Filomat 29:7 (2015), 1491–1499
DOI 10.2298/FIL1507491M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Reverse Proper Splittings of Rectangular Matrices

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Abstract. In this article, we introduce a new splitting for rectangular matrices called reverse proper splitting. We then propose several subclasses of this splitting and also discuss convergence results for these splittings.

1. Introduction

A class of matrices known as M-matrices were introduced by Ostrowski [21], with reference to the work of Minkowski [15, 16]. At least 50 different but equivalent characterizations of M-matrices are given by Berman and Plemmons [4]. This class of matrices arise in many areas of mathematics and statistics, such as the convergence analysis of iterative processes for the solution of linear systems and the analysis of Markov chains, to name a few. A real square matrix A is called a Z-matrix if its off-diagonal entries are nonpositive [10]. A Z-matrix A is called an M-matrix if it is of the form $A = sI - B,\ \ B \geq 0$ and $s \geq \rho(B)$, where the inequality is considered entry-wise and $\rho(X)$ denotes the spectral radius of the matrix X. In the above definition, if the condition $s > \rho(B)$ is satisfied, then it is called as nonsingular M-matrix, otherwise we call it a singular M-matrix. One of the most important properties of a nonsingular M-matrix is that its inverse is nonnegative. There are several works that have considered generalizations of some of the important properties of M-matrices in the literature. But here we are interested in singular M-matrices. In particular, finding solution of a singular system of linear equations where the co-efficient matrix is a singular M-matrix. Besides these, the proposed technique also deals with the problem for finding solution of a rectangular system of linear equations. A brief description related to these notion is as follows.

Many real world problems involve solving a system of linear equations in n unknowns

$$Ax = b \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. In a wide variety of such problems, including the Neumann problem and those for elastic bodies with free surfaces, the finite difference formulations lead to a singular, consistent linear system (1) where A is large and sparse. Often, it becomes difficult to determine the exact solution, even if it exists. Therefore, one is interested in various computational techniques to solve the above system, and thus the notion of proper splitting (a splitting $A = U - V$ of a real rectangular matrix A is called a *proper splitting ([3])* if $R(A) = R(U)$ and $N(A) = N(U)$, where $R(A)$ and $N(A)$ denote range and null space of the

2010 *Mathematics Subject Classification*. Primary 15A09; Secondary 65F15, 65F20
Keywords. Moore-Penrose inverse; Group inverse; Proper splittings; Nonnegativity; Convergence theorem; Comparison theorem.
Received: 19 December 2013; Accepted: 30 June 2014
Communicated by Dragana Cvetković Ilić
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matrix A) of a matrix was proposed so that one can attempt to solve the above problem iteratively with the following iterative scheme

$$x^{(i+1)} = U^{\dagger}Vx^{(i)} + U^{\dagger}b, \tag{2}$$

where $U^{\dagger}$ is the Moore-Penrose inverse of U (see next section for this definition). In connection with iterative techniques for solving singular linear systems, various types of matrix splittings and their convergence results are introduced by many authors (see [5–7, 12–14, 17–20, 23, 25, 26]). The iteration scheme (1) is said to be *convergent* if the spectral radius of $U^{\dagger}V$ is less than 1. As convergence result determines, the smaller is the spectral radius of $U^{\dagger}V$, the quicker is the convergence. For a proper splitting, the authors of [3] have shown that $x = A^{\dagger}b$ for any initial vector x^0 if and only if (2) is convergent.

Very recently, Jena *et al*. [14] proposed the following splittings with a aim to solve the rectangular systems of linear equations, iteratively. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper regular splitting* if it is a proper splitting such that $U^{\dagger} \geq 0$ and $V \geq 0$. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper weak regular splitting* if it is a proper splitting such that $U^{\dagger} \geq 0$ and $U^{\dagger}V \geq 0$. Thereafter, Mishra [17] introduced another splitting and is recalled next. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper nonnegative splitting* if it is a proper splitting such that $U^{\dagger}V \geq 0$. Then the convergence of the above splittings are examined. From the above definitions, it is clear that all matrices (A) may not have the above splittings. Now a question arises how to deal with the rest of matrices. In this direction, the main results of this article may be useful.

In this paper, we suggest a method for the solution of rectangular linear systems using iteration method by introducing a new splitting called *reverse proper splitting* for rectangular matrices. The definition of this splitting is motivated by the ideas of proper splittings ([3]) and of reverse splittings ([28]). The proposed iteration method associated with these splittings is convergent if the spectral radius of the iteration matrix is less than one. Here, we provide some theoretical convergence results mainly giving conditions when the corresponding iteration matrix has a spectral radius less than one.

The remainder of the paper is organized as follows. In Section 2 we set up notation and terminology. Furthermore, we collect some facts about nonnegative matrices. Section 3 contains the main results of the paper concerning the convergence results for different matrix reverse splittings. Finally, we end up this paper with a few concluding remarks.

2. Preliminaries

The set of all $m \times n$ matrices over the real numbers $\mathbb{R}$ is denoted by $\mathbb{R}^{m\times n}$. We denote the transpose of a matrix $A \in \mathbb{R}^{m\times n}$ by A^T. Let K, L be complementary subspaces of $\mathbb{R}^n$, i.e., $K \oplus L = \mathbb{R}^n$. Then $P_{K,L}$ denotes the (not necessarily orthogonal) projection of $\mathbb{R}^n$ onto K along L. So, we have $P^2_{K,L} = P_{K,L}$, $R(P_{K,L}) = K$ and $N(P_{K,L}) = L$. If in addition $K \perp L$, $P_{K,L}$ will be replaced by P_K. In that case, we also have $P^T_K = P_K$. For $K \subseteq \mathbb{R}^n$, $K^{\perp}$ will denote the orthogonal complement of K in $\mathbb{R}^n$. For real rectangular matrices A and B, $A = [a_{ij}] \geq 0$ means $a_{ij} \geq 0$ for all i, j and $A \leq B$ means $B - A \geq 0$. Let $A \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then, the *spectral radius* $\rho(A)$ of A is defined by $\rho(A) = \max\limits_{1\leq i\leq n} |\lambda_i|$ and $\lambda_i(A)$ means eigenvalues of A.

Before proceeding further, we shall outline briefly some more notation and definitions used throughout. The *Moore-Penrose inverse* of a matrix $A \in \mathbb{R}^{m\times n}$, denoted by $A^{\dagger}$ is the unique solution X of the equations

$$\begin{aligned} AXA &= A && (1)\\ XAX &= X && (2)\\ (AX)^T &= AX && (3)\\ (XA)^T &= XA && (4) \end{aligned}$$

Equivalently, it was shown in [9] that $A^{\dagger}$ is the unique matrix X which satisfies $XAx = x$ for all $x \in N(A)^{\perp}$ and $Xy = 0$ for all $y \in R(A)^{\perp}$. We list some of the well known properties of $A^{\dagger}$ that will be frequently used in this paper: $R(A^T) = R(A^{\dagger})$; $N(A^T) = N(A^{\dagger})$; $AA^{\dagger} = P_{R(A)}$; $A^{\dagger}A = P_{R(A^T)}$. In particular, if $x \in R(A^T)$ then $x = A^{\dagger}Ax$.

The *group inverse* of a matrix $A \in \mathbb{R}^{n\times n}$ (if it exists), denoted by $A^{\#}$ is the unique matrix X satisfying $A = AXA$, $X = XAX$ and $AX = XA$. Equivalently, $A^{\#}$ is the unique matrix X which satisfies $XAx = x$ for all $x \in R(A)$ and $Xy = 0$ for all $y \in N(A)$. The *index* of a real square matrix A is the least nonnegative integer k such that rank(A^{k+1})=rank(A^k). $A^{\#}$ exists if and only if index of A is 1. Let $A \in \mathbb{R}^{n\times n}$ be of index k. Then, the *Drazin inverse* of A is the unique matrix $A^D \in \mathbb{R}^{n\times n}$ which satisfies the equations $A^{k+1}A^D = A^k$, $A^DAA^D = A^D$ and $AA^D = A^DA$. (See the book by Ben-Israel and Greville, [1] for more properties.)

The next theorem is a part of Perron−Frobenius theorem.

Theorem 2.1. *(Theorem 2.20, [24])*
Let $A \geq 0$. Then A has a nonnegative real eigenvalue equal to its spectral radius.

Another result which relates spectral radius of two nonnegative matrices is given below.

Theorem 2.2. *(Theorem 2.21, [24])*
Let $A \geq B \geq 0$. Then $\rho(A) \geq \rho(B)$.

We conclude this section with the following result which is well known in the theory of nonnegative matrices.

Theorem 2.3. *(Theorem 3.16, [24])*
Let $X \in \mathbb{R}^{n\times n}$ and $X \geq 0$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq 0$.

3. Main Results

Influenced by the definition of proper splitting introduced by Berman and Plemmons [3], and reverse splitting for nonsingular matrices [28], we here propose another splitting called reverse proper splitting of rectangular(singular) matrices. Here on words all our matrices are considered to be real rectangular matrices unless otherwise mentioned.

Definition 3.1. *(Reverse proper splitting:)*
A splitting of the form $A = U - V$ is called a reverse proper splitting of A if $R(V) = R(U)$ and $N(V) = N(U)$.

Berman and Plemmons proved the following result which collects some of the properties of a proper splitting.

Theorem 3.2. *(Theorem 1, [3])*
Let $A = U - V$ be a proper splitting. Then
(a) $A = U(I - U^{\dagger}V)$;
(b) $I - U^{\dagger}V$ is invertible;
(c) $A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}$.

Now we present a similar result for the reverse proper splitting.

Theorem 3.3. *Let $A = U - V$ be a reverse proper splitting. Then*
(i) $VV^{\dagger} = UU^{\dagger}$; $V^{\dagger}V = U^{\dagger}U$.
(ii) $V = U(I - U^{\dagger}A)$.
(iii) $I - U^{\dagger}A$ is invertible.
(iv) $V^{\dagger} = (I - U^{\dagger}A)^{-1}U^{\dagger}$.

Proof. (i) $VV^\dagger = P_{R(V)} = P_{R(U)} = UU^\dagger$. The second identity is similar.

(ii) Since $R(V) = R(U)$, it follows that $R(A) \subseteq R(U)$. Hence $UU^\dagger A = A$ and we then have $V = U - A = U - UU^\dagger A = U(I - U^\dagger A)$.

(iii) Let $(I - U^\dagger A)x = 0$. Then $x = U^\dagger Ax \in R(U^\dagger) = R(U^T) = R(V^T)$ and $x = V^\dagger Vx = U^\dagger Ux$. Therefore $Ux = UU^\dagger Ax = Ax$. So $(U - A)x = 0$ i.e., $Vx = 0$. So $x \in N(V)$. Hence $x = 0$. Thus $I - U^\dagger A$ is invertible.

(iv) Set $X = (I - U^\dagger A)^{-1}U^\dagger$. Let $x \in R(V^T)$. So $x = V^\dagger Vx = U^\dagger Ux$. Then $XVx = (I - U^\dagger A)^{-1}U^\dagger(U - A)x = (I - U^\dagger A)^{-1}(U^\dagger Ux - U^\dagger Ax) = (I - U^\dagger A)^{-1}(x - U^\dagger Ax) = (I - U^\dagger A)^{-1}(I - U^\dagger A)x = x$. Again, suppose that $y \in N(V^T) = N(U^T) = N(U^\dagger)$. Then $Xy = (I - U^\dagger A)^{-1}U^\dagger y = 0$. Hence $V^\dagger = (I - U^\dagger A)^{-1}U^\dagger$. □

We next recall a result of Mishra and Sivakumar, [18] which is for proper splittings.

Lemma 3.4. *(Lemma 2.6, [18])*
Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m\times n}$. Let μ_i, $1 \le i \le s$ and λ_j, $1 \le j \le s$ be the eigenvalues of the matrices $U^\dagger V$ ($VU^\dagger$) and $A^\dagger V$ ($VA^\dagger$), respectively. Then for every j, we have $1 + \lambda_j \ne 0$. Also, for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ and for every j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.

An analogous result for reverse proper splitting is stated next. The proof is very similar and is omitted.

Lemma 3.5. *Let $A = U - V$ be a reverse proper splitting of $A \in \mathbb{R}^{m\times n}$. Let μ_i, $1 \le i \le s$ and λ_j, $1 \le j \le s$ be the eigenvalues of the matrices $U^\dagger A$ ($AU^\dagger$) and $V^\dagger A$ ($AV^\dagger$), respectively. Then for every j, we have $1 + \lambda_j \ne 0$. Also, for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$ and for every j, there exists i such that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.*

Next result adds a few more properties of a reverse proper splitting with the assumption of two other conditions.

Theorem 3.6. *Let $A = U - V$ be a reverse proper splitting. Suppose that $U^\dagger \ge 0$ and $A \ge 0$. Then the following are equivalent:*
(i) $V^\dagger \ge 0$;
(ii) $V^\dagger A \ge 0$;
(iii) $\rho(U^\dagger A) < 1$.

Proof. $(i) \Rightarrow (ii)$: The conditions $V^\dagger \ge 0$ and $A \ge 0$ imply $V^\dagger A \ge 0$.

$(ii) \Rightarrow (iii)$: Suppose that $V^\dagger A \ge 0$. Also from the given data, we have $U^\dagger A \ge 0$. Let λ and μ be any nonnegative eigenvalues of $V^\dagger A$ and $U^\dagger A$, respectively. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, $\lambda \ge 0$. Then f is a strictly increasing function. Then by Lemma 3.5, $\mu = \frac{\lambda}{1+\lambda}$. So, μ attains its maximum when λ is maximum. But λ is maximum when $\lambda = \rho(V^\dagger A)$. As a result, the maximum value of μ is $\rho(U^\dagger A)$. Hence, $\rho(U^\dagger A) = \frac{\rho(V^\dagger A)}{1+\rho(V^\dagger A)} < 1$.

$(iii) \Rightarrow (i)$: Let $\rho(U^\dagger A) < 1$. Then by Theorem 2.3, we have $(I - U^\dagger A)^{-1} = \sum_{k=0}^{\infty}(U^\dagger A)^k \ge 0$. Since $A \ge 0$, so

$$V^\dagger = (I - U^\dagger A)^{-1}U^\dagger = \sum_{k=0}^{\infty}(U^\dagger A)^k U^\dagger \ge 0. \quad \square$$

We now introduce a subclass of a reverse splitting called reversible proper splitting and the definition is as follows.

Definition 3.7. *A reverse splitting $A = U - V$ is called a reversible proper splitting of A if $\lambda_i(U^\dagger A) \ge 0$ for $i = 1, 2, \cdots, n$.*

Example 3.8. *Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$. Set $U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then, $U^\dagger = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{pmatrix}$, $U^\dagger A = \begin{pmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\lambda_i(U^\dagger A) = \{0, 0.5, 1.5\}$ for $i = 1, 2, 3$. So $\lambda_i(U^\dagger A) \ge 0$. Hence A possess a reversible proper splitting.*

A convergence result for a reversible proper splitting is presented next.

Theorem 3.9. *Let $A = U - V$ be a reversible proper splitting of A and $\rho(U^{\dagger}A) < 1$, then $\rho(U^{\dagger}V) < 1$.*

Proof. Since $A = U - V$ is a reversible proper splitting of A, then $\lambda_i(U^{\dagger}A) \geq 0$ for $i = 1, 2, \cdots, n$. From Theorem 3.3 (ii) $V = U(I - U^{\dagger}A)$. So $U^{\dagger}V = U^{\dagger}U(I - U^{\dagger}A)$, we then have $U^{\dagger}V + U^{\dagger}A = U^{\dagger}U$ and $\lambda_i(U^{\dagger}V) + \lambda_i(U^{\dagger}A) = \lambda_i(U^{\dagger}U) \leq 1$ for $i = 1, 2, \cdots, n$. The condition $\rho(U^{\dagger}A) < 1$ implies $\lambda_i(U^{\dagger}A) < 1$, $i = 1, 2, \cdots, n$ (since $\lambda_i(U^{\dagger}A) \geq 0$). Hence, $\rho(U^{\dagger}V) = \max\limits_{1 \leq i \leq n}\{\lambda_i(U^{\dagger}V)\} < 1$. □

In case of square nonsingular matrices Theorem 1, [28] follows from the above result as a corollary, and is obtained below.

Corollary 3.10. *(Theorem 1, [28])*
Let $A = U - V$ be a reversible splitting of $A \in \mathbb{R}^{n \times n}$ and $\rho(U^{-1}A) < 1$, then $\rho(U^{-1}V) < 1$.

The authors of [14] and [17] studied the convergence and comparison results for proper regular splittings, proper weak regular splittings and proper nonnegative splittings. Unlike the above splittings, we now introduce a few new splittings which are subclasses of reverse proper splittings.

Definition 3.11. *A splitting $A = U - V$ is called*
(i) a reversible proper regular splitting of A if A possess a reversible proper splitting with $U^{\dagger} \geq 0$ and $A \geq 0$;
(ii) a reversible proper weak regular splitting of A if A possess a reversible proper splitting with $U^{\dagger} \geq 0$ and $U^{\dagger}A \geq 0$;
(iii) a reversible proper nonnegative splitting of A if A possess a reversible proper splitting with $U^{\dagger}A \geq 0$.

Example 3.12. *Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. Set $U = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $U^{\dagger} = \begin{pmatrix} 0.05 & 0.05 \\ 0.15 & 0.15 \end{pmatrix} \geq 0$ and $U^{\dagger}A = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.6 \end{pmatrix}$ and $\lambda_i(U^{\dagger}A) = \{0, 0.7\} \geq 0$. Since $A \geq 0$, $U^{\dagger} \geq 0$ and $\lambda_i(U^{\dagger}A) \geq 0$, then A has a reversible proper regular splitting.*

The above example also possess a reversible proper weak regular splitting and a reversible proper nonnegative splitting. Since $A = U - V$ is reversible proper regular splitting, so $A = U - V$ is a reverse proper splitting with $U^{\dagger} \geq 0$ and $A \geq 0$. Therefore $\rho(U^{\dagger}A) < 1$, by Theorem 3.6 and then it follows from Theorem 3.9 that $\rho(U^{\dagger}V) < 1$. Thus we remark that *the reversible proper regular splitting is always convergent.* Hence we will focus on convergence of other proposed splittings, but before that we establish an analogous result to Theorem 3.6.

Theorem 3.13. *Let $A = U - V$ be a reversible proper regular splitting with $V^{\dagger} \geq 0$. Then the following conditions holds.*
(i) $V^{\dagger} \geq U^{\dagger}$.
(ii) $\rho(V^{\dagger}A) \geq \rho(U^{\dagger}A)$.
(iii) $\rho(U^{\dagger}A) = \frac{\rho(V^{\dagger}A)}{1+\rho(V^{\dagger}A)} < 1$.

Proof. (i): Theorem 3.3 yields $V^{\dagger} = (I - U^{\dagger}A)^{-1}U^{\dagger}$. So, we have $U^{\dagger} = (I - U^{\dagger}A)V^{\dagger} = V^{\dagger} - U^{\dagger}AV^{\dagger}$. Therefore $U^{\dagger}AV^{\dagger} = V^{\dagger} - U^{\dagger}$. Since A possess reversible proper regular splitting, we have $U^{\dagger} \geq 0$ and $A \geq 0$. Hence $V^{\dagger} \geq 0$. So $U^{\dagger}AV^{\dagger} \geq 0$. Thus $V^{\dagger} - U^{\dagger} \geq 0$, i.e., $V^{\dagger} \geq U^{\dagger}$.

(ii): Post-multiplying A to the inequality $V^{\dagger} \geq U^{\dagger}$, and then Theorem 2.2 yields $\rho(V^{\dagger}A) \geq \rho(U^{\dagger}A)$.

(iii): We have $A = U - V$ be a reverse proper splitting with $U^{\dagger} \geq 0$ and $A \geq 0$. So $V^{\dagger}A \geq 0$ and $U^{\dagger}A \geq 0$. Let λ and μ be any nonnegative eigenvalues of $V^{\dagger}A$ and $U^{\dagger}A$, respectively. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, $\lambda \geq 0$. Then f is a strictly increasing function. Then by Lemma 3.5, $\mu = \frac{\lambda}{1+\lambda}$. So, μ attains its maximum when λ is maximum. But λ is maximum when $\lambda = \rho(V^{\dagger}A)$. As a result, the maximum value of μ is $\rho(U^{\dagger}A)$. Hence, $\rho(U^{\dagger}A) = \frac{\rho(V^{\dagger}A)}{1+\rho(V^{\dagger}A)} < 1$. □

When two splittings of A are given, it is of interest to compare the spectral radii of the corresponding iteration matrices. The comparison of asymptotic rates of convergence of the iteration matrix induced by two splittings of a given square nonsingular matrix, has been studied by many authors, Csordas and Varga [8], Elsner [11], Song [22], Varga [24] and Woźnicki [27], to name a few. In case of square nonsingular matrices, U is also nonsingular in the splitting $A = U - V$. Woźnicki [27] has considered different types of splittings (such as regular, weak regular and weak nonnegative splittings of different types) of a given monotone matrix and has proved corresponding comparison theorems. Elsner [11] considered weak regular splitting and multi splittings, and proved comparison results. Song [22] studied a comparison theorem for nonnegative splittings and then applied it to study different basic iterative methods. Recently, Mishra and Sivakumar [18], Jena and Mishra [12, 13], and Jena *et al.* [14] have proved different comparison results for splittings of rectangular (singular) matrices. Next result is in this direction.

Theorem 3.14. *Let $A = U_1 - V_1 = U_2 - V_2$ be two reversible proper regular splittings such that $V_2^\dagger \geq V_1^\dagger$ and $V_i^\dagger \geq 0$ for $i = 1, 2$. If $A \geq 0$, then*

$$\rho(U_1^\dagger A) \leq \rho(U_2^\dagger A) < 1.$$

Proof. We have $\rho(U_i^\dagger A) < 1$ for $i = 1, 2$ by Theorem 3.6. Since $A \geq 0$ and $V_i^\dagger \geq 0$ for $i = 1, 2$, then $V_i^\dagger A \geq 0$. Let λ_i be the eigenvalues of $V_i^\dagger A$ for $i = 1, 2$. Similarly, the fact $V_2^\dagger \geq V_1^\dagger$ implies $(V_2^\dagger - V_1^\dagger)A \geq 0$. Since $f(\lambda) = \frac{\lambda}{1+\lambda}$ is a strictly increasing function for $\lambda \geq 0$ and $V_2^\dagger A \geq V_1^\dagger A \geq 0$, so Theorem 2.2 yields $\rho(V_2^\dagger A) \geq \rho(V_1^\dagger A)$. Hence

$$\frac{\rho(V_2^\dagger A)}{1 + \rho(V_2^\dagger A)} \geq \frac{\rho(V_1^\dagger A)}{1 + \rho(V_1^\dagger A)}$$

i.e.

$$\rho(U_1^\dagger A) \leq \rho(U_2^\dagger A) < 1.$$

□

From the proof of Theorem 3.9, we get $\lambda_i(U^\dagger V) + \lambda_i(U^\dagger A) = \lambda_i(U^\dagger U) \leq 1$ for $i = 1, 2, \cdots, n$ and then by the above Theorem, we obtain the following comparison result as a corollary.

Corollary 3.15. *Let $A = U_1 - V_1 = U_2 - V_2$ be two reversible proper regular splittings such that $V_2^\dagger \geq V_1^\dagger$ and $V_i^\dagger \geq 0$ for $i = 1, 2$. If $A \geq 0$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

From Theorem 3.9, we conclude that showing a convergence of a reversible proper splitting is equivalent to show that $\rho(U^\dagger A) < 1$. Hence for all other different subclasses of reversible proper splitting, we will focus on $\rho(U^\dagger A)$. The next result discusses convergence of reversible proper weak regular splitting together with some more other equivalent conditions.

Theorem 3.16. *Let $A = U - V$ be a reversible proper weak regular splitting. Then (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (g) where*
(a) $V^\dagger U \geq 0$;
(b) $\rho(U^\dagger A) = \frac{\rho(V^\dagger U) - 1}{\rho(V^\dagger U)}$;
(c) $\rho(U^\dagger A) < 1$;
(d) $(I - U^\dagger A)^{-1} \geq 0$;
(e) $V^\dagger A \geq 0$;
(f) $V^\dagger A \geq U^\dagger A$;
(g) $\rho(U^\dagger A) = \frac{\rho(V^\dagger A)}{1 + \rho(V^\dagger A)}$.

Proof. Since A has a reversible proper weak regular splitting, so A possess reverse proper splitting with $U^\dagger \geq 0$, $U^\dagger A \geq 0$.

(a) $\Rightarrow$ (b): Since $V^\dagger U \geq 0$, then by Theorem 2.1, there exists a nonnegative vector x ($x \neq 0$) such that $U^\dagger Ax = \rho(U^\dagger A)x$. Hence $x \in R(U^\dagger) = R(U^T)$ so that $U^\dagger Ux = x$. By Theorem 3.3 (iv), we also have $V^\dagger = (I - U^\dagger A)^{-1}U^\dagger$ it implies $V^\dagger U = (I - U^\dagger A)^{-1}U^\dagger U$. Then

$$V^\dagger Ux = (I - U^\dagger A)^{-1}U^\dagger Ux = (I - U^\dagger A)^{-1}x = \frac{1}{1 - \rho(U^\dagger A)}x.$$

So $\frac{1}{1-\rho(U^\dagger A)} \geq 0$ and is an eigenvalue of $V^\dagger U$. Hence $0 \leq \frac{1}{1-\rho(U^\dagger A)} \leq \rho(V^\dagger U)$, i.e., $\rho(U^\dagger A) \leq \frac{\rho(V^\dagger U)-1}{\rho(V^\dagger U)}$. Again, the condition $V^\dagger U \geq 0$ yields existence of a nonnegative vector y ($y \neq 0$) such that $V^\dagger Uy = \rho(V^\dagger U)y$. Then $y \in R(V^\dagger) = R(V^T) = R(U^T) = R(U^\dagger)$. So $U^\dagger Uy = y$. Also $U^\dagger U = (I - U^\dagger A)V^\dagger U$. Therefore $y = (I - U^\dagger A)V^\dagger Uy$ which implies $y - V^\dagger Uy = -U^\dagger AV^\dagger Uy$. Hence $U^\dagger Ay = \frac{V^\dagger U - I}{V^\dagger U}y$, again gives $U^\dagger Ay = \frac{\rho(V^\dagger U)-1}{\rho(V^\dagger U)}y$. So $\frac{\rho(V^\dagger U)-1}{\rho(V^\dagger U)} \leq \rho(U^\dagger A)$. Hence $\rho(U^\dagger A) = \frac{\rho(V^\dagger U)-1}{\rho(V^\dagger U)}$.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (d): The condition $U^\dagger A \geq 0$ and Theorem 2.3 yields $(I - U^\dagger A)^{-1} = \sum_{k=0}^{\infty}(U^\dagger A)^k \geq 0$.

(d) $\Rightarrow$ (e): By Theorem 3.3 (iv), we also have $V^\dagger = (I - U^\dagger A)^{-1}U^\dagger$. Since $(I - U^\dagger A)^{-1} \geq 0$, $U^\dagger A \geq 0$, then $V^\dagger A = (I - U^\dagger A)^{-1}U^\dagger A$ implies $V^\dagger A \geq 0$.

(e) $\Rightarrow$ (f): Since $V^\dagger A = (I - U^\dagger A)^{-1}U^\dagger A$. So $(I - U^\dagger A)V^\dagger A = U^\dagger A$, i.e., $V^\dagger A - U^\dagger A = U^\dagger AV^\dagger A$. Again $V^\dagger A \geq 0$ and $U^\dagger A \geq 0$ implies $V^\dagger A - U^\dagger A \geq 0$. Hence $V^\dagger A \geq U^\dagger A$.

(f) $\Rightarrow$ (g): We have $V^\dagger A \geq 0$ and $U^\dagger A \geq 0$. Let λ and μ be any nonnegative eigenvalues of $V^\dagger A$ and $U^\dagger A$, respectively. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, $\lambda \geq 0$. Then f is a strictly increasing function. Then by Lemma 3.5, $\mu = \frac{\lambda}{1+\lambda}$. So, μ attains its maximum when λ is maximum. But λ is maximum when $\lambda = \rho(V^\dagger A)$. As a result, the maximum value of μ is $\rho(U^\dagger A)$. Hence, $\rho(U^\dagger A) = \frac{\rho(V^\dagger A)}{1+\rho(V^\dagger A)}$. □

A convergence result for the reversible proper nonnegative splitting is presented next.

Theorem 3.17. *Let $A = U - V$ be a reversible proper nonnegative splitting. Then $V^\dagger A \geq 0$ if and only if $\rho(U^\dagger A) = \frac{\rho(V^\dagger A)}{1+\rho(V^\dagger A)} < 1$.*

Proof. Suppose that $V^\dagger A \geq 0$. The fact $A = U - V$ is a reversible proper nonnegative splitting yields that A possess a reverse proper splitting with $U^\dagger A \geq 0$. Now proceeding as in the proof of Theorem 3.16 (f) $\Rightarrow$ (g), we have the desired result.

Conversely, let $\rho(U^\dagger A) < 1$. Then by Theorem 2.3, we get $(I - U^\dagger A)^{-1} = \sum_{k=0}^{\infty}(U^\dagger A)^k \geq 0$. Since $V^\dagger A = (I - U^\dagger A)^{-1}U^\dagger A$. So, we have $V^\dagger A = (I - U^\dagger A)^{-1}U^\dagger A = \sum_{k=0}^{\infty}(U^\dagger A)^{k+1} \geq 0$. □

Using the above one, the convergence result for reversible proper nonnegative splitting is presented next.

Corollary 3.18. *Let $A = U - V$ be a reversible proper nonnegative splitting. If $V^\dagger A \geq 0$, then $\rho(U^\dagger V) < 1$.*

Theorem 3.19. *Let $A = U - V$ be a reversible splitting with $V^\dagger U \geq 0$ ($UV^\dagger \geq 0$). Then,*

$$\rho(U^\dagger A) = \frac{\rho(V^\dagger U) - 1}{\rho(V^\dagger U)} < 1.$$

Conversely, if $\rho(U^\dagger A) < 1$, then $V^\dagger U \geq 0$.

Proof. The proof is similar to $(b) \Rightarrow (c)$ of Theorem 3.16. □

By using the above Theorem and Theorem 3.9, we obtain the following result.

Corollary 3.20. *Let $A = U - V$ be a reversible proper nonnegative splitting. If $V^\dagger U \geq 0$, then $\rho(U^\dagger V) < 1$.*

4. Conclusions

In this section, we mention a result for the group inverse, analogous to the case of the Moore-Penrose inverse. The proof can be extracted analogously from the proof of Theorem 3.3.

Theorem 4.1. *Let $A = U - V$ be a reverse proper splitting of $A \in \mathbb{R}^{n\times n}$. Suppose that $V^\#$ exists. Then*
(a) $U^\#$ exists.
(b) $VV^\# = UU^\# = U^\# U$.
(c) $V = U(I - U^\# A)$.
(d) $I - U^\# A$ is invertible.
(e) $V^\# = (I - U^\# A)^{-1} U^\#$.

We conclude that all other results can also be written in terms of group inverse of a matrix. But in order to obtain the Drazin inverse analog of all those results in the last section, we have to introduce a few more generalizations of reverse proper splittings using similar idea from index splitting [25] and index-proper splitting [5, 13]. Not only these ideas, many other problems which were solved using proper splittings may also be studied using reverse proper splittings. Finally, convergence rate for both the splittings may be compared. Berman and Neumann [2] and Mishra and Sivakumar [20] obtained different methods of constructions for proper splittings. Hence the same case may be examined for the reverse proper splittings.

From the discussed results in last section, we now finish this section with a few concluding remarks. First of all, we have not here compared the proposed splitting with the existing splittings. Nevertheless, we defer this for future investigation. At this point, we can say only that these splittings may be useful whenever the co-efficient matrix A in (1) fails to have any subclass of proper splittings. Besides these, the idea of reverse proper splitting may also be used for other theoretical problems.

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