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Limit Formulas Related to the p-Gamma and p-Polygamma Functions at Their Singularities

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Abstract. In this paper, we mainly give much simple proof to Theorem 1.1 and (1.6) of Theorem 1.2 posed in the paper " F. Qi, Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities, Filomat 27 (2013) 601-604."

1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $\mathrm{Re} z > 0$ by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-z}dt$. The digamma (or psi) function is defined as the logarithmic derivative of Euler's gamma function for positive real numbers z, that is $\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Moreover, Euler also gave another equivalent definition for the $\Gamma(z)$ (see [1][15]),

$$\Gamma_p(z) = \frac{p!p^z}{z(z+1)\dots(z+p)} = \frac{p^z}{z(1+\frac{z}{1})\dots(1+\frac{z}{p})}, \tag{1}$$

where p is a positive integer, and

$$\lim_{p\to\infty}\Gamma_p(z) = \Gamma(z). \tag{2}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function in [5], that is,

$$\psi_p(z) = \frac{d}{dz}\ln\Gamma_p(z) = \frac{\Gamma_p'(z)}{\Gamma_p(z)}. \tag{3}$$

The function ψ_p defined in (3) has the following series representation

$$\psi_p(z) = \ln p - \sum_{k=0}^{p}\frac{1}{z+k} \tag{4}$$

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in [6]. Its derivatives are given by

$$\psi_p^{(i)}(z) = \sum_{k=0}^{p} \frac{(-1)^{i-1} i!}{(z+k)^{i+1}}. \tag{5}$$

In [8] and [9], the limit formulas

$$\lim_{z\to -k} \frac{\Gamma(nz)}{\Gamma(qz)} = (-1)^{(n-q)k} \frac{q}{n} \frac{(qk)!}{(nk)!} \tag{6}$$

and

$$\lim_{z\to -k} \frac{\psi(nz)}{\psi(qz)} = \frac{q}{n} \tag{7}$$

for any non-negative integer k and all positive integers n and q were established by A. Prabhu and H. M. Srivastava. Later, by using explicit formula for the n-th derivative of the cotangent function, F. Qi obtained the following formulas

$$\lim_{z\to -k} \frac{\psi^{(i)}(nz)}{\psi^{(i)}(qz)} = \left(\frac{q}{n}\right)^{i+1} \tag{8}$$

and

$$\lim_{z\to -k} \frac{\Gamma^{(i)}(nz)}{\Gamma^{(i)}(qz)} = (-1)^{(n-q)k} \left(\frac{q}{n}\right)^{i+1} \frac{(qk)!}{(nk)!} \tag{9}$$

for any non-negative integer k and all positive integers n and q in [10][11] and [12]. İ. Ege, and E. Yýldýrým got some equalities of the $\Gamma_p(z)$ for $0 < p < 1$ by the neutrix and neutrix limit in [3]. In addition, V. B. Krasniqi, H. M. Srivastava and S. S. Dragomir obtained some properties related to convexity, log-convexity and complete monotonicity by defined $(p,q)-$gamma and $(p,q)-$psi functions in [7]. In particular, the $(p,q)-$gamma function coincides with the classical $p-$gamma function $\Gamma_p(z)$ when $q \to 1$. For more results, we refer the reader to the papers [16]-[17].

It is easily known that the p-gamma function $\Gamma_p(z)$ is single valued and analytic over the entire complex plane, except for the points $z = 0, -1, -2, \ldots, -p$. Motivated by limit formulas (1.6)-(1.9), we present some limit formulas related to ratios of derivatives of the $p-$gamma function.

2. Main Results

Theorem 2.1. *For every positive integer n, we have*

$$\lim_{z\to 0} \frac{\Gamma_p(nz)}{\Gamma_p(z)} = \frac{1}{n}. \tag{10}$$

Proof. By a simple computation we have

$$\begin{aligned}\lim_{z\to 0} \frac{\Gamma_p(nz)}{\Gamma_p(z)} &= \lim_{z\to 0} \frac{p!p^{nz}}{nz(nz+1)\ldots(nz+p)} \frac{z(z+1)\ldots(z+p)}{p!p^{z}} \\ &= \lim_{z\to 0} \frac{p^{(n-1)z}(z+1)\ldots(z+p)}{n(nz+1)\ldots(nz+p)} = \frac{1}{n}.\end{aligned}$$

The proof is completed. □

Remark 2.2. *Let $p \to \infty$ at the both sides of the limit equality (10), we obtain Theorem 1 in [8].*

Theorem 2.3. *For any non-negative integer k and all positive integers n, q satisfying $nk = m \le p$, $qk = l \le p$, the following equality holds*

$$\lim_{z\to -k} \frac{\Gamma_p(nz)}{\Gamma_p(qz)} = (-1)^{l-m} \left(\frac{q}{n}\right)^2 \frac{p^{l-m}(l-1)!(p-l)!}{(m-1)!(p-m)!}. \tag{11}$$

Proof. Using expression of the function $\Gamma_p(z)$, it follows that

$$\begin{aligned}
&\lim_{z\to -k} \frac{\Gamma_p(nz)}{\Gamma_p(qz)} = \lim_{z\to -k} \frac{p!p^{nz}}{nz(nz+1)\dots(nz+p)} \frac{qz(qz+1)\dots(qz+p)}{p!p^{qz}} \\
&= \lim_{z\to -k} \frac{qp^{(n-q)z}}{n} \frac{(qz+1)\dots(qz+l-1)(qz+l)(qz+l+1)\dots(qz+p)}{(nz+1)\dots(nz+m-1)(nz+m)(nz+m+1)\dots(nz+p)} \\
&= \frac{qp^{(n-q)(-k)}}{n} \lim_{z\to -k} \frac{(qz+1)\dots(qz+l-1)q(z+k)(qz+l+1)\dots(qz+p)}{(nz+1)\dots(nz+m-1)n(z+k)(nz+m+1)\dots(nz+p)} \\
&= (-1)^{l-m} \left(\frac{q}{n}\right)^2 \frac{p^{l-m}(l-1)!(p-l)!}{(m-1)!(p-m)!}.
\end{aligned}$$

□

Remark 2.4. *It is obvious that the following limit equality holds*

$$\lim_{p\to\infty} \frac{p^{l-m}(p-l)!}{(p-m)!} = 1. \tag{12}$$

In fact, without loss of generality, we suppose $q \le n$. Applying Stirling's formula $n! \sim \sqrt{2\pi n} n^n e^{-n}$, we have

$$\begin{aligned}
&\lim_{p\to\infty} \frac{p^{l-m}(p-l)!}{(p-m)!} = \lim_{p\to\infty} \frac{p^{l-m}\sqrt{2\pi(p-l)}(p-l)^{(p-l)}e^{-(p-l)}}{\sqrt{2\pi(p-m)}(p-m)^{(p-m)}e^{-(p-m)}} \\
&= \lim_{p\to\infty} \frac{1}{e^{m-l}} \sqrt{\frac{p-l}{p-m}} \left(\frac{p-l}{p-m}\right)^{p-m} \left(\frac{p-l}{p}\right)^{m-l} = 1.
\end{aligned}$$

Let $p \to \infty$ at the both sides of the limit equality (11), *we obtain limit formula* (6).

Remark 2.5. *Taking $k = 0$ and $q = 1$, the formula (11) becomes (10). In fact,*

$$\lim_{z\to 0} \frac{\Gamma_p(nz)}{\Gamma_p(z)} = (-1)^{0-0} \left(\frac{1}{n}\right)^2 \frac{m}{l} \frac{p^{0-0}0!(p-0)!}{0!(p-0)!} = \frac{1}{n}$$

where $nk = m, qk = l$.

Theorem 2.6. *For any non-negative integer k and all positive integers n, q satisfying $nk = m \le p$, $qk = l \le p$, it holds*

$$\lim_{z\to -k} \frac{\psi_p^{(i)}(nz)}{\psi_p^{(i)}(qz)} = \left(\frac{q}{n}\right)^{i+1}. \tag{13}$$

Proof. An easy computation results in

$$\lim_{z\to -k}\frac{\psi_p^{(i)}(nz)}{\psi_p^{(i)}(qz)} = \lim_{z\to -k}\frac{\sum_{k=0}^{p}\frac{(-1)^{i-1}i!}{(nz+k)^{i+1}}}{\sum_{k=0}^{p}\frac{(-1)^{i-1}i!}{(qz+k)^{i+1}}}$$

$$= \lim_{z\to -k}\frac{\frac{(-1)^{i-1}i!}{(nz)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(nz+m)^{i+1}}+\frac{(-1)^{i-1}i!}{(nz+m+1)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(nz+p)^{i+1}}}{\frac{(-1)^{i-1}i!}{(qz)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(qz+l-1)^{i+1}}+\frac{(-1)^{i-1}i!}{(qz+l)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(qz+p)^{i+1}}}$$

$$= \lim_{z\to -k}\frac{\left(\frac{(-1)^{i-1}i!}{(nz)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(nz+m)^{i+1}}+\frac{(-1)^{i-1}i!}{(nz+m+1)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(nz+p)^{i+1}}\right)(z+k)^{i+1}}{\left(\frac{(-1)^{i-1}i!}{(qz)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(qz+l-1)^{i+1}}+\frac{(-1)^{i-1}i!}{(qz+l)^{i+1}}+\ldots+\frac{(-1)^{i-1}i!}{(qz+p)^{i+1}}\right)(z+k)^{i+1}}$$

$$= \left(\frac{q}{n}\right)^{i+1}$$

where we use (5). □

Remark 2.7. *Taking the limit both sides of the limit equality (13) as $p \to \infty$, we obtain Theorem 1.1 in [10]. If adding up $i = 0$, we can get (1.2) in [11].*

Remark 2.8. *We give simply new proofs of (6)-(8) by limit formulas related to ratios of derivatives of the $p-$gamma function $\Gamma_p(z)$.*

Finally, we pose a conjecture.

Conjecture 2.9. *For any non-negative integer k and all positive integers $n > 2, q$ satisfying $nk = m \le p$, $qk = l \le p$, then*

$$\lim_{z\to -k}\frac{\Gamma_p^{(i)}(nz)}{\Gamma_p^{(i)}(qz)} = (-1)^{l-m}\left(\frac{q}{n}\right)^{i+2}\frac{p^{l-m}(l-1)!(p-l)!}{(m-1)!(p-m)!}. \tag{14}$$

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