Invertible Harmonic and Harmonic Quasiconformal Mappings

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Abstract. Recently G. Alessandrini - V. Nesi and Kalaj generalized a classical result of H. Kneser (RKC-Theorem). Using a new approach we get some new results related to RKC-Theorem and harmonic quasiconformal (HQC) mappings. We also review some results concerning bi-Lipschitz property for HQC-mappings between Lyapunov domains and related results in planar case using some novelty.

1. Introduction

G. Alessandrini and V. Nesi prove necessary and sufficient criteria of invertibility for planar harmonic mappings which generalize a classical result of H. Kneser, also known as the Radó-Kneser-Choquet theorem (RKC-Theorem), cf. [1].

Let $\mathbb{S}^1$ denote the unit disk and let γ be a closed Jordan curve, and $f_0 : \mathbb{S}^1 \overset{\text{onto}}{\longrightarrow} tr(\gamma)$, where tr denotes the trace of a curve. The basic question that they address in this paper is under which conditions on f_0 we have that Poisson integral of f_0, $F = P[f_0]$ is a homeomorphism of the unit disk $\mathbb{B}$ onto D, where D denotes the bounded open, simply connected set for which $\partial D = \gamma$. The fundamental result for this issue is a classical theorem, first conjectured by T. Radó in 1926, which was proved immediately after by H. Kneser, and subsequently rediscovered, with a different proof, by G. Choquet, cf. [1]. Let us recall the result.

Theorem 1.1 (H. Kneser). *If D is convex, then F is a homeomorphism of $\mathbb{B}$ onto D.*

We first state G. Alessandrini and V. Nesi results using their notation, [1].

Recall, let $\mathbb{B} := \{(x; y) \in R^2 : x^2 + y^2 < 1\}$ denote the unit disk. Given a homeomorphism f_0 from the unit circle $\partial \mathbb{B}$ onto a simple closed curve $\gamma \subset \mathbb{R}^2$, let us consider the solution $f \in C^2(\mathbb{B}; \mathbb{R}^2) \cap C(\bar{B}; \mathbb{R}^2)$ to the following Dirichlet problem

$$\Delta f = 0 \quad \text{in} \quad \mathbb{B}; \quad f = f_0 \quad \text{on} \quad \partial \mathbb{B}. \tag{1.1}$$

Following the notation in [1], denote by Df the derivative of f and note that $\det Df$ is Jacobian of f.

Theorem 1.2 (Theorem AN1 [1]). *Let $f_0 : \partial\mathbb{B} \to \gamma \subset \mathbb{R}^2$ be an orientation preserving diffeomorphism of class C^1 onto a simple closed curve γ. Let D be the bounded domain such that $\partial D = \gamma$. Let $f \in C^2(\mathbb{B}; R^2) \cap C(\overline{\mathbb{B}}; R^2)$ be the solution to* (1.1) *and assume, in addition, that $f \in C^1(\overline{\mathbb{B}}; \mathbb{R}^2)$. The mapping f is a diffeomorphism of $\overline{\mathbb{B}}$ onto $\overline{D}$ if and only if*

$$\det Df > 0 \text{ everywhere on } \partial\mathbb{B}. \tag{1.2}$$

In order to compare this statement with Kneser's Theorem, it is worth noticing that, when γ is convex, (1.2) is automatically satisfied. Indeed it is proved, see Lemma 5.3 [1], that $\det Df > 0$ always holds true on the points of $\partial\mathbb{B}$ which are mapped through f_0 on the part of γ which agrees with its convex hull, see also Definition 5.1 [1]. As a consequence it is possible to refine the statement of Theorem 1.3 [1], by requiring (1.2) on a suitable proper subset of $\partial\mathbb{B}$. This is the content of Theorem 5.2 [1].

In [1], with next result the authors return to the original issue for homeomorphisms. Unfortunately, in this case, the characterization of the parameterizations f_0, which give rise to homeomorphic harmonic mappings $f = P[f_0]$, is less transparent. It involves the following classical notion of local homeomorphism.

Definition 1.1. *Given $P \in \overline{\mathbb{B}}$, a mapping $f \in C(\overline{\mathbb{B}}; \mathbb{R}^2)$ is a local homeomorphism at P if there exists a neighborhood G of P such that U is one-to- one on $G \cap \overline{\mathbb{B}}$.*

Theorem 1.3 ([1], Theorem AN2). *Let $f_0 : \partial\mathbb{B} \to \gamma \subset \mathbb{R}^2$ be an orientation preserving diffeomorphism of class C^1 onto a simple closed curve γ. Let D be the bounded domain such that $\partial D = \gamma$. Let* (i): *$f \in W^{1,2}_{loc}(\mathbb{B}; \mathbb{R}^2) \cap C(\overline{\mathbb{B}}; \mathbb{R}^2)$ be the solution to* (1.1), *and assume, in addition, that $f \in C^1(\overline{\mathbb{B}}; \mathbb{R}^2)$. The mapping f is a homeomorphism of $\overline{\mathbb{B}}$ onto $\overline{D}$ if and only if, for every $P \in \partial\mathbb{B}$, the mapping f is a local homeomorphism at P.*

By Exercise 2.9 in [7]: if u is continuous weak harmonic (subharmonic, superharmonic) then it is harmonic (subharmonic, superharmonic). Hence if f satisfies the hypothesis (i), it is harmonic and therefore $f = P[f_0]$. We can restate the result if we set here $f = P[f_0]$ instead of Sobolev hypthesis (i).

In view of a better appreciation of the strength and novelty of Theorem 1.2 we refer the reader to Remark 1.5 [1]. Until the appearance [1], the so-called method of shear construction introduced by Clunie and Sheil-Small has been known as the only other general means for construction of invertible harmonic mappings, besides Kneser's Theorem. In fact, it is shown in [1] that Theorem 1.2, and the arguments leading to its proof, yield a new and extremely wide generalization of the shear construction. We refer the reader to Theorem 7.3 and Corollary 7.4 in Section 7 [1], where the shear construction of Clunie and Sheil-Small is reviewed and their new version is demonstrated.

Kalaj [14] also has extended the Rado-Choquet-Kneser theorem to mappings between the unit circle and Lyapunov closed curves with Lipschitz boundary data and essentially positive Jacobian at the boundary (but without restriction on the convexity of image domain). The proof is based on the above mentioned extension of the Rado-Choquet-Kneser theorem by Alessandrini and Nesi and an approximation scheme is used in it.

In [5, 25] we used so called E-function which is related to the boundary data of the radial derivative of harmonic maps and the normal unit vector of the boundary of codomain (see Definition 2.4, Section 2). Motivated by an approach described in Kalaj's Studia paper[14] and using the continuity of E-function, the author found a new proof of Kalaj result[1], but had not published it officially at that time. A version of that proof is outlined in this paper, cf. also [23, 28].

Recently Iwaniec, cf. [9], has also communicated an interesting analytic proof of Rado-Kneser-Choquet theorem[2]; cf. also Iwaniec- Onninen [10] and see [27, 28] for more details.

While writing this paper, Kalaj put his new considerations on the arxiv [16] [3]. In this manuscript, Kalaj extends the version of Rado-Choquet-Kneser (G. Alessandrini - V. Nesi) theorem obtained in his

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[1] Roughly speaking around 2010
[2] V. Manojlović informed me about Iwaniec's lecture [9].
[3] Mar 2015 arxiv paper

Studia paper for the mappings with weak homeomorphic Lipschitz boundary data f and Dini's smooth boundary but without restriction on the convexity of image domain, provided that the Jacobian of $F = P[f]$ (Poisson integral of f) is essentially positive at the boundary. The proof is based again on the extension of Rado-Choquet-Kneser theorem by Alessandrini and Nesi and a new version of the approximation method previously used in his Studia paper. Also an important fact here is that in this setting Kalaj proves that: there is a continuous function T on $\mathbb{T}$ such that $J(F, z) = |\partial_t F(z)| T(z)$ a.e. $t \in [0, 2\pi]$, where $z = e^{it}$ and $\mathbb{T}$ denotes the unit circle. [4]

G. Alessandrini - V. Nesi and Kalaj also have discussed connections between the subject related to RKC-theorem and HQC mappings. Note that the subject related to HQC mappings was intensively studied by the participants of Belgrade Analysis Seminar, see for example [17, 18, 25, 27–29] and the literature cited there. In particular, in [28], we review some results from Božin-Mateljević manuscript [5] concerning bi-Lipschitz property for HQC-mappings between Lyapunov domains in planar case. The main result in [5] is:

Theorem 1.4. *Let Ω and Ω_1 be Jordan Lyapunov domains (i.e. in $\mathcal{D}_1$ class, see Definition 2.3), and let $h : \Omega \to \Omega_1$ be a harmonic quasiconformal (q.c.) homeomorphism. Then h is bi-Lipschitz.*

In view of a better appreciation of this result, we will give a few comments. Let D and G be Jordan domains with Dini's smooth boundaries and and let $f : D \to G$ be a harmonic homeomorphism. In [15] it is proved the following result: If f is quasiconformal, then f is Lipschitz. The method developed in [5, 28] shows that it is by-Lipschitz. This extends some recent results, where stronger assumptions on the boundary are imposed, and somehow it is optimal, since it coincides with the best known conditions for Lipschitz behavior of conformal mappings in the plane and conformal parametrization of minimal surfaces (see for instance Example 1).

It seems that using our approach outlined in this paper one can get further results, in particular local versions of RKC-Theorem and of Kellogg and Warschawski theorem for harmonic maps.

The content of the paper is as follows. In Section 2 we give some definitions and notation we need in this paper. In Section 3 we consider various characterizations of HQC (Theorems 3.1, 3.2, 3.3 and 3.5). Invertible harmonic mappings are subject of Section 4 (Theorems 4.1, 4.3, 4.4). We outline a proof of Theorem 1.4 in Section 5.

2. Definitions and Notations

Throughout this paper, $\mathbb{U}$ (or $\mathbb{D}$) will denote the unit disc $\{z : |z| < 1\}$, $\mathbb{T}$ the unit circle, $\{z : |z| = 1\}$ and we will use frequently notation $z = re^{i\theta}$ or $z = re^{i\varphi}$.

By $\partial_\theta h$ and $\partial_r h$ (or sometimes by h'_r and h'_θ), h'_x and h'_y we denote partial derivatives with respect to θ and r, x and y respectively. Let

$$P_r(t) = \frac{1 - r^2}{1 - 2r \cos(t) + r^2}$$

denote the Poisson kernel.

If $\psi \in L^1[0, 2\pi]$ and

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) \, \psi(t) \, dt, \tag{2.1}$$

then the function $h = P[\psi]$ so defined is called Poisson integral of ψ.

If ψ is of bounded variation, define $T_\psi(x)$ as variation of ψ on $[0, x]$, and let $V(\psi)$ denote variation of ψ on $[0, 2\pi]$ (see, for example, [30] p.171).

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[4] It seems that the function T which appears in [16] is the function E defined in our paper [5] (see also [26]) and that there is some overlaps of obtained results.

Define

$$h_*(\theta) = h^*(e^{i\theta}) = \lim_{r \to 1} h(re^{i\theta})$$

when this limit exists.

Definition 2.1 (Cauchy and Hilbert transform). *If $\psi \in L^1[0, 2\pi]$ (or $L^1[\mathbb{T}]$), then the Cauchy transform $C(\psi)$ is defined as*

$$C(\psi)(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{\psi(t)e^{it}}{e^{it} - z} \, \mathrm{d}t \tag{2.2}$$

with its kernel

$$K(z, t) = \frac{e^{it}}{e^{it} - z}.$$

While the Hilbert transform $H(\psi)$ is defined as

$$H(\psi)(\varphi) = -\frac{1}{2\pi} \int\limits_{0_+}^{\pi} \frac{\psi(\varphi + t) - \psi(\varphi - t)}{\tan t/2} \, \mathrm{d}t,$$

where we abuse notation by extending ψ to be 2π periodic, or consider it to be a function from $L^1(\mathbb{T})$. The following property of the Hilbert transform is also sometimes taken as the definition:
If $u = P[\psi]$ and v is the harmonic conjugate of u, then $v_* = H(\psi)$ a.e.
   Note that, if ψ is 2π-periodic, absolutely continuous on $[0, 2\pi]$ (and therefore $\psi' \in L^1[0, 2\pi]$), then

$$h'_\theta = P[\psi']. \tag{2.3}$$

Hence, since rh'_r is the harmonic conjugate of h'_θ, we find

$$rh'_r = P[H(\psi')], \tag{2.4}$$
$$(h'_r)^*(e^{i\theta}) = H(\psi')(\theta) \text{ a.e.} \tag{2.5}$$

It is clear that

$$K(z, t) + \overline{K(z, t)} - 1 = P_r(\theta - t).$$

Recall, for $f : \mathbb{U} \to \mathbb{C}$, we define

$$f_*(\theta) = f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

when this limit exists. For $f : \mathbb{T} \to \mathbb{C}$, define $f(\theta) = f(e^{i\theta})$ (we also use notation f_* instead of $\underline{f}$).

   If f is a bounded harmonic map defined on the unit disc $\mathbb{U}$, then f^* exists a.e., f^* is a bounded integrable function defined on the unit circle $\mathbb{T}$, and f has the following representation

$$f(z) = P[f^*](z) = \frac{1}{2\pi} \int\limits_0^{2\pi} P(r, t - \varphi)f^*(e^{it})\mathrm{d}t, \tag{2.6}$$

where $z = re^{i\varphi}$.

Definition 2.2 (Quasiconformal mappings). *A homeomorphism $f: D \mapsto G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-q.c or k-q.c), $K \geq 1$, if f is absolutely continuous on a.e. horizontal and a.e. vertical line in D and there is $k \in [0, 1)$ such that*

$$|f_{\bar{z}}| \leq k|f_z| \quad a.e. \text{ on } D, \tag{2.7}$$

where $K = \frac{1+k}{1-k}$, i.e. $k = \frac{K-1}{K+1}$.
  Note that the condition (2.7) can be written as

$$D_f := \frac{\Lambda}{\lambda} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq K, \tag{2.8}$$

where $\Lambda = |f_z| + |f_{\bar{z}}|$, $\lambda = |f_z| - |f_{\bar{z}}|$ and $K = \frac{1+k}{1-k}$, i.e. $k = \frac{K-1}{K+1}$.

Note that if γ is 2π-periodic absolutely continuous on $[0, 2\pi]$ (and therefore $\gamma' \in L^1[0, 2\pi]$) and $h = P[\gamma]$, then

$$(h'_r)^*(e^{i\theta}) = H(\gamma')(\theta) \, a.e.,$$

where H denotes the Hilbert transform.

Definition 2.3 (Lyapunov and Dini curves). *If X is a topological space, a path in X is a continuous mapping γ of a compact interval $[\alpha, \beta] \subset \mathbb{R}$ (here $\alpha < \beta$) into X. We call $[\alpha, \beta]$ the parameter interval of γ and denote the range of γ by $\mathrm{tr}(\gamma)$. Thus γ is a mapping, and $\mathrm{tr}(\gamma)$ is the set of all points $\gamma(t)$, for $\alpha \leq t \leq \beta$. A curve Γ is a class of equivalent paths. It is convenient to identify a curve with a path γ from the class. If $X = \mathbb{C}$ we say that curve Γ is planar.*
  Suppose that Γ is a planar curve and there is a rectifiable planar path γ which is representative of Γ. For $t \in [\alpha, \beta]$ denote by $s = s(t) = s_\gamma(t)$ the length of the curve γ^t which is the restriction of γ on $[\alpha, t]$. Then $l = s(\beta)$ is the length of γ and there exists a function $g = \breve{\gamma}$ such that $\gamma(x) = g(s(x))$ for all $x \in [\alpha, \beta]$. We call $\breve{\gamma}$ an arc-length parameterization (natural parametrization) of γ and $s = s_\gamma$ an arc-length parameter function associated to γ. Note that an arc-length parameterization $\breve{\gamma}$ is independent of representative γ and we can denote it by $\breve{\Gamma}$. Sometimes it is convenient to abuse notation and identify a curve with its arc-length parameterization (natural parametrization) and to denote it by $\Gamma(s)$.
  Suppose γ is a rectifiable, oriented, differentiable planar curve given by its arc-length parameterization.
  If γ is differentiable, then $|g'(s)| = 1$, $s = \int_0^s |g'(t)| dt$, for all $s \in [0, l]$ and $\gamma'(x) = g'(s(x))s'(x)$.
  We say that a rectifiable planar curve $\Gamma \in C^{1,\mu}$, $0 < \mu \leq 1$, if $g = \breve{\Gamma} \in C^1[0, l]$ and

$$\sup_{t,s \in [0,l]} \frac{|g'(t) - g'(s)|}{|t - s|^\mu} < \infty.$$

$C^{1,\mu}$ curves are also known as Lyapunov (we say of order μ or μ-Lyapunov) curves.
  Let $f : [a, b] \to \mathbb{C}$ be a continuous function. The modulus of continuity of f is $\omega_f(t) = \sup\{|f(x) - f(y)| : x, y \in [a, b], |x - y| \leq t\}$. The function f is called Dini-continuous if $\int_{0^+} \frac{\omega(t)}{t} dt$ is finite. Here $\int_{0^+} \frac{\omega(t)}{t} dt = \int_0^\delta \frac{\omega(t)}{t} dt$ for some positive constant δ. A C^1 Jordan curve γ with the length $l = |\gamma|$, is said to be Dini smooth if g' is Dini continuous.
  We say that a bounded Jordan domain is Lyapunov (or in $\mathcal{D}_1$ class) respectively Dini if its boundary is Lyapunov respectively Dini curve.

We also need definitions of so called *E*-function and related functions, which play important role in our approach here.

Definition 2.4 (E-function). *If γ is 2π-periodic and L^1 on $[0, 2\pi]$, recall by $P[\gamma]$ we denote Poisson integral of γ. Note that if γ is absolutely continuous on $[0, 2\pi]$ (and therefore $\gamma' \in L^1[0, 2\pi]$) and $h = P[\gamma]$, then*

$$(h'_r)^*(e^{i\theta}) = H(\gamma')(\theta) \, a.e.,$$

where H denotes the Hilbert transform. Let Γ be a curve of $C^{1,\mu}$ class (Lyapunov curve of order μ) and $\gamma : \mathbb{R} \to tr(\Gamma)$ be arbitrary topological (homeomorphic) parametrization of Γ. If γ is absolutely continuous we define $s(\varphi) = s_\gamma(\varphi) = \int_0^\varphi |\gamma'(t)|dt$. Sometimes it is convenient to abuse notation and to denote by Γ(s) natural parametrization.

For $\varphi \in \mathbb{R}$, we define $n = n_\gamma(\varphi) = i\Gamma'\big(s(\varphi)\big)$ (normal vector at the point $w = \Gamma(s) = \gamma(\varphi)$) and

$$R_\gamma(\varphi, t) = (\gamma(t) - \gamma(\varphi), n_\gamma(\varphi)).$$

For $\theta \in \mathbb{R}$ and $h = P[\gamma]$, define

$$E_\gamma(\theta) = \big((h'_r)^*(e^{i\theta}), n_\gamma(\theta)\big) = \big(H(\gamma')(\theta), n_\gamma(\theta)\big) \, a.e. \tag{2.9}$$

$$v(z, \theta) = v_\gamma(z, \theta) = \big(rh'_r(z), n_\gamma(\theta)\big), \quad z \in \mathbb{D}. \tag{2.10}$$

We also write $E(\gamma)$ instead of E_γ. Note that $v_*(t, \theta) = \big(H(\gamma'_*)(t), n_\gamma(\theta)\big)$ a.e. Define

$$e_\gamma = e_\gamma(\theta, t) = \frac{1}{4\pi} \frac{\big(\gamma(\theta + t) - \gamma(\theta), n_\gamma(\theta)\big)}{\sin^2(t/2)}.$$

Then the formula $E_\gamma(\theta) = \int_{-\pi}^\pi e_\gamma(\theta, t)dt$ plays an important role in the subject.
Define

$$E_\gamma^+(\theta) = \overline{E}_\gamma^\mu(\theta) = \frac{1}{4\pi} \int_{-\pi}^\pi \frac{|\gamma(\theta + t) - \gamma(\theta)|^{1+\mu}}{\sin^2(t/2)} \, dt,$$

$F_\gamma(\theta, \eta) = \int_{-\eta}^\eta e_\gamma(\theta, t) \, dt$, $G(\theta, \eta) = \int_{-\pi}^{-\eta} e(\theta, t) \, dt + \int_\eta^\pi e(\theta, t) \, dt$, $\eta > 0$.

In general h'_r can not be extended to be continuous on $\overline{\mathbb{U}}$ if h is a harmonic quasiconformal (abbreviated HQC) mapping between $\mathbb{U}$ and a smooth domain. However E is continuous and $(h'^*_r, n_h) = E$ a.e. on $\mathbb{T}$ and therefore the function E plays an important role in the subject. We also prove in [5] that $E \neq 0$.

3. HQC

3.1. Bi-Lipschitz property of HQC

Recall that harmonic quasiconformal (abbreviated by HQC) mappings are now very active area of investigation (see for example [17, 18, 25, 27–29]). Let $\mathcal{D}_1$ (respectively $\mathcal{D}_2$) be the family of all Jordan domains in the plane which are of class $C^{1,\mu}$ (res $C^{2,\mu}$) for some $0 < \mu < 1$.

In [12] the following result is proved:

Theorem A. Let Ω and Ω_1 be Jordan domains, let $\mu \in (0, 1]$, and let $f : \Omega \to \Omega_1$ be a harmonic homeomorphism.
Then
(a) If f is q.c. and $\partial\Omega, \partial\Omega_1 \in \mathcal{D}_1$, then f is Lipschitz;
(b) if f is q.c., $\partial\Omega, \partial\Omega_1 \in \mathcal{D}_1$ and Ω_1 is convex, then f is bi-Lipschitz; and
(c) if Ω is the unit disk, Ω_1 is convex, and $\partial\Omega_1 \in C^{1,\mu}$, then f is quasiconformal if and only if its boundary function is bi-Lipschitz and the Hilbert transform of its derivative is in L^∞.
In [13] it is proved that the convexity hypothesis can be dropped if codomain is in $\mathcal{D}_2$:
(b1) if f is q.c., $\partial\Omega \in \mathcal{D}_1$ and $\partial\Omega_1 \in \mathcal{D}_2$, then f is bi-Lipschitz.
Similar results were announced in [23]. These extend the results obtained in [11, 20, 29].

The proof of the part (*a*) of Theorem A in [12] is based on an application of Mori's theorem on quasi-conformal mappings, which has also been used in [29] in the case $\Omega_1 = \Omega = \mathbb{U}$, and Lemma 3.1 (below). In [17], we prove a version of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain

estimate concerning their Laplacian. As an application of this estimate, we show that quasi-conformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz. Our discussion in [17] includes harmonic mappings with respect to (a) spherical and Euclidean metrics (which are approximately analytic) as well as (b) the metric induced by the holomorphic quadratic differential.

We also need the following lemma in Section 4.

Lemma 3.1 ([12, 26]). *Let* Γ *be a curve of class* $C^{1,\mu}$ *and* $\gamma : \mathbb{T} \to tr(\Gamma)$ *be arbitrary topological (homeomorphic) parameterization of* Γ. *Then*

$$|R_\gamma(\varphi, t)| \le A |\gamma((e^{i\varphi}) - \gamma(e^{it})|^{1+\mu}, \tag{3.1}$$

where $A = A(\Gamma)$.

For a rectifiable planar path γ let $g = g(s)$ be an arc-length parameterization. We define

$$K(s, t) = Re[\overline{g(t) - g(s)} \cdot ig'(s)]$$

for $(s, t) \in [0, l] \times [0, l]$, where l is the length of γ.

Lemma 3.2 ([12, 28]). *Let* γ *be a Jordan closed rectifiable curve,* $\Gamma : [0, l] \to tr(\gamma)$ *be its natural parametrization and let* $f : [0, 2\pi] \to tr(\gamma)$ *be arbitrary topological parametrization of* $tr(\gamma)$. *Suppose that* γ *is a* $C^{1,\mu}$ *at* $w_0 = \Gamma(s_0)$, *where* $s_0 = s_f(\varphi_0)$. *Then*

$$|K(s_0, t)| \le C_\gamma(w_0) \min\{|s_0 - t|^{1+\mu}, (l - |s_0 - t|)^{1+\mu}\} \tag{3.2}$$

for all t and

$$|R_f(\varphi_0, x)| \le C_\gamma(w_0) \min\{|s(\varphi_0) - s(x)|^{1+\mu}, (l - |s(\varphi_0) - s(x)|)^{1+\mu}\}, \tag{3.3}$$

for all x, where recall $s = s_f$ *is an arc-length parameter function associated to f and*

$$C_\gamma(w_0) = \frac{1}{1+\mu} \sup_{0 \le t \ne s_0 \le \ell} \frac{|\Gamma'(t) - \Gamma'(s_0)|}{|t - s_0|^\mu}.$$

More generally if Γ *is Dini's smooth Jordan curve, then* $|K(s_0, t)| \le c|t - s_0|\omega(|t - s_0|)$, *where* $\omega = \omega_{\Gamma'}$.

3.2. Cauchy and Hilbert transform of HQC

Every harmonic function h in $\mathbb{D}$ can be written in the form (i) $h = f + \bar{g}$, where f and g are holomorphic functions in $\mathbb{D}$. Then an easy calculation shows

$\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)})$, $h'_r = e^{i\theta} f' + \overline{e^{i\theta} g'}$, $h'_\theta + irh'_r = 2izf'$ and therefore rh'_r is the harmonic conjugate of h'_θ. We also use notation $p = f', q = g', \Lambda_h = |f'| + |g'|, \lambda_h = |f'| - |g'|$ and $\mu_h = q/p$.

Together with the form (i) we also use the following form:

(ii) There are analytic functions F_1 and F_2 on $\mathbb{D}$ such that $Re\, h = Re\, F_1$ and $Im\, h = Im\, F_2$.

Under the condition $F_1(0) = F_2(0) = h(0)$ the form (ii) is unique and we find $F_1 = f + g$ and $F_2 = f - g$.

The form (i) is unique up to a constant. For example it is unique under the condition (iii): $f(0) = h(0)$ and $g(0) = 0$. The decomposition (i) $h = f + \bar{g}$, which satisfies the condition (iii) we call normalized decomposition and use notation $\check{h}$ for f and $\check{h}$ for g.

It is clear that if $f, g \in H^1$, then $C(h_*) = f$ on $\mathbb{U}$. Here H^1 denotes Hardy class on the unit disk.

Let $\ln z = \ln |z| + i\theta$ be a branch of logaritam in $\mathbb{H}$ determined by $0 < \theta < \pi$ and $f = -\frac{i}{2} \ln z$. Then $\theta = f + \bar{f}$ on $\mathbb{H}$ and in particular θ is bounded function on $\mathbb{H}$ while f is not bounded function on $\mathbb{H}$. Set $A(z) = i\frac{1+z}{1-z}$, $h(z) = \theta(A(z))$ and $f \circ A$ we get decomposition of h on $\mathbb{U}$. Thus in general some known spaces are not invariant under the operator $h \to \check{h}$. However, h is Lipschitz if and only if f and g are Lipschitz (ie. $f', g' \in H^\infty$).

Theorem 3.1 ([28]). *Suppose h is harmonic on $\mathbb{U}$ and $h = f + \bar{g}$ is normalized decomposition. The following condition are equivalent.*

1. *h is Lipschitz on $\mathbb{U}$*
2. *f and g are Lipschitz on $\mathbb{U}$*
3. *$f', g' \in H^\infty(\mathbb{U})$.*
4. *$f'_* \in L^\infty[0, 2\pi]$ and $g'_* \in L^\infty[0, 2\pi]$.*

Note that characterization of HQC of the unit disk onto itself by Hilbert transformations of derivative of boundary mapping first appears in Pavlović [29] and then it has been stated in Kalaj [12] and M. Mateljević, Božin and M. Knežević [24, 25] for Lyapunov co-domains.

Theorem 3.2. *Let h be a Lipschitz harmonic injective map of the unit disc and h_* bi-Lipschitz. If $\operatorname{ess\,inf}\{J(h,z) : z \in \mathbb{T}\} \geq j_0 > 0$, then h is qc.*

Since h is injective Lipschitz map, $|\mu| < 1$ and $|p| \leq M_0$ on $\mathbb{U}$.

Using $J_* = |p_*|^2 - |q_*|^2 \geq j_0$, we find $|\mu_*| \leq k_0$, where $k_0 = \sqrt{1 - j_0/M_0^2}$. An application of Maximum Principle shows that $|\mu| \leq k_0$ on $\mathbb{U}$.

Theorem 3.3 ([25]). *Suppose that D is a Lyapunov $C^{1,\alpha}$ domain. Let h be a harmonic orientation preserving map of the unit disc onto D and homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{D}$. The following conditions are equivalent*
a1) h is K-qc mapping
a2) the boundary function h_* is absolutely continuous, $\operatorname{ess\,sup}|h'_*| < +\infty$, $Hh'_* \in L^\infty$ and $s_0 = \operatorname{ess\,inf}|(Hh'_*, ih'_*)| > 0$.

We only outline the proof of this theorem.

Proof. Put $\mu = \mu_h$. Clearly $a2)$ implies $\operatorname{ess\,inf}|h'_*| > 0$. We leave to the reader to check that

$$2zp^* = H(h'_*) - ih'_*, \quad 2zq^* = H(h'_*) + ih'_* \quad J_h^* = (h_r^*, ih_*^*) = (H(h'_*), ih'_*) \geq 0$$

a.e. on $\mathbb{T}$ and $J_h > 0$ on $\mathbb{D}$. Hence $|\mu| < 1$ and $\Lambda_h^* \lambda_h^* = J_h^* \geq s_0 > 0$. Similarly like in the proof of the main characterization theorem $a2)$ implies $|\mu^*|_\infty = k < 1$ and so we have $a1)$. The converse is straightforward. $\square$

We need the following result related to convex codomains.

Theorem 3.4 ([22, 27]). *Suppose that h is a Euclidean harmonic mapping from $\mathbb{D}$ onto a bounded convex domain $D = h(\mathbb{D})$, which contains the disc $B(h(0); R_0)$. Then*
(1) $d(h(z), \partial D) \geq (1 - |z|)R_0/2, \quad z \in \mathbb{D}$.
(2) Suppose that $\omega = h^*(e^{i\theta})$ and $h_r^* = h_r'(e^{i\theta})$ exist at a point $e^{i\theta} \in \mathbb{T}$, and there exists the unit inner normal $n = n_\omega$ at $\omega = h^*(e^{i\theta})$ with respect to ∂D.
Then $E = (h_r^*, n_{h_*}) \geq c_0$, where $c_0 = \dfrac{R_0}{2}$.
(3) In addition to the hypothesis stated in the item (2), suppose that h'_* exists at the point $e^{i\theta}$. Then $|J_h| = |(h_r^*, N)| = \left|(h_r^*, n)\right||N| \geq c_0|N|$, where $N = ih'_*$ and the Jacobian is computed at the point $e^{i\theta}$ with respect to the polar coordinates.

If in addition D is of $C^{1,\mu}$ class and h qc, using the result that the function E is continuous, we find in [5] that
(4) $|E| \geq c_0$.

Theorem 3.5 ([24, 25]). *Suppose that Lyapunov $C^{1,\alpha}$ domain D is convex and denote by γ positively oriented boundary of D. Let $h_0 : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism and $h = P[h_0]$.*
The following conditions are then equivalent

(a) *h is K-qc mapping.*

(b) *h is bi-Lipschitz in the Euclidean metric.*

(c) *The boundary function h_* is bi-Lipschitz in the Euclidean metric and the Cauchy transform $C[h'_*]$ of its derivative is in L^∞.*

(d) *The boundary function h_* is absolutely continuous, $\operatorname{ess\,inf} |h'_*| > 0$ and the Cauchy transform $C[h'_*]$ of its derivative is in L^∞.*

(e) *The boundary function h_* is bi-Lipschitz in the Euclidean metric and the Hilbert transform $H[h'_*]$ of its derivative is in L^∞.*

(f) *The boundary function h_* is absolutely continuous, $\operatorname{ess\,sup} |h'_*| < +\infty$, $\operatorname{ess\,inf} |h'_*| > 0$ and the Hilbert transform $H[h'_*]$ of its derivative is in L^∞.*

Note that, by our notation, here $\underline{h_0} = h_*$ and $\overline{h_0} = h^*$.

Proof. By the fundamental theorem of Rado, Kneser and Choquet, h is an orientation preserving harmonic mapping of the unit disc onto D.

If D is $C^{1,\alpha}$, it has been shown in [5] that (a) implies (b) even without hypothesis that D is a convex domain. Note that an arbitrary bi-Lipschitz mapping is quasiconformal. Hence the conditions (a) and (b) are equivalent.

The Hilbert transform of a derivative of HQC boundary function will be in L^∞, and hence (a) implies (e).

Recall, we use notation $p = f'$, $q = g'$, $\Lambda_h = |f'| + |g'|$, $\lambda_h = |f'| - |g'|$.

If h_* is absolutely continuous, since $h'_\theta(z) = i(zf'(z) - \overline{zg'(z)})$, we find $C[h'_*](z) = izf'(z)$. It follows that (a) implies (c) and (d).

Since bi-Lipschitz condition implies absolute continuity, (c) implies (d) and (e).

Let us show that (d) implies (a).

Hypothesis $C[h'_*] \in L^\infty$ implies that $f' \in L^\infty$ and therefore since h is orientation preserving and $|f'| \geq |g'|$, we find $g' \in L^\infty$.

This shows that Λ_h is bounded from above.

We will show that $|p^*|$ is bounded from above, $\lambda_h^* = |p^*|(1 - |\mu^*|)$ is bounded from below, and therefore that $(1 - |\mu^*|)$ is bounded from below.

Let $N = i h'_*$ and $N = n|N|$.

Since D is a convex domain $|f'|$ and (h_r^*, n) are bounded from below with positive constant (for an outline of proof see [21, 22]).

Condition $C[h'_*] \in L^\infty$ implies that $f' \in H^\infty$. Hence, since $|f'|$ is bounded from below with positive constant, it follows that Λ_h is bounded from above and below with two positive constants.

By assumption (d), $|h'_*|$ is bounded essentially from below. Since, $J_h = \Lambda_h \lambda_h$ and by Theorem 3.4

$$J_h^* = (h_r^*, N) = (h_r^*, n)|N| \geq c_0|N|,$$

where $n = n_{h_*}$ and $N = n|N|$ and $N = i h'_*$, we conclude that λ_h^* is bounded from above and below with two positive constants. It follows from $\lambda_h^* = |p^*|(1 - |\mu^*|)$, that $(1 - |\mu^*|)$ is bounded from below with positive constant c_1 and therefore $k_1 = (1 - c_1) \geq |\mu^*|$. By the maximum principle, $\|\mu\|_\infty \leq k_1$.

Note that hypothesis (d) implies that $|h'_*|$ is bounded from above and therefore the boundary functions h_* is bi-Lipschitz. Thus, we have that (a) and (b) follow from (d).

Lets prove that (f) implies (d). This will finish the proof, since (e) implies (f) and we have already established that (d) implies (a).

Since the boundary function h_* is absolutely continuous, recall that, by (2.3), we have

$$\partial_\theta h(z) = P[h'_*](z) = i(zf'(z) - \overline{zg'(z)}),$$

and, by (2.4), that its harmonic conjugate is

$$zf'(z) + \overline{zg'(z)} = rh'_r(z) = P[H(h'_*)].$$

Thus if h_* is Lipschitz and $H(h'_*)$ is bounded, then $\partial_\theta h$ and $irh'_r(z)$ are bounded on $\mathbb{D}$ so by adding these two together we conclude that $h'_\theta + irh'_r = 2izf' = 2C[h'_*]$ is bounded and therefore the Cauchy transform $C[h'_*]$ is bounded, and (d) follows.

Note that we have here $|f'|$ is bounded and therefore all partial derivatives of h are bounded, and

$$H(h'_*) = zp^* + \overline{zq^*} \quad \text{a.e. on } \mathbb{T},$$

where $p = f'$ and $q = g'$. $\square$

A version of the part (a) equivalent to (f) of the main characterization has been stated in [12] and for homeomorphism of unit circle onto itself in [29].

Theorem 3.6 ([12]). *Let $f : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism of the unit circle onto the Jordan convex curve $\gamma = \partial\Omega \in C^{1,\mu}$.*
Then $h = P[f]$ is a quasiconformal mapping if and only if

$$0 < \operatorname{ess\,inf} |f'(\varphi)|, \tag{3.4}$$

$$\operatorname{ess\,sup} |f'(\varphi)| < \infty, \tag{3.5}$$

and

$$\operatorname{ess\,sup}_\varphi |H(f')(\varphi)| < \infty, \tag{3.6}$$

where

$$H(f')(\varphi) = -\frac{1}{2\pi} \int\limits_{0_+}^{\pi} \frac{f'(\varphi + t) - f'(\varphi - t)}{\tan t/2} dt,$$

denotes the Hilbert transformations of f'.

The hypothesis that f is absolutely continuous function was omitted in [12], but it seems to be needed to justify the proof from that paper [5] Indeed, it is easy to find an example of a function f satisfying conditions (3.4), (3.5) and (3.6), such that the corresponding harmonic map $h = P[f]$ is not q.c., cf [4].

4. Invertible Harmonic Mappings

In this section, we extend some recent results of Alessandrini- Nesi and Kalaj concerning invertibility for planar harmonic mappings. In particular, we prove:

Theorem 4.1 ([5, 28]). *Let Γ be a curve of $C^{1,\mu}$ class.*
Suppose that
    (a1) $\gamma : \mathbb{T} \to tr(\Gamma)$ *is a Lipschitz mapping.*
If (a1) *holds then*
    (A1) E_γ *and* $\overline{E}_\gamma^\mu$ *are continuous.*

If $h = P[\gamma]$ and if we use definition (2.9) for E_h one can consider this result as a version of Kellogg and Warschawski theorem for harmonic maps. Then
(I1) J_h^* exists a.e. and there continuous function E such that $J_h^*(e^{it}) = |\underline{\gamma}'(t)|E(t)$ a.e. $t \in [0, 2\pi]$.
(I2) If $\underline{\gamma}'$ is continuous at t_0, then J_h is continuous at $z_0 = e^{it_0}$.

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[5] As far as I remember in communication with Šarić and Anić, Mateljević first proved ACL property of boundary value of hqc mapping between disk and a domain bounded with rectifiable boundary, [21, 23]; see also [29].

Proof. Define $\chi(t, \delta) = |\gamma_*(\theta_0 + \delta + t) - \gamma_*(\theta_0 + \delta)|^{1+\mu}|t|^{-2}$. Since γ is L-Lipshitz, by Lemma 3.1, $\chi(t, \delta) \le cL^{1+\mu}|t|^{\mu-1}$. Hence, by Lebesgue dominated theorem, $\overline{E}_\gamma^\mu$ is continuous. Since γ is L-Lipshitz, by Lemma 3.1, $|e(\theta, t)| \le |R(\theta, \theta + t)||t|^{-2}/2 \le A2^\mu L^{1+\mu}|t|^{\mu-1}$. Hence, for given $\varepsilon > 0$, there is $\delta_1 > 0$ such that $|F(\theta, \delta_1)| < \varepsilon/4$, for every θ, where $F(\theta, \delta) = \int_{-\delta}^{\delta} e(t, \theta) dt$. Let $I_\delta = \{t : \delta \le |t| \le \pi\}$ and $I = [-\pi, \pi]$. Since e is continuous on $I_\delta \times I$, it is uniformly continuous on $I_\delta \times I$. Hence, for given θ_0, there is $\delta_2 > 0$ such that $2\pi|e(\theta, \theta) - e(t, \theta_0)| \le \varepsilon/2$ for $t \in I_{\delta_1}$ and $|\theta - \theta_0| \le \delta_2$. More generally if Γ is Dini's smooth Jordan curve, then we use $|e(\theta, t)| \le |R(\theta, \theta + t)||t|^{-2}/2 \le A\omega(|t|)/|t|$. $\square$

Now we consider extended (2π-periodic) parameterizations which are convenient for Poisson's transformation.

Theorem 4.2 ([28]). *Let Γ be a closed curve of $C^{1,\mu}$ class* (*more generally Dini's smooth Jordan curve*). *Suppose*

(a1) $\gamma, \gamma_n : \mathbb{R} \to \Gamma^*$ *are L-Lipschitz extended parameterization of Γ, $h_n = P[\gamma_n]$ and $h = P[\gamma]$;*
(a2) γ_n *converges uniformly to γ on $\mathbb{R}$.*

Then

(A2) $E(\gamma_n)$ *converges uniformly to $E(\gamma)$ on $\mathbb{R}$.*

We can extend Lemma 2.5 [14]:

Lemma 4.1. *If* (i1) $s : \mathbb{R} \to \mathbb{R}$ *is M -Lipschitz homeomorphism (M- Lipschitz weak homeomorphism), such that $s(x + a) = s(x) + b$ for every x, then there exist a sequence of diffeomoprhisms $s_n : \mathbb{R} \to \mathbb{R}$ such that*
(I) s_n *converges uniformly to s, s_n is M-Lipschitz homeomorphism, $s_n(x + a) = s_n(x) + b$; and*
(II) s_n *converges in Sobolev norm $H^1(0, a)$ to s and $s'_n(x)$ converges to $s'(x)$ a.e.*

We call s_n a I-mollifier sequence for s if it satisfies (I). If in addition s_n satisfies (II), we call it II-mollifier.

Proof. We outline a proof of this lemma. We introduce appropriate mollifiers: Fix a smooth function $\rho : \mathbb{R} \to [0, 1]$ which is compactly supported in the interval $[-1, 1]$ and satisfies $\int_{\mathbb{R}} \rho(z) dz = 1$. For n consider the mollifier $\rho_n(t) := n\rho(nt)$, $l_n = s * \rho_n$ and $s_n(x) = \frac{n-1}{n} l_n(x) + \frac{b}{na} x$. We call s_n a mollifier sequence for function s. Then $l'_n(x) = \int_{\mathbb{R}} s'(x - z/n)\rho(z) dz$ and $l'_n(x) \ge 0$ for every $x \in \mathbb{R}$. Therefore $s'_n(x) \ge \frac{b}{na}$ for every $x \in \mathbb{R}$, $l'_n(x) - s'(x) = \int_{\mathbb{R}} (s'(x - z/n) - s'(x))\rho(z) dz$.

Since s belongs Sobolev space $H^1(0, a)$, it is known that s_n converges in Sobolev norm $H^1(0, a)$ to s. In particular, $s'_n(x)$ converges to $s'(x)$ in $L^2(0, a)$ and therefore there is a subsequence of s'_n such that $s'_n(x)$ converges to $s'(x)$ a.e. $\square$

Remark 4.1. *If a function s satisfies (i1), then $s' \in L^\infty$ and its mollifier sequence s_n satisfies (I) and in addition*
(II') s_n *converges in Sobolev norm $H^1(0, a)$ to s.*
In particular, there is a subsequence of $s'_n(x)$ which converges to $s'(x)$ a.e.

Theorem 4.3. *[14, 16, 28]. Suppose that*
(b1) Γ *is a $C^{1,\mu}$ smooth Jordan closed curve* (*more generally Dini's smooth Jordan closed curve*) *and that D is the domain bounded by Γ;*
(b2) $\gamma : \mathbb{T} \to \partial D$ *is an orientation preserving weak homeomorphic Lipschitz mapping of the unit circle onto ∂D, $h = P[\gamma]$ and*
(b3) $j_0 = \text{essinf}\{J_h(e^{it}) : t \in [0, 2\pi]\} > 0$.
Then
(B1) *the mapping h is a diffeomorphism of $\mathbb{U}$ onto D and γ is bi-Lipschitz continuous.*

Proof. Here we outline short and new proof of Nesi-Alessandrini and Kalaj result. It is convenient to denote by $\Gamma(s)$ natural parameterization of Γ. Let $s = s_\gamma$ be an arc-length parameter function associated to γ. We can extend s to $\mathbb{R}$ such that $s(x + 2\pi) = s(x) + l$, where l is the length of Γ. Then $\gamma(\varphi) = \Gamma(s(\varphi))$ and

$J_h(e^{i\tau}) = s'(\tau)E(\tau)$. Since E is continuous $e_+ = \max E$ is finite, $e_+ s'(\tau) \geq s'(\tau)E(\tau) \geq j_0$ and $s'(\tau) \geq s^0 > 0$ a.e., where $s^0 = j_0/e_+ > 0$. Note $s' \geq 0$ a.e. and $\bar{s} = ess \sup s'$ is finite positive; hence $\bar{s} E \geq s'(\tau)E(\tau) \geq j_0$ and $E \geq \underline{e}$ a.e. , where $\underline{e} = j_0/\bar{s} > 0$. Since E is continuous then $E \geq \underline{e}$ on $\mathbb{R}$.

Suppose that s_n is given by Lemma 4.1 such that $s_n(x + 2\pi) = s_n(x) + l$, $x \in \mathbb{R}$, and that it satisfies only (I). Define $\gamma_n(\varphi) = \Gamma(s_n(\varphi))$ and $h_n = P[\gamma_n]$. One can check that γ_n converges uniformly to γ on $\overline{\mathbb{U}}$ and therefore h_n converges uniformly to h on $\overline{\mathbb{U}}$. Now, by Theorem 4.2, E_n converges uniformly to E. Hence for $n > n_0$, $E_n \geq \underline{e}/2$ on $\mathbb{R}$ and $J_{h_n} > 0$ on $\mathbb{T}$ and h_n satisfies Nesi- Alessandrini condition. Next we conclude that h_n is a diffeomorphism of $\mathbb{U}$ onto D and therefore h is a diffeomorphism. $\quad\square$

We call the sequence h_n which appears in the above proof a I-mollifier sequence of harmonic functions associated to γ (or to its arc-length parameter s). Recall by Lemma 4.1 we can choose s_n such that it satisfies (I) and (II); in particular it satisfies condition: (b1) $s'_n \to s'$ a.e.

A I-mollifier sequence of harmonic functions we call II-mollifier sequence if the corresponding s_n satisfies (I) and (b1). In this setting, we can prove J_{h_n} converges a.e. on $\mathbb{T}$. More precisely, we have:

Theorem 4.4. *Suppose that γ is Lipschitz parametrization of Lyapunov closed Jordan curve and $s = s_\gamma$ an arc-length parameter function associated to γ and that h_n is a II-mollifier sequence of harmonic functions associated to γ. Then $J_{h_n}(e^{i\tau}) = s'_n(\tau)E_n(\tau) \to s'(\tau)E(\tau)$ a.e. on $\mathbb{T}$.*

Motivated by Theorem 4.3 Kalaj states the following conjecture.
Conjecture. Let f be a homeomorphism of the unit circle onto a rectifiable Jordan closed curve C and let D be the domain bounded by C. The mapping $w = P[f]$ is a diffeomorphism of $\mathbb{U}$ onto D if and only if $ess \inf\{J_w(z) : z \in \mathbb{T}\} \geq 0$.

As first step in trying to settle this conjecture we can ask:
Question. If, in Theorem 4.3, we relax the hypothesis $(b3)$ with the hypothesis $(b3')$: $j_0 \geq 0$, whether $(B1)$ holds.

Using the Argument Principle, we can give a simple proof of the following result:

Theorem 4.5 ([28]). *Let D and G are bounded Jordan domains. Suppose that $h : D \to G$ is harmonic and continuous on D, $h|_\mathbb{T}$ is injective and partial derivatives have continuous extension to $\overline{D}$. If $J_h > 0$ on ∂D, then (I1): h is a harmonic difeomorphism of $\mathbb{D}$ onto G.*

5. Bi-Lipschitz Property for HQC between Lyapunov Domains

Let h be a harmonic quasiconformal map from the unit disk onto D in Lyapunov class $\mathcal{D}_1$. Examples show that a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case. In [5] it is proved that the corresponding functions E_{h_*} are continuous on the boundary and for fixed θ_0, $v_{h_*}(z, \theta_0)$ is continuous in z at $e^{i\theta_0}$ on $\mathbb{D}$.

We can compute the quasihyperbolic metric k on $\mathbb{C}^*$ by using the covering $\exp : \mathbb{C} \to \mathbb{C}^*$, where $\exp$ is exponential function. Let $z_1, z_2 \in \mathbb{C}^*$, $z_1 = r_1 e^{it_1}, z_2 = r_2 e^{it_2}$ and $\theta = \theta(z_1, z_2) \in [0, \pi]$ the measure of convex angle between z_1, z_2. We use

$$k(z_1, z_2) = \sqrt{\left|\ln \frac{r_2}{r_1}\right|^2 + \theta^2}.$$

This well-known formula is due to Martin and Osgood.

Let $\ell = \ell(z_1)$ be line defined by 0 and z_1. Then z_2 belongs to one half-plane, say M, on which $\ell = \ell(z_1)$ divides $\mathbb{C}$.

Locally denote by $\ln$ a branch of Log on M. Note that $\ln$ maps M conformally onto horizontal strip of with π. Since $w = \ln z$, we find the quasi-hyperbolic metric

$$|dw| = \frac{|dz|}{|z|}.$$

Note that $\rho(z) = \dfrac{1}{|z|}$ is the quasi-hyperbolic density for $z \in \mathbb{C}^*$ and therefore

$$k(z_1, z_2) = |w_1 - w_2| = |\ln z_1 - \ln z_2|.$$

Let $z_1, z_2 \in \mathbb{C}^*$, $w_1 = \ln z_1 = \ln r_1 + it_1$. Then $z_1 = r_1 e^{it_1}$; there is $t_2 \in [t_1, t_1 + \pi)$ or $t_2 \in [t_1 - \pi, t_1)$ and $w_2 = \ln z_2 = \ln r_2 + it_2$. Hence

$$k(z_1, z_2) = \sqrt{\left| \ln \frac{r_2}{r_1} \right|^2 + (t_2 - t_1)^2} \, ,$$

and therefore as a corollary of the Gehring-Osgood inequality, we have

Proposition 5.1. *Let f be a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$ (respectively $\theta^* \in [0, \pi]$) is the measure of convex angle between z_1, z_2 (respectively $f(z_1), f(z_2)$), then*

$$\theta^* \le c \max\{\theta^\alpha, \theta\},$$

where $c = c(K)$. In particular, if $\theta \le 1$, then $\theta^* \le c\theta^\alpha$.

We now discuss some results obtained in [5]. The results make use of Proposition 5.1, which is a corollary of the Gehring-Osgood inequality [6], as we are going to explain. Recall the main result in [5] is:

Theorem 5.1. *Let Ω and Ω_1 be Jordan domains in $\mathcal{D}_1$, and let $h : \Omega \to \Omega_1$ be a harmonic q.c. homeomorphism. Then h is bi-Lipschitz.*

It seems that a new idea is used here. Let Ω_1 be a $C^{1,\mu}$ curve. We reduce proof to the case when $\Omega = H$. Suppose that $h(0) = 0 \in \Omega_1$. We show that there is a convex domain $D \subset \Omega_1$ in $\mathcal{D}_1$ such that $\gamma_0 = \partial D$ touch the boundary of Ω_1 at 0 and that the part of γ_0 near 0 is a curve $\gamma(c) = \gamma(c, \mu)$. Since there is a qc extension h_1 of h to $\mathbb{C}$, we can apply Proposition 5.1 to $h_1 : \mathbb{C}^* \to \mathbb{C}^*$. This gives estimate for $arg\gamma_1(z)$ for z near 0, where $\gamma_1 = h^{-1}(\gamma(c))$, and we show that there exist constants $c_1 > 0$ and μ_1 such that the graph of the curve $h^{-1}(\gamma(c))$ is below of the graph of the curve $\gamma(c_1) = \gamma(c_1, \mu_1)$. Therefore there is a domain $D_0 \subset \mathbb{H}$ in $\mathcal{D}_1$ such that $h(D_0) \subset D$. Finally, we combine the convexity type argument and noted continuity of functions E and v to finish the proof.

The next example which is shortly discussed in [8], shows that there is a conformal map of unit disk onto C^1 domain which is not bi-Lipschitz. Here we will give more details.

Example 1. *(i) Let $l = \{iy : y \le 0\}$ and $\ln z = \ln |z| + i arg z$ defined on $O = \mathbb{C} \setminus l$ by $-\pi/2 < arg < 3\pi/2$. Set $\ln \frac{1}{z} = -\ln z$, $w = A(z) = z \ln \frac{1}{z}$, and $U_r^+ = \{z : |z| < r, y > 0\}$. For r small enough, A maps interval $(0, r)$ onto interval $(0, r \ln \frac{1}{r})$, interval $(-r, 0)$ onto arc $\gamma_1(x) = x \ln \frac{1}{x} = x \ln \frac{1}{|x|} - i\pi x$, semicircle $C_r^+ = \{re^{it} : 0 \le t \le \pi\}$ onto a curve close the semicircle of radius $r \ln \frac{1}{r}$. For r small enough A is univalent in U_r^+. We can check that there is a smooth domain $D \subset U_r^+$ such that interval $(-r_0, r_0)$, $r_0 > 0$, is a part of the boundary of D, $D^* = A(D)$ is C^1 domain and A is not bi-Lipschitz on D. Since $A(z)$ tends 0 if z throughout O tends 0, we can set $A(0) = 0$. Thus $A(1) = A(0) = 0$, and therefore $r_0 \in (0, 1)$. Let us outline that $A(D)$ is C^1 domain. Let $c_0(x) = -x \ln |x|$ and $C(x) = x \ln \frac{1}{x}$, $-1 < x < 1$, $x \ne 0$, $C(0) = 0$, and $\Gamma(s)$ natural parameterization of C, where a branch of $\ln$ is defined by $0 \le arg z \le \pi$ and s denotes length element; so for $-1 < x < 0$, $C(x) = x(\ln \frac{1}{|x|} - i\pi)$. Note that the restriction of $Im\, C'$ on $(-1, 1)$ has a jump discontinuity (or step discontinuity) at 0, c_0 is odd function and therefore $c_0'(x) = -1 - \ln |x|$ is even and $c_0'(x) = |c_0'(x)| \to +\infty$ if $x \to 0$. Then $C'(x) = c_0'(x) = \ln \frac{1}{x} - 1$, $x > 0$, and $C'(x) = c_0'(x) - i\pi = \ln \frac{1}{x} - 1 - i\pi$ for $x \in (-1, 0)$. Therefore $\Gamma'(s) = C'(x)/|C'(x)| = e^{i\varphi(x)}$, $\sin \varphi(x) = -\pi/|C'(x)|$. Hence, if $x \to 0_-$, $|C'(x)| \to \infty$, $\varphi(x) \to 0$ and $\Gamma'(s) \to 1$. Since $\Gamma'(s) = 1$ for $x \in (0, r_0)$ and therefore $\Gamma'(s)$ is continuous for $-r_0 < x < r_0$. Set $s_* = s_*(x) = \int_0^x |C'(t)|dt$ and denote by $x = x(s_*)$ the inverse function. Since*

$$\Gamma'(|s_*(x)|) - 1 = \frac{1}{|\ln |x||} + o\left(\frac{-i\pi}{|\ln |x||}\right), \quad x \to 0,$$

and $|s_*| = |s_*(x)| = |x|\big|\ln|x|\big| + o(|x|\big|\ln|x|\big|)$ for x around 0, we find

$$\frac{1}{\big|\ln|x|\big|} \geq \frac{1}{2\big|ln|s_*|\big|}$$

for s_* around 0, where $x = x(s_*)$. Hence we conclude that for s_* near 0,

$$\omega_{\Gamma'}(|s_*|) \geq \frac{1}{2\big|ln|s_*|\big|}$$

and therefore the boundary of D^* is not Dini's smooth curve.
(ii) Consider also

$$w = \frac{z}{\ln\frac{1}{z}}, \quad w(0) = 0.$$

Note $\ln\frac{1}{z} = -\ln z$, $w'(z) = -(\ln z)^{-1} + (\ln z)^{-2}$ and $w'(z) \to 0$ if $z \to 0$ throughout $\mathbb{H}$.

Finally, we give an illustrative example mentioned in the beginning of this section, and given in our previous paper, [4].

Example 2 ([4]). *Consider $h : \mathbb{H} \to \mathbb{H}$, a harmonic function given by the following expression*

$$h(z) = h^\phi(z) = \Phi_1(z) + icy + c_1,$$

where $c > 0$, $c_1 \in \mathbb{R}$, $\Phi(z) = \int_i^z \phi(\zeta)d\zeta$, and $\Phi_1 = \operatorname{Re}\Phi$. Note that $h'_x(z) = \operatorname{Re}\phi(z)$ and $h'_y(z) = -\operatorname{Im}\phi(z) + ic$. Let $\phi(z) = 2 + e^{-i/z} = 2 + e^{-y/|z|^2}(\cos\frac{x}{|z|^2} - i\sin\frac{x}{|z|^2})$ and $h = h^\phi$. Then $h'_x(z) = \operatorname{Re}\phi(z) = 2 + e^{-y/|z|^2}\cos\frac{x}{|z|^2}$. Hence $h'(x) = \operatorname{Re}\phi(x) = 2 + \cos\frac{1}{x}$; so h' is not continuous at 0. In polar coordinates, $h'_y(z) = -\operatorname{Im}\phi(z) + ic = e^{-\sin\theta/\rho}\sin(\cos\theta/\rho) + ic$; hence $h'_y(z) \to \sin(1/\rho) + ic$ when $\theta \to 0$ for fixed $\rho > 0$.

Let $G \subset \mathbb{H}$ be a smooth domain such that $\partial G \cap \mathbb{R} = [-a, a]$, $a > 0$, ϕ be conformal mapping of $\mathbb{U}$ onto G, $\phi(1) = 0$, $z = \phi(\zeta)$, and $\check{h} = h \circ \phi$. One can check that $\check{h}_r$ is not continuous at 1. However $\check{h}'$ is bi-Lipschitz.

We can give a more delicate example. Let $\phi_k(z) = 2 + e^{i/(x_k - z)}$ and x_k, $k \in \mathbb{N}$, be a sequence of real numbers. Define

$$\phi(z) = 2 + \sum_{k=1}^{\infty} 2^{-k}\phi_k(z).$$

For example if x_k is a sequence of all rational numbers, i.e. enumerating $\mathbb{Q}$, then h_y will have no continuous extension to $\mathbb{Q}$, where $h = h^\phi$.

These examples can also be translated to the unit disc.

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