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On Distance Spectral Radius of Trees with Fixed Maximum Degree

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Abstract. We determine the unique trees with minimum distance spectral radius in the class of all trees on n vertices with a fixed maximum degree bounded below by $\lceil \frac{n}{2} \rceil$, and in the class of all trees on $2m$ vertices with perfect matching and a fixed maximum degree bounded below by $\lceil \frac{m}{2} \rceil + 1$.

1. Introduction

We consider simple and undirected graphs. Let G be a connected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, the distance between u and v, denoted by $d_G(u, v)$, is the length of a shortest path from u to v in G. The distance matrix of G is the $n \times n$ matrix $D(G) = (d_G(u, v))_{u,v \in V(G)}$. Since $D(G)$ is real symmetric, all its eigenvalues are real. The distance spectral radius of G, denoted by $\rho(G)$, is the largest eigenvalue of $D(G)$. By the Perron-Frobenius Theorem, there is a unique unit positive eigenvector of $D(G)$ corresponding to $\rho(G)$, which is called the distance Perron vector of G.

The distance spectral radius has received much attention. Ruzieh and Powers [3] and Stevanović and Ilić [4] showed that the n-vertex path P_n is the unique n-vertex connected graph with maximum distance spectral radius. Stevanović and Ilić [4] showed that the star S_n is the unique n-vertex tree with minimum distance spectral radius, and determined the unique n-vertex tree with maximum distance spectral radius when the maximum degree is fixed. Ilić [5] determined the unique n-vertex tree with minimum distance spectral radius when the matching number is fixed. Wang and Zhou [6] determined the unique n-vertex tree with minimum (maximum, respectively) distance spectral radius when the domination number is fixed. More results in this line may be found in, e.g., [1, 2, 7].

In this paper, we determine the unique n-vertex tree with minimum distance spectral radius when the maximum degree is at least $\lceil \frac{n}{2} \rceil$, and the unique $2m$-vertex perfect matching tree with minimum distance spectral radius when the maximum degree is at least $\lceil \frac{m}{2} \rceil + 1$.

Let T be a tree. For $u \in V(T)$, $N_T(u)$ denotes the set of neighbors of u in T, and $d_T(u)$ denotes the degree of u in T, i.e., $d_T(u) = |N_T(u)|$. Let $\Delta(T)$ be the maximum degree of T. Let $|T| = |V(T)|$.

Let G be a graph with complement $\overline{G}$. For $E \subseteq E(G)$, let $G - E$ be the graph obtained from G by deleting all edges of E. For $F \subseteq E(\overline{G})$, let $G + F$ be the graph obtained from G by adding all edges of F.

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A path $u_1u_2\dots u_r$ (with $r \geq 2$) in a graph G is called a pendent path (of length $r-1$) at u_1 if $d_G(u_1) \geq 3$, the degrees of $u_2,\dots,u_{r-1}$ (if any exists) are all equal to 2 in G, and $d_G(u_r) = 1$.

If x is the distance Perron vector of a (connected) graph G, then x_u denotes the component of x corresponding to vertex u in G, and $s(W) = \sum_{u\in W} x_u$ for $W \subseteq V(G)$.

2. Distance Spectral Radius of Trees with Fixed Maximum Degree

We give several lemmas that will be used in our proof.

Lemma 2.1. *Let T be a tree and u_1v, u_2v be two non-pendent edges of T. Let $T' = T-\{u_2w : w \in N_T(u_2)\setminus\{v\}\}+\{u_1w : w \in N_T(u_2)\setminus\{v\}\}$. Then $\rho(T') < \rho(T)$.*

Proof. Let T_1 (T_2, T_3, respectively) be the component of $T - \{u_1v, u_2v\}$ containing u_1 (u_2, v, respectively), and let $V_i = V(T_i)$ for $i = 1,2,3$. Let x be the distance Perron vector of T'. Then

$$\rho(T) - \rho(T') \geq x^TD(T)x - x^TD(T')x = 4s(V_2 \setminus \{u_2\})(s(V_1) - x_{u_2}).$$

Since u_2v is a non-pendent edge of T, $|V_2| \geq 2$ and thus $s(V_2 \setminus \{u_2\}) > 0$.

Next we show that $s(V_1) - x_{u_2} > 0$. Since u_1v is a non-pendent edge of T, $|V_1| \geq 2$. Let z be a neighbor of u_1 in T_1.

Case 1. $d_T(z) = 1$. Then

$$\begin{aligned}
\rho(T')(s(V_1) - x_{u_2}) &\geq \rho(T')(x_z + x_{u_1} - x_{u_2})\\
&= \sum_{w\in V(T')\setminus\{z,u_1,u_2\}} (d_{T'}(z,w) + d_{T'}(u_1,w) - d_{T'}(u_2,w))x_w\\
&\quad +5x_{u_2} - x_{u_1} - 2x_z\\
&= \sum_{w\in V_1\cup V_2\setminus\{u_1,u_2,z\}} (d_{T'}(z,w) - 2)x_w\\
&\quad + \sum_{w\in V_3} d_{T'}(z,w)x_w + 5x_{u_2} - x_{u_1} - 2x_z\\
&> 2x_{u_2} - 2x_{u_1} - 2x_z\\
&\geq 2x_{u_2} - 2s(V_1).
\end{aligned}$$

So we have $(\rho(T') + 2)(s(V_1) - x_{u_2}) > 0$, and thus $s(V_1) > x_{u_2}$.

Case 2. $d_T(z) \geq 2$. Let z_1 be a neighbor of z different from u_1 in T_1. Then

$$\begin{aligned}
\rho(T')(s(V_1) - x_{u_2}) &\geq \rho(T')(x_z + x_{z_1} + x_{u_1} - x_{u_2})\\
&= \sum_{w\in V(T')\setminus\{z,z_1,u_1,u_2\}} (d_{T'}(z,w) + d_{T'}(z_1,w))x_w\\
&\quad + \sum_{w\in V(T')\setminus\{z,z_1,u_1,u_2\}} (d_{T'}(u_1,w) - d_{T'}(u_2,w))x_w\\
&\quad +9x_{u_2} + x_{u_1} - x_z - x_{z_1}\\
&= \sum_{w\in V_1\cup V_2\setminus\{z,z_1,u_1,u_2\}} (d_{T'}(z,w) + d_{T'}(z_1,w) - 2)x_w\\
&\quad + \sum_{w\in V_3} (d_{T'}(z,w) + d_{T'}(z_1,w))x_w\\
&\quad +9x_{u_2} + x_{u_1} - x_z - x_{z_1}\\
&> x_{u_2} - x_{u_1} - x_z - x_{z_1}\\
&\geq x_{u_2} - s(V_1).
\end{aligned}$$

So we have $(\rho(T') + 1)(s(V_1) - x_{u_2}) > 0$, and thus $s(V_1) > x_{u_2}$.

Combining Cases 1 and 2, we have $s(V_1) > x_{u_2}$, and thus $\rho(T') < \rho(T)$. □

Lemma 2.2. *[6] Let G be a connected graph and uv a non-pendent cut edge of G. Let G' be the graph obtained from G by contracting uv and attaching a new pendent vertex to u (v). Then $\rho(G') < \rho(G)$.*

Let $\mathcal{T}_n^{\Delta}$ be the set of trees on n vertices with maximum degree Δ.

Let $q(T)$ be the number of non-pendent vertices of a tree T. Let $\mathcal{T}_n^{\Delta}(q) = \{T \in \mathcal{T}_n^{\Delta} : q(T) = q\}$.

Lemma 2.3. *Let $T \in \mathcal{T}_n^{\Delta}(q)$, where $\lceil \frac{n}{2} \rceil \le \Delta \le n-1$ and $q \ge 3$. Then there is a tree T' in $\mathcal{T}_n^{\Delta}(q-1)$ such that $\rho(T') < \rho(T)$.*

Proof. Let v be a vertex of T such that $d_T(v) = \Delta$.

Case 1. Each non-pendent edge of T is incident with v. Let u_1, u_2 be two distinct non-pendent vertices different from v, and let

$$T' = T - \{u_2 y : y \in N_T(u_2) \setminus \{v\}\} + \{u_1 y : y \in N_T(u_2) \setminus \{v\}\}.$$

Obviously, $T' \in \mathcal{T}_n^{\Delta}$ as $\Delta \ge \lceil \frac{n}{2} \rceil$, and the non-pendent edge vu_2 of T becomes pendent in T', and thus $T' \in \mathcal{T}_n^{\Delta}(q-1)$. By Lemma 2.1, $\rho(T') < \rho(T)$.

Case 2. There is a non-pendent edge uw of T, where u and w are different from v. Suppose without loss of generality that $d_T(v, u) < d_T(v, w)$. Let

$$T' = T - \{wz : z \in N_T(w) \setminus \{u\}\} + \{uz : z \in N_T(w) \setminus \{u\}\}.$$

Obviously, $T' \in \mathcal{T}_n^{\Delta}(q-1)$. By Lemma 2.2, $\rho(T') < \rho(T)$. □

Let $S_{n,i}$ be the double star obtained by attaching $i-1$ and $n-i-1$ pendent vertices to the two end vertices of P_2 respectively, where $\lceil \frac{n}{2} \rceil \le i \le n-1$. In particular, $S_{n,n-1} = S_n$.

Theorem 2.4. *Let $T \in \mathcal{T}_n^{\Delta}$, where $\lceil \frac{n}{2} \rceil \le \Delta \le n-1$. Then $\rho(T) \ge \rho(S_{n,\Delta})$ with equality if and only if $T \cong S_{n,\Delta}$.*

Proof. Let T be a tree in $\mathcal{T}_n^{\Delta}$ with minimal distance spectral radius. We only need to show that $T \cong S_{n,\Delta}$.

The case $\Delta = n-1$ is trivial as $\mathcal{T}_n^{n-1} = \{S_n\}$.

Suppose that $\Delta \le n-2$. Then $q(T) \ge 2$. By Lemma 2.3, $q(T) = 2$, and then $T \cong S_{n,\Delta}$. □

Stevanović and Ilić [4] conjectured that a complete Δ-ary tree has the minimum distance spectral radius among trees $\mathcal{T}_n^{\Delta}$. Theorem 2.4 shows that this is true for $\Delta \ge \lceil \frac{n}{2} \rceil$.

3. Distance Spectral Radius of Perfect Matching Trees with Fixed Maximum Degree

It is well known that if a tree has a perfect matching, then it is unique. Let $\mathcal{T}_{2m}$ be the set of trees on $2m$ vertices with a perfect matching. For $T \in \mathcal{T}_{2m}$, let $M(T)$ be the unique perfect matching of T. For $0 \le j \le m-2$, let $X_{2m}^{j} = \{T \in \mathcal{T}_{2m}$: there are exactly j non-pendent edges in $M(T)\}$. Obviously, $\mathcal{T}_{2m} = \cup_{j=0}^{m-2} X_{2m}^{j}$.

Let A_m be the tree with $2m$ vertices obtained from the star S_{m+1} by attaching a pendent vertex to each of certain $m-1$ non-central vertices. The center of the star S_{m+1} is also the center of A_m. Obviously, $A_m \in \mathcal{T}_{2m}$, and all edges in $M(A_m)$ are pendent in A_m. Let $\mathcal{H} = \{A_k$: k is a positive integer$\}$.

Lemma 3.1. *$T \in X_{2m}^{0}$ if and only if T is a tree with $2m$ vertices obtainable from the union of some graphs in $\mathcal{H}$ by joining centers with edges.*

Proof. Suppose that T is a tree with $2m$ vertices obtained from union of $H_1, H_2, \dots, H_t \in \mathcal{H}$ by joining centers with edges. Then T has a unique perfect matching $M(T) = \cup_{i=1}^{t} M(H_i)$ and all edges in $M(T)$ are pendent edges of T. Thus $T \in X_{2m}^0$.

Suppose that $T \in X_{2m}^0$. If $m = 1$, then $T = P_2 = A_1 \in \mathcal{H}$, and if $m = 2$, then $T = P_4 = A_2 \in \mathcal{H}$. Suppose that $m \geq 3$. Let $N = \{v \in V(T) : d_T(v) \geq 3\}$ and $P = \{uv \in E(T) : u, v \in N\}$. Note that $P \cap M(T) = \emptyset$. Obviously, $T - P$ is a forest on $2m$ vertices. Let C be a component of $T - P$. If there are two vertices, say u and v with degree at least 3 in C, then each internal vertex (if any exists) in the path connecting u and v is of degree at least 3 (because each non-pendent vertex has a pendent neighbor), and thus all edges in this path should be in P, a contradiction. Then C contains at most one vertex with degree at least 3, and thus $C \in \mathcal{H}$. Obviously, the vertices in N are their centers of components $T - P$. □

Lemma 3.2. *Let $T \in \mathcal{T}_{2m}$ with $u, v \in V(T)$ and $u \neq v$. Then $d_T(u) + d_T(v) \leq m + 2$.*

Proof. Let T_1 be the subgraph of T induced by $N_T(u) \cup N_T(v) \cup \{u, v\}$. Obviously, $|E(T_1)| \geq d_T(u) + d_T(v) - 1$ and $E(T_1)$ contains at most 2 edges in $M(T)$. Thus there are at most $2m - 1 - (d_T(u) + d_T(v) - 1)$ edges outside T_1. If $d_T(u) + d_T(v) > m + 2$, then there are at most $2m - 1 - (m + 3 - 1) = m - 3$ edges outside T_1, and thus $|M(T)| \leq 2 + m - 3 = m - 1$, a contradiction. □

Lemma 3.3. *Let $T \in X_{2m}^j$, where $1 \leq j \leq m - 2$. If $\Delta(T) \geq \lceil \frac{m}{2} \rceil + 1$, then there is a tree $T' \in X_{2m}^{j-1}$ with $\Delta(T') = \Delta(T)$ such that $\rho(T') < \rho(T)$.*

Proof. Let v be a vertex of T with $d_T(v) = \Delta(T)$.
Case 1. v is not incident with any non-pendent edge in $M(T)$. Let uw be a non-pendent edge in $M(T)$. Let $T' = T - \{wy : y \in N_T(w) \backslash \{u\}\} + \{uy : y \in N_T(w) \backslash \{u\}\}$. Obviously, $M(T') = M(T)$ and $T' \in X_{2m}^{j-1}$. By Lemma 3.2 and the fact that $\Delta(T) \geq \lceil \frac{m}{2} \rceil + 1$, we have

$$\begin{aligned} d_{T'}(u) &\leq m + 2 - d_{T'}(v) = m + 2 - d_T(v) = m + 2 - \Delta(T) \\ &\leq m + 2 - \left(\left\lceil \frac{m}{2} \right\rceil + 1 \right) = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ &\leq \Delta(T). \end{aligned}$$

Then $\Delta(T') = \max\{d_{T'}(u), d_{T'}(v)\} = \Delta(T)$. By Lemma 2.2, $\rho(T') < \rho(T)$.
Case 2. v is incident with some non-pendent edge in $M(T)$, say vw is a non-pendent edge in $M(T)$. Let z be a neighbor of v different from w. Since $vw \in M(T)$, zv is also a non-pendent edge of T. Let $T' = T - \{wy : y \in N_T(w) \backslash \{v\}\} + \{zy : y \in N_T(w) \backslash \{v\}\}$. Obviously, $M(T') = M(T)$ and $T' \in X_{2m}^{j-1}$. By Lemma 3.2 and the fact that $\Delta(T) \geq \lceil \frac{m}{2} \rceil + 1$, we have $d_{T'}(z) \leq m + 2 - d_{T'}(v) = m + 2 - \Delta(T) \leq \Delta(T)$, and thus $\Delta(T') = \max\{d_{T'}(v), d_{T'}(z)\} = \Delta(T)$. By Lemma 2.1, $\rho(T') < \rho(T)$. □

For $T \in X_{2m}^0$ with $m \geq 3$, let $P = \{uv \in E(T) : d_T(u), d_T(v) \geq 3\}$. By the proof of Lemma 3.1, $T - P$ is a forest, whose components are trees in $\mathcal{H}$. Let H_i be the component of $T - P$ and v_i be the center of H_i for $i = 1, 2, \dots, t$, where $t \geq 1$. The contracted tree of T, denoted by $\widehat{T}$, is defined to be the tree obtained from T by replacing H_i with v_i for $i = 1, 2, \dots, t$, i.e., $V(\widehat{T}) = \{v_1, v_2, \dots, v_t\}$ and $v_i v_j \in \widehat{T}$ if and only if $v_i v_j \in T$. For $T \in X_{2m}^0$ with $m = 1, 2$, let $\widehat{T} = K_1$.

Lemma 3.4. *Let $T \in X_{2m}^0$ with $\Delta(T) \geq \lceil \frac{m}{2} \rceil + 1$. If $|\widehat{T}| \geq 3$, then there is a tree $T' \in X_{2m}^0$ with $\Delta(T') = \Delta(T)$ and $|\widehat{T'}| = |\widehat{T}| - 1$ such that $\rho(T') < \rho(T)$.*

Proof. Let v be a vertex of T with $d_T(v) = \Delta(T)$. Obviously, $\widehat{T}$ has at least two pendent edges.
Case 1. $\widehat{T}$ has a pendent edge uy, where $u \neq v$ and $d_{\widehat{T}}(y) = 1$. Let z be a neighbor of u in $\widehat{T}$ different from y, and yy_1, uu_1, zz_1 pendent edges of T. Let T_1 (T_2, respectively) be the component of $T - \{uy\}$ containing

u (y, respectively), and $V_i = V(T_i)$ for $i = 1, 2$. Note that $d_T(y) \geq 3$. Then $|N_T(y) \setminus \{u, y_1\}| \geq 1$. Let $T' = T - \{yw : w \in N_T(y) \setminus \{u, y_1\}\} + \{uw : w \in N_T(y) \setminus \{u, y_1\}\}$. Let x be the distance Perron vector of T'. Then

$$\rho(T) - \rho(T') \geq x^T D(T)x - x^T D(T')x = 2s(V_2 \setminus \{y, y_1\})(s(V_1) - x_y - x_{y_1}).$$

Note that

$$\begin{aligned}
\rho(T')(s(V_1) - x_y - x_{y_1}) &\geq \rho(T')(x_z + x_{z_1} + x_u + x_{u_1} - x_y - x_{y_1}) \\
&= \sum_{w \in V_1 \setminus \{z, z_1, u, u_1\}} (d_{T'}(z, w) + d_{T'}(z_1, w) - 2)x_w \\
&\quad + \sum_{w \in V_2 \setminus \{y, y_1\}} (d_{T'}(u, w) + d_{T'}(u_1, w))x_w \\
&\quad + 7x_y + 11x_{y_1} - x_z - x_{z_1} + x_u + x_{u_1} \\
&> x_y + x_{y_1} - x_z - x_{z_1} - x_u - x_{u_1} \\
&\geq x_y + x_{y_1} - s(V_1).
\end{aligned}$$

So $s(V_1) > x_y + x_{y_1}$, and thus $\rho(T') < \rho(T)$.

Case 2. All pendent edges of $\widehat{T}$ are incident with v. Obviously, $\widehat{T} = S_t$ with center v. Let vv_1, vv_2 be two edges in $\widehat{T}$. Then $d_T(v_1), d_T(v_2) \geq 3$. Let z be a non-pendent neighbor of v_1 different from v in T, and v_1z_1, v_2z_2, zz_3 pendent edges of T. Let T_1 (T_2, T_3, respectively) be the component of $T - \{vv_1, vv_2\}$ containing v_1 (v_2, v, respectively), and $V_i = V(T_i)$ for $i = 1, 2, 3$. Obviously, $|N_T(v_2) \setminus \{v, z_2\}| \geq 1$. Let $T' = T - \{v_2w : w \in N_T(v_2) \setminus \{v, z_2\}\} + \{v_1w : w \in N_T(v_2) \setminus \{v, z_2\}\}$. Let x be the distance Perron vector of T'. Then

$$\rho(T) - \rho(T') \geq x^T D(T)x - x^T D(T')x = 4s(V_2 \setminus \{v_2, z_2\})(s(V_1) - x_{v_2} - x_{z_2}).$$

Note that

$$\begin{aligned}
\rho(T')(s(V_1) - x_{v_2} - x_{z_2}) &\geq \rho(T')(x_z + x_{z_1} + x_{z_3} + x_{v_1} - x_{v_2} - x_{z_2}) \\
&= \sum_{w \in V_1 \setminus \{z, z_1, z_3, v_1\}} (d_{T'}(v_1, w) + d_{T'}(z_1, w) - 2)x_w \\
&\quad + \sum_{w \in V_2 \setminus \{z_2, v_2\}} (d_{T'}(v_1, w) + d_{T'}(z_1, w) - 2)x_w \\
&\quad + \sum_{w \in V_3} (d_{T'}(z, w) + d_{T'}(z_3, w))x_w \\
&\quad + 11x_{v_2} + 15x_{z_2} - x_{v_1} - x_{z_1} - 3x_z - 3x_{z_3} \\
&> 3x_{v_2} + 3x_{z_2} - 3x_{v_1} - 3x_{z_1} - 3x_z - 3x_{z_3} \\
&\geq 3(x_{v_2} + x_{z_2} - s(V_1)).
\end{aligned}$$

So $s(V_1) > x_{v_2} + x_{z_2}$, and thus $\rho(T') < \rho(T)$.

In either case, $M(T') = M(T)$, all edges in $M(T')$ are pendent edges of T', and thus $T' \in X^0_{2m}$. Moreover, $|\widehat{T'}| = |\widehat{T}| - 1$ and $\Delta(T') = d_{T'}(v) = d_T(v) = \Delta(T)$ since $\Delta(T) \geq \lceil \frac{m}{2} \rceil + 1$. □

Let $S^*_{2m,i}$ be the tree in $\mathcal{T}_{2m}$ obtained by attaching a new pendent edge at each vertex of $S_{m,i-1}$, where $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m$.

Theorem 3.5. *Let $T \in \mathcal{T}_{2m}$ with $\Delta(T) = \Delta$, where $\lceil \frac{m}{2} \rceil + 1 \leq \Delta \leq m$. Then $\rho(T) \geq \rho(S^*_{2m,\Delta})$ with equality if and only if $T \cong S^*_{2m,\Delta}$.*

Proof. Let T be a tree in $\mathcal{T}_{2m}$ with $\Delta(T) = \Delta$ having minimal distance spectral radius. We only need to show that $T \cong S^*_{2m,\Delta}$.

By Lemma 3.3, $T \in X^0_{2m}$. If $\Delta = m$, then $T \cong A_m \cong S^*_{2m,\Delta}$, and thus the result holds trivially. Suppose that $\Delta \leq m-1$. Then $|\widehat{T}| \geq 2$. By Lemma 3.4, $|\widehat{T}| = 2$, and thus $\widehat{T} = P_2$, or equivalently, $T \cong S^*_{2m,\Delta}$. □

For a graph G with $v \in V(G)$ and nonnegative integers k and l with $k \geq \max\{l, 1\}$, let $G_v(k,l)$ be the graph obtained from G by attaching a path of length k and a path of length l at v (if $l = 0$, then only a path of length k is attached).

Lemma 3.6. *[4, 7] Let G be a connected graph with at least two vertices and $v \in V(G)$. If $k \geq l \geq 1$, then $\rho(G_v(k,l)) < \rho(G_v(k+1, l-1))$.*

Let $B^*_{2m,i}$ be the tree in $\mathcal{T}_{2m}$ obtained by adding an edge between the center of $S^*_{2(i-1),i-1}$ and a pendent vertex of $P_{2(m-i+1)}$, where $2 \leq i \leq m$. In particular, $B^*_{2m,2} = P_{2m}$. For a graph G with $W \subseteq V(G)$, $G[W]$ denotes the subgraph of G induced by W. The following theorem was given in [5]. For completeness, however, we include a proof here.

Theorem 3.7. *Let $T \in \mathcal{T}_{2m}$ with $\Delta(T) = \Delta$, where $2 \leq \Delta \leq m$. Then $\rho(T) \leq \rho(B^*_{2m,\Delta})$ with equality if and only if $T \cong B^*_{2m,\Delta}$.*

Proof. Let T be a tree in $\mathcal{T}_{2m}$ with $\Delta(T) = \Delta$ having maximal distance spectral radius. We only need to show that $T \cong B^*_{2m,\Delta}$. The case $\Delta = 2$ is trivial. Suppose that $\Delta \geq 3$. Let $u \in V(G)$ with $d_T(u) = \Delta$.

Suppose that there are at least two vertices with degree at least 3 in T. Choose a vertex v with degree at least 3 such that the distance between u and v is as large as possible. There are at least two pendent paths, say $P_1 = vu_1 \ldots u_k$ and $P_2 = vv_1 \ldots v_l$ at v in T, where $k \geq l \geq 1$. Let $G = T[V(T) \setminus \{u_1, \ldots, u_k, v_1, \ldots, v_l\}]$. Then $T \cong G_v(k,l)$. Let $T' = T - vu_1 + v_1u_1$ if $l = 1$ and $T' = T - v_{l-2}v_{l-1} + u_kv_{l-1}$ if $l \geq 2$ (where $v_{l-2} = v$ for $l = 2$). Then $M(T') = M(T)$, $T' \in \mathcal{T}_{2m}$, and $\Delta(T') = \Delta$. Note that $T' \cong G_v(k+1, 0)$ if $l = 1$ and $T' \cong G_v(k+2, l-2)$ if $l \geq 2$. By Lemma 3.6, $\rho(T') > \rho(T)$, a contradiction. Thus u is the unique vertex of T with degree at least 3, i.e., T consists of Δ pendent paths at u.

Suppose that there are at least two pendent paths at u in T with length at least 3, say $Q_1 = uw_1 \ldots w_k$ and $Q_2 = uz_1 \ldots z_l$, where $k \geq l \geq 3$. Then $T = H_u(k,l)$ with $H = T[V(T) \setminus \{w_1, \ldots, w_k, z_1, \ldots, z_l\}]$. Let $T'' = T - z_{l-2}z_{l-1} + w_kz_{l-1}$. Then $M(T'') = M(T)$, $T'' \in \mathcal{T}_{2m}$, and $\Delta(T'') = \Delta$. Note that $T'' \cong H_u(k+2, l-2)$. By Lemma 3.6, $\rho(T'') > \rho(T)$, a contradiction. Thus there is exactly one pendent path at u with length at least 3. Since $T \in \mathcal{T}_{2m}$, we have $T \cong B^*_{2m,\Delta}$. □

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