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Sequence Spaces of Fuzzy Numbers Defined by a Musielak-Orlicz Function

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Abstract. The purpose of this paper is to introduce some sequence spaces of fuzzy numbers defined by a Musielak-Orlicz function. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

1. Introduction and Preliminaries

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [15] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming etc. Matloka [6] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0,1]$ which satisfies the following four conditions:

1. X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;
2. X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $X(\lambda x + (1-\lambda)y) \geq \min[X(x), X(y)]$;
3. X is upper semi-continuous;
4. the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is compact and convex }\}$. The space $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\mu A = \{\mu a, \mu \in A\}$$

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for $A, B \in C(\mathbb{R}^n)$ and $\mu \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \ \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (non separable) metric space. For $0 < \alpha \leq 1$, the α-level set

$$X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$$

is a non-empty compact convex, subset of $\mathbb{R}^n$, as is the support $[X]^0$. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X + Y$ and scalar multiplication $\mu X, \mu \in \mathbb{R}$, in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\mu X]^\alpha = \mu[X]^\alpha$$

for each $0 \leq \alpha \leq 1$. Define for each $1 \leq q < \infty$

$$d_q(X, Y) = \left\{\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\right\}^{\frac{1}{q}}$$

and $d_\infty(X, Y) = \sup_{0\leq\alpha\leq1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X, Y) = \lim_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(L(\mathbb{R}^n), d_\infty)$ is a complete metric space. We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

For more details about fuzzy sequence spaces (see [1–4, 14]) and references therein. Recently, some double sequences spaces related to the concepts of Musielak-Orlicz functions, bounded-regular and invariant mean have been defined and studied in [8, 9].

Let C denote the space whose elements are the sets of distinct positive integers. Given any elements σ of C, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma, c_n(\sigma) = 0$ otherwise. Further

$$C_s = \left\{\sigma \in C : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\},$$

the set of those σ whose support has cardinality at most s, and

$$\Phi = \left\{\varphi = \{\varphi_k\} \in \ell^0 : \varphi_1 > 0, \ \ \Delta\varphi_k \geq 0 \ \text{ and } \ \Delta\Big(\frac{\varphi_k}{k}\Big) \leq 0 \ \ (k = 1, 2, \cdots)\right\},$$

where $\Delta\varphi_k = \varphi_k - \varphi_{k-1}$, where $\{\varphi_k\}$ is a real sequences (see [6]). For $\varphi \in \Phi$, Sargent [13] define the following sequence space

$$m(\varphi) = \left\{x = \{x_k\} \in \ell^0 : \sup_{s\geq1} \sup_{\sigma\in C_s} \left(\frac{1}{\varphi_s} \sum_{k\in\sigma} |x_k|\right) < \infty\right\}.$$

In [13], Sargent studied some of its properties and obtained its relationship with the space l_p. Mursaleen and Noman (see [10, 11]) introduced the notion of λ-convergent and λ-bounded sequences as follows :
Let w be the set of all complex sequences $x = (x_k)$. Let $\lambda = (\lambda_k)_{k=1}^\infty$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \ \text{ and } \ \lambda_k \to \infty \ \text{ as } \ k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [8] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \frac{1}{\lambda_m}\left(\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})|x_k - a|\right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [5] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function [7, 12]. A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$, $k = 1, 2, 3, \cdots$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. In this paper we define the following classes of sequences of fuzzy numbers:

$$l_\infty^F(\mathcal{M}, \Lambda, \sigma, p) = \left\{X = (X_k) \in w(F) : \sup_{k,n} \left[M_k\left(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\},$$

$$l_1^F(\mathcal{M}, \Lambda, \sigma, p) = \left\{X = (X_k) \in w(F) : \sup_n \sum_k \left[M_k\left(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\}$$

and

$$m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) = \left\{X = (X_k) \in w(F) : \sup_n \sup_{s \geq 1, \sigma \in C_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} \left[M_k\left(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\}.$$

When $\sigma(n) = n + 1$, we obtain the classes of sequences of fuzzy numbers as follows:

$$l_\infty^F(\mathcal{M}, \Lambda, p) = \left\{X = (X_k) \in w(F) : \sup_{k,n} \left[M_k\left(\frac{d(\Lambda_k X_{k+n}, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\},$$

$$l_1^F(\mathcal{M}, \Lambda, p) = \left\{X = (X_k) \in w(F) : \sup_n \sum_k \left[M_k\left(\frac{d(\Lambda_k X_{k+n}, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\}$$

and

$$m^F(\mathcal{M}, \Lambda, \varphi, p) = \left\{X = (X_k) \in w(F) : \sup_n \sup_{s \geq 1, \sigma \in C_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} \left[M_k\left(\frac{d(\Lambda_k X_{k+n}, \bar{0})}{\rho}\right)\right]^{p_k} < \infty\right\}.$$

If we take $p = (p_k) = 1$, we obtain the classes of sequences of fuzzy numbers as follows:

$$l_\infty^F(\mathcal{M}, \Lambda, \sigma) = \left\{X = (X_k) \in w(F) : \sup_{k,n} M_k\left(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\right) < \infty\right\},$$

$$l_1^F(\mathcal{M}, \Lambda, \sigma) = \left\{X = (X_k) \in w(F) : \sup_n \sum_k M_k\left(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\right) < \infty\right\}$$

and

$$m^F(\mathcal{M}, \Lambda, \varphi, \sigma) = \left\{X = (X_k) \in w(F) : \sup_n \sup_{s \geq 1, \sigma \in C_s} \frac{1}{\varphi_s} \sum_{k \in \sigma} M_k\left(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\right) < \infty\right\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a bounded sequence of positive real numbers with $0 < p_k \leq \sup_k = H$ and let $D = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1}$$

The main aim of this paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then the spaces $l_\infty^F(\mathcal{M}, \Lambda, \sigma, p)$, $l_1^F(\mathcal{M}, \Lambda, \sigma, p)$ and $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $X = (X_k), Y = (Y_k) \in m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$ and $\alpha, \beta \in \mathbb{C}$, then there exist positive numbers ρ_1, ρ_2 such that

$$\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho_1}\Big)\Big]^{p_k} < \infty$$

and

$$\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k Y_{\sigma^k(n)}, \bar{0})}{\rho_2}\Big)\Big]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing, convex and so by using inequality (1), we have

$$\begin{aligned}
&\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k(\alpha X_{\sigma^k(n)} + \beta Y_{\sigma^k(n)}), \bar{0})}{\rho_3}\Big)\Big]^{p_k} \\
&\leq \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{\alpha d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho_3} + \frac{\beta d(\Lambda_k Y_{\sigma^k(n)}, \bar{0})}{\rho_3}\Big)\Big]^{p_k} \\
&\leq \frac{1}{2} \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho_1}\Big)\Big]^{p_k} \\
&+ \frac{1}{2} \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k Y_{\sigma^k(n)}, \bar{0})}{\rho_2}\Big)\Big]^{p_k} \\
&< \infty.
\end{aligned}$$

This proves that $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$ is a linear space. Similarly, we can prove that $l_1^F(\mathcal{M}, \Lambda, \sigma, p)$ and $l_\infty^F(\mathcal{M}, \Lambda, \sigma, p)$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then the space $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$ is a complete metric space, with the metric defined by*

$$g(X, Y) = \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k(X_{\sigma^k(n)}, Y_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k}.$$

Proof. Let (X^i) be a Cauchy sequence in $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Then,

$$g(X^i, X^j) = \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k(X^i_{\sigma^k(n)}, X^j_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k} \to 0 \text{ as } i, j \to \infty.$$

Hence

$$\Big[M_k\Big(\frac{d(\Lambda_k(X^i_{\sigma^k(n)}, X^j_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k} \to 0 \text{ as } i, j \to \infty, \text{ for all } n.$$

Therefore (X^i) is a Cauchy sequence in $L(\mathbb{R}^n)$. Since $L(\mathbb{R}^n)$ is complete, it is convergent so that $\lim_{i\to\infty} X_k^i = X_k$, for each $k \in \mathbb{N}$. Since (X^i) is a Cauchy sequence for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that

$$g(X^i, X^j) < \epsilon, \text{ for all } i, j \geq n_0.$$

So, we have

$$\begin{aligned}
&\lim_j \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k(X^i_{\sigma^k(n)}, X^j_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k} \\
&= \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k(X^i_{\sigma^k(n)}, X_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k} \\
&< \epsilon, \text{ for all } i \geq n_0.
\end{aligned}$$

This implies that $g(X^i, X) < \epsilon$, for all $i \geq n_0$, i.e. $X^i \to X$ as $i \to \infty$, where $X = (X_k)$. Since

$$\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k(X_{\sigma^k(n)}, X_0))}{\rho}\Big)\Big]^{p_k}$$

$$\leq \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k(X^{n_0}_{\sigma^k(n)}, X_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k}$$

$$+ \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k(X_{\sigma^k(n)}, X_0))}{\rho}\Big)\Big]^{p_k}$$

then we obtain $X = (X_k) \in m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Therefore $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$ is a complete metric space. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then we have the following*
(i) the space $l_1^F(\mathcal{M}, \Lambda, \sigma, p)$ is a complete metric space, with the metric defined by

$$g(X, Y) = \sup_n \sum_k \Big[M_k\Big(\frac{d(\Lambda_k(X_{\sigma^k(n)}, Y_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k},$$

(ii) the space $l_\infty^F(\mathcal{M}, \Lambda, \sigma, p)$ is a complete metric space, with the metric defined by

$$g(X, Y) = \sup_{k,n} \Big[M_k\Big(\frac{d(\Lambda_k(X_{\sigma^k(n)}, Y_{\sigma^k(n)}))}{\rho}\Big)\Big]^{p_k}.$$

Proof. It is easy to prove in view of Theorem 2.2, so we omit the details. □

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) \subset m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$ if and only if $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$.*

Proof. Let $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$ and $(X_k) \in m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Then

$$\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty,$$

$$\Longrightarrow \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\psi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k}$$

$$\leq \Big\{\sup_{s\geq 1}\frac{\varphi_s}{\psi_s}\Big\} \sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty.$$

Therefore $(X_k) \in m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$. Hence $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) \subset m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$.
Conversely, let $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) \subset m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$. Suppose that $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} = \infty$, then there exists a sequence of natural numbers (s_i) such that $\lim_{i\to\infty} \frac{\varphi_{s_i}}{\psi_{s_i}} = \infty$. Let $(X_k) \in m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Then,

$$\sup_n \sup_{s\geq 1,\sigma\in C_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}\Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty.$$

Now, we have

$$\Longrightarrow \sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\psi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k}$$

$$\geq \Big\{\sup_{i\geq 1} \frac{\varphi_{s_i}}{\psi_{s_i}}\Big\} \sup_n \sup_{i\geq 1, \sigma\in C_s} \frac{1}{\varphi_{s_i}} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} = \infty.$$

Therefore $(X_k) \notin m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$, which is a contradiction. Therefore, $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$. This completes the proof of the theorem. □

Theorem 2.5. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) = m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$ if and only if $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$ and $\sup_{s\geq 1} \frac{\psi_s}{\varphi_s} < \infty$.*

Proof. The proof directly follows from Theorem 2.4. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then $l_1^F(\mathcal{M}, \Lambda, \sigma, p) \subset m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) \subset l_\infty^F(\mathcal{M}, \Lambda, \sigma, p)$.*

Proof. Let $(X_k) \in l_1^F(\mathcal{M}, \Lambda, \sigma, p)$, then we have

$$\sup_n \sum_k \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty.$$

Since (φ_n) is monotonic increasing, so we have

$$\begin{aligned}\frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} &\leq \frac{1}{\varphi_1} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} \\ &\leq \frac{1}{\varphi_1} \sum_{k} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty.\end{aligned}$$

Hence

$$\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty.$$

Thus $(X_k) \in m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Therefore $l_1^F(\mathcal{M}, \Lambda, \sigma, p) \subset m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Next, let $(X_k) \in m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p)$. Then, we have

$$\sup_n \sup_{s\geq 1, \sigma\in C_s} \frac{1}{\varphi_s} \sum_{k\in\sigma} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty.$$

Thus,

$$\sup_{k,n} \frac{1}{\varphi_1} \Big[M_k\Big(\frac{d(\Lambda_k X_{\sigma^k(n)}, \bar{0})}{\rho}\Big)\Big]^{p_k} < \infty, \quad \text{(on taking cardinality of } \sigma \text{ to be 1).}$$

Thus $(X_k) \in l_\infty^F(\mathcal{M}, \Lambda, \sigma, p)$. Hence $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) \subset l_\infty^F(\mathcal{M}, \Lambda, \sigma, p)$. This completes the proof of the theorem. □

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) = l_1^F(\mathcal{M}, \Lambda, \sigma, p)$ if and only if $\sup_{s\geq 1} \varphi_s < \infty$.*

Proof. It is clear that $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) = l_1^F(\mathcal{M}, \Lambda, \sigma, p)$ when $\psi_s = 1$ for all $s \in \mathbb{N}$. By Theorem 2.4, $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) \subset m^F(\mathcal{M}, \Lambda, \psi, \sigma, p)$ if and only if $\sup_{s\geq 1} \frac{\varphi_s}{\psi_s} < \infty$ i.e $\sup_{s\geq 1} \varphi_s < \infty$. Therefore by Theorem 2.6, $m^F(\mathcal{M}, \Lambda, \varphi, \sigma, p) = l_1^F(\mathcal{M}, \Lambda, \sigma, p)$ if and only if $\sup_{s\geq 1} \varphi_s < \infty$. □

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