ has a graded dinhd system $(N_{\mathcal{T}}, M_{\mathcal{T}})$ and $(\mathcal{T}, \mathcal{K}) = (\mathcal{T}_{N_{\mathcal{T}}}, \mathcal{K}_{M_{\mathcal{T}}})$ (Proposition 3.2 and Corollary 3.6). Each graded dinhd system (N, M) satisfies the properties $(N1)-(N5)$ and $(M1)-(M5)$ and generates a graded ditopology $(\mathcal{T}_N, \mathcal{K}_M)$ (Theorem 3.3 and Proposition 3.5). Under the plainness condition, $(N, M) = (N_{\mathcal{T}_N}, M_{\mathcal{K}_M})$ is valid. Without plainness condition, it is an open problem that whether this equality valid or not. Graded dinhd systems have been investigated and the open problem mentioned above has been represented in categorical aspects.

Moreover, an alternative neighborhood structure "graded Q-nhd systems" has been presented by using the concept "quasi coincidence" on the complemented textures. Contrary to graded dinhd systems, in this structure the equalities $(\mathcal{T}, \mathcal{K}) = (\mathcal{T}_{\widetilde{N}_{\mathcal{T}}}, \mathcal{K}_{\widetilde{M}_{\mathcal{T}}})$ and $(\widetilde{N}, \widetilde{M}) = (\widetilde{N}_{\mathcal{T}_{\widetilde{N}}}, \widetilde{M}_{\mathcal{K}_{\widetilde{M}}})$ are valid. But there are two handicaps of this structure such that graded Q-dinhd systems are not a generalization of dinhd systems and they are defined only on complemented textures.

It can be possible to construct other topological structures such as filters, uniformities, etc on graded ditopological texture spaces and we continue to study on this structures.

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References

[1] J. Adámek, H. Herrlich, G.E. Strecer, Abstract and Concrete Categories (John Wiley & Sons, Inc., 1990).
[2] L.M. Brown, M. Diker, Ditopological texture spaces and intuitionistic sets, Fuzzy Sets and Systems 98 (1998) 217–224.
[3] L.M. Brown, R. Ertürk, Fuzzy sets as texture spaces, I. Representation theorems, Fuzzy Sets and Systems 110 (2000) 227–236.
[4] L.M. Brown, R. Ertürk, Fuzzy sets as texture spaces, II. Subtextures and quotient textures, Fuzzy Sets and Systems 110 (2000) 237–245.
[5] L.M. Brown, R. Ertürk, Ş. Dost, Ditopological texture spaces and fuzzy topology, I. Basic concepts, Fuzzy Sets and Systems 147 (2) (2004) 171–199.
[6] L.M. Brown, R. Ertürk, Ş. Dost, Ditopological texture spaces and fuzzy topology, II. Topological considerations, Fuzzy Sets and Systems 147 (2) (2004) 201–231.
[7] L.M. Brown, R. Ertürk, Ş. Dost, Ditopological texture spaces and fuzzy topology, III. Separation axioms, Fuzzy Sets and Systems 157 (14) (2006) 1886–1912.
[8] C.L. Chang, Fuzzy topological spaces, Journal of Mathematical Analysis and Applications 24 (1968) 182–190.
[9] K C. Chattopadhyay, R.N. Hazra, Gradation of openness: fuzzy topology, Fuzzy Sets and Systems 49 (1992) 237–242.
[10] M. Demirci, Neighborhood structures of smooth topological spaces, Fuzzy Sets and Systems 92 (1997) 123–128.
[11] R. Ertürk, Seperation axioms in fuzzy topology characterized by bitopologies, Fuzzy Sets and Systems 58 (1993) 206–209.
[12] R. Ertürk, Ş Dost, S. Özçağ, Generalization some fuzzy separation axioms to ditopological texture spaces, The Journal of Nonlinear Sciences and Applications 2 (2009) 234–242.
[13] T. Kubiak, On fuzzy topologies, Ph.D. Thesis, A. Mickiewicz University Poznan, Poland, 1985.
[14] S. Özçağ, L.M. Brown, The prime dicompletion of a di-uniformity on a plain texture, Topology and its Applications 158 (2011) 1584–1594.
[15] A.A. Ramadan, Smooth topological spaces, Fuzzy Sets and Systems 48 (1992) 371–375.
[16] A. Šostak, On a fuzzy topological structure, Rendiconti Circolo Matematico Palermo Serie II No. 11 (1985) 89–103.
[17] A. Šostak, Two decates of fuzzy topology: basic ideas, notions and results, Russian Mathematical Surveys 44:6 (1989) 125–186.
[18] A. Šostak, L.M. Brown, Categories of fuzzy topology in the context of graded ditopologies on textures, Iranian Journal of Fuzzy Systems 11:6 (2014) 1–20.
[19] İ.U. Tiryaki, L.M. Brown, Plain ditopological texture spaces, Topology and its Applications 158 (2011) 2005–2015.