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A Homotopy Fixed Point Theorem in 0-Complete Partial Metric Space

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Abstract. We generalize a result of Feng and Liu, on multi-valued contractive mappings, for studying the relationship between fixed point sets and homotopy fixed point sets. The presented results are discussed in the generalized setting of 0-complete partial metric spaces. An example and a nonlinear alternative of Leray-Schauder type are given to support our theorems.

1. Introduction and Preliminaries

Homotopy theory, which is the main part of algebraic topology, is devoted to studying topological objects up to homotopy equivalence. Over the last century, deep connections have emerged between this theory and many other branches of mathematics. For instance, this direction of research contributes to promote connections between homotopy theory and category theory (higher-dimensional), which received an increasing attention in recent years, see [23]. Also, homotopy theory is useful in quantum mechanics for dealing with Hamiltonian manifolds.

Briefly, we recall that two continuous functions from one topological space to another are said to be homotopic if one can be continuously deformed into the other. Such a deformation is called a homotopy between the two functions. More formally we have:

Definition 1.1. *Let X, Y be two topological spaces, and let $G, S : X \to Y$ be two continuous mappings. Then, a homotopy from G to S is a continuous function $\mathcal{H} : X \times [0,1] \to Y$ such that $\mathcal{H}(x,0) = Gx$ and $\mathcal{H}(x,1) = Sx$, for all $x \in X$. Also, G and S are called homotopic mappings.*

In the same decades, the theory of multi-valued mappings received much attention, because of its applications in mathematical economics, generalized dynamical systems and differential inclusions. This theory considers different forms of continuity of multi-valued mappings, then investigates differentiable and measurable multi-valued mappings, also considers single-valued continuous approximations of multi-valued mappings, finally studies fixed points of multi-valued mappings and their data dependence. A powerful tool in this study is the following Nadler's theorem [17].

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Theorem 1.2 ([17]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multi-valued mapping such that $H(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, where $k \in (0, 1)$ and $CB(X)$ denotes the family of non-empty closed and bounded subsets of X. Then T has a fixed point, that is, there exists a point $u \in X$ such that $u \in Tu$.*

Later on, Feng and Liu [12] discussed the existence of fixed points for multi-valued mappings in the classical setting of metric spaces. Precisely, they proved fixed point theorems, which generalize known results in the literature, by using a suitable semi-continuous function. Successively, Chifu and Petruşel [10] gave a local version of the main result in [12]. For more recent results, the reader is referred to [2, 4, 21] and references therein.

We can say that the basic idea of the above (homotopy and multi-valued mappings) theories is to let emerge a new approach to the development of mathematics, on the basis of constructive techniques. In such a type of approach, the concept of metric space provides a general framework usable in many areas of research, without restriction. Also for this reason, metric spaces were largely studied and generalized in many directions. In this paper, we consider one of the most interesting generalization, called partial metric space, which is due to Matthews [16]. After this, Aydi et al. [8] introduced the concept of partial Hausdorff metric and extended Theorem 1.2 in the setting of partial metric spaces; see also [1, 6, 7, 9, 11, 14, 15, 18, 19, 24].

Now, notice that a fixed point theory may be considered "perturbation stable" if, when its theorems hold true with respect to a mapping T, then the same theorems remain true with respect to any small perturbation of T (data dependence). It is clear that we are more interested in establishing if this property not only holds true for small perturbations but also for any deformation of the mapping T. In other words, we would like to get that if the hypotheses of a theorem are satisfied for T, then these hypotheses remain verified for all mappings which are homotopic to T. In view of the above considerations, first we investigate the possibility to extend the results in [10, 12] to the setting of partial Hausdorff metric spaces, then we discuss the data dependence of the fixed points for two multi-valued mappings, finally we prove a homotopy fixed point result. Clearly, our theorems generalize and complement various results in the literature. Also, an example and a nonlinear alternative of Leray-Schauder type are given to support our theorems.

2. Partial Metric Spaces

In this section, we collect some concepts and known results needed in the successive sections of the paper. In particular, the new definition of T-0-lower semi-continuity is given. In the sequel, $\mathbb{R}^+$ denotes the set of all non-negative real numbers and $\mathbb{N}$ the set of all positive integers.

Matthews [16] introduced the concept of partial metric as a generalization of classical metrics, where self-distances are not necessarily zero. Matthews' idea was that the relation: "partial metric = metric + self-distance" could turn the concept of metric into a more logic and suitable based construct. Indeed, he applied this notion to studying denotational semantics of dataflow networks. Then, we start with the definition of partial metric.

Definition 2.1 ([16]). *A partial metric on a non-empty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$, the following conditions are satisfied:*

(p1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(p2) $p(x, x) \leq p(x, y)$;

(p3) $p(x, y) = p(y, x)$;

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A non-empty set X equipped with a partial metric p is called partial metric space. We shall denote it by a pair (X, p).

If $p(x, y) = 0$, then (p1) and (p2) imply that $x = y$, but the converse does not hold true always. Also, each partial metric p on X generates a T_0 topology γ_p on X which has as a base, the family of the open balls (p-balls) $\{B_p(x, R) : x \in X, R > 0\}$ where

$$B_p(x, R) = \{y \in X : p(x, y) < p(x, x) + R\},$$

for all $x \in X$ and $R > 0$.

Moreover, if (X, p) is a partial metric space and $(X, \preceq)$ is a partially ordered set, then $(X, p, \preceq)$ is called an ordered partial metric space. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.2 ([5, 16]). *Let (X, p) be a partial metric space. Then a sequence $\{x_n\}$ is called:*

(i) convergent, with respect to γ_p, if there exists some x in X such that $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$;

(ii) Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to γ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

Moreover, notice that, according to Romaguera [20], a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$. Also, we say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to the partial metric p, to a point $x \in X$ such that $p(x, x) = 0$.

Let (X, p) be a partial metric space. Let $C^p(X)$ denote the collection of all non-empty closed subsets of X and $CB^p(X)$ the collection of all non-empty closed and bounded subsets of X with respect to the partial metric p. Consistent with Aydi et al. [8], closedness is taken from (X, γ_p). Moreover, boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$. Then, for $A, B \in CB^p(X)$, $x \in X$, $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ define

$$p(x, A) = \inf\{p(x, a) : a \in A\}, \quad p(A, B) = \inf\{p(x, y) : x \in A, y \in B\},$$

$$\delta_p(A, B) = \sup\{p(a, B) : a \in A\}, \quad \delta_p(B, A) = \sup\{p(b, A) : b \in B\}$$

and

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Proposition 2.3 ([8]). *Let (X, p) be a partial metric space. For all $A, B, C \in CB^p(X)$, we have the following:*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;

(ii) $\delta_p(A, A) \leq \delta_p(A, B)$;

(iii) $\delta_p(A, B) = 0$ implies that $A \subseteq B$;

(iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c\in C} p(c, c)$.

Proposition 2.4 ([8]). *Let (X, p) be a partial metric space. For all $A, B, C \in CB^p(X)$, we have the following:*

(h1) $H_p(A, A) \leq H_p(A, B)$;

(h2) $H_p(A, B) = H_p(B, A)$;

(h3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$;

(h4) $H_p(A, B) = 0 \Longrightarrow A = B$.

The mapping $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ is called the partial Hausdorff metric induced by p. Every Hausdorff metric is a partial Hausdorff metric but the converse is not true, see Example 2.6 in [8]. We note that $H_p : C^p(X) \times C^p(X) \to \mathbb{R}^+ \cup \{+\infty\}$ is called the generalized partial Hausdorff metric induced by p.

Lemma 2.5 ([5]). *Let (X, p) be a partial metric space and A any non-empty set in (X, p), then*

$$a \in \overline{A} \iff p(a, A) = p(a, a),$$

where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Notice that A is closed in (X, p) if and only if $A = \overline{A}$.

Theorem 2.6 ([8]). *Let (X, p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have $H_p(Tx, Ty) \leq kp(x, y)$, where $k \in (0, 1)$, then T has a fixed point, that is, there exists a point $u \in X$ such that $u \in Tu$.*

Lemma 2.7. *Let (X, p) be a partial metric space and $T : X \to C^p(X)$ be a multi-valued mapping. If $\{x_n\} \subset X$ is a sequence, $x_n \to u$ and $p(u, u) = 0$, then*

$$\lim_{n\to+\infty} p(x_n, Tu) = p(u, Tu).$$

Remark 2.8. *Notice that the proof of Lemma 2.7 is an immediate consequence of the fact that the inequality*

$$p(u, Tu) - p(u, x_n) \leq p(x_n, Tu) \leq p(x_n, u) + p(u, Tu)$$

holds for all $n \in \mathbb{N}$.

Now, we introduce the following definition.

Definition 2.9. *Let (X, p) be a partial metric space and $T : X \to C^p(X)$ be a multi-valued mapping. A function $f : X \to \mathbb{R}$ is called T-0-lower semi-continuous, if for any $\{x_n\} \subset X$ with $x_{n+1} \in Tx_n$ and $x \in X$ with $p(x, x) = 0$,*

$$\lim_{n\to+\infty} x_n = x \qquad \text{implies} \qquad fx \leq \liminf_{n\to+\infty} fx_n.$$

In the case $x_{n+1} \notin Tx_n$, but retaining the rest, f is called 0-lower semi-continuous.

Moreover, on $X \times X$, we consider the partial metric p^* defined by

$$p^*((x, y), (u, v)) = p(x, u) + p(y, v) \qquad \text{for all } (x, y), (u, v) \in X \times X.$$

Definition 2.10. *Let (X, p) be a partial metric space and let $T : X \to C^p(X)$ be a multi-valued mapping. The graph of T is the subset $\{(x, y) : x \in X, y \in Tx\}$ of $X \times X$; we denote the graph of T by $G(T)$. Then, T is a closed multi-valued mapping if the graph $G(T)$ is a closed subset of $(X \times X, p^*)$.*

3. Main Results

First fix our notation as follows. Let (X, p) be a partial metric space and let $T : X \to C^p(X)$ be a multi-valued mapping. Let $Fix(T) := \{x \in X : x \in Tx\}$ denote the fixed point set of T. Also, define the function $f_T : X \to \mathbb{R}$ as $f_T x = p(x, Tx)$. Then, for a positive constant $\alpha \in (0, 1)$ and each $x \in X$, define the set

$$I_\alpha^x := \{y \in Tx : \alpha\, p(x, y) \leq p(x, Tx)\}.$$

Now, inspired by [12], we state and prove the following theorems.

Theorem 3.1. *Let (X, p) be a 0-complete partial metric space and let $T : X \to C^p(X)$ be a multi-valued mapping. Suppose that there exists $r \in (0, \alpha)$, with $\alpha \in (0, 1)$, such that for any $x \in X$ there is $y \in I_\alpha^x$ satisfying*

$$p(y, Ty) \leq r\, p(x, y). \tag{1}$$

Then T has a fixed point in X provided that one of the following conditions holds:

(i) f_T *is* T*-0-lower semi-continuous,*

(ii) T *is closed.*

Proof. Since Tx is a non-empty closed set for any $x \in X$, I_α^x is non-empty for any constant $\alpha \in (0,1)$. Now, for a fixed point $x_0 \in X$, there exists $x_1 \in I_\alpha^{x_0}$ such that

$$p(x_1, Tx_1) \le rp(x_0, x_1).$$

If x_1 is not a fixed point of T, we choose $x_2 \in I_\alpha^{x_1}$ such that

$$p(x_2, Tx_2) \le rp(x_1, x_2).$$

Again, if x_2 is not a fixed point of T (and so on), by iterating this procedure, we can get an iterative sequence $\{x_n\}$, where $x_{n+1} \in I_\alpha^{x_n}$ and

$$p(x_{n+1}, Tx_{n+1}) \le rp(x_n, x_{n+1}) \qquad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2}$$

On the other hand, $x_{n+1} \in I_\alpha^{x_n}$ implies

$$\alpha\, p(x_n, x_{n+1}) \le p(x_n, Tx_n) \qquad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3}$$

The next step of the proof is to show that the sequence $\{x_n\}$ is a 0-Cauchy sequence. Using (2) and (3), we get

$$p(x_{n+1}, x_{n+2}) \le \frac{r}{\alpha} p(x_n, x_{n+1}) \qquad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

This implies

$$p(x_n, x_{n+1}) \le k^n p(x_0, x_1) \qquad \text{for all } n \in \mathbb{N} \cup \{0\},$$

where $k = \frac{r}{\alpha} < 1$.

Now, for each $q \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n, x_{n+q}) &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+q}) - p(x_{n+1}, x_{n+1}) \\ &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+q}) - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) \\ &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+q-2}, x_{n+q-1}) + p(x_{n+q-1}, x_{n+q}) - \sum_{j=n+1}^{n+q-1} p(x_j, x_j) \\ &\le k^n p(x_0, x_1) + k^{n+1} p(x_0, x_1) + \cdots + k^{n+q-2} p(x_0, x_1) + k^{n+q-1} p(x_0, x_1) \\ &= k^n p(x_0, x_1)\Big[1 + k + k^2 + \cdots + k^{q-1}\Big] \\ &\le \frac{k^n}{1-k}\, p(x_0, x_1). \end{aligned}$$

Consequently, since

$$\frac{k^n}{1-k}\, p(x_0, x_1) \to 0 \quad \text{as} \quad n \to +\infty,$$

we deduce that $\{x_n\}$ is a 0-Cauchy sequence and so, by 0-completeness of the space (X, p), $x_n \to x$ for some $x \in X$ with $p(x, x) = 0$. Now we claim that x is a fixed point of T. Therefore, we distinguish two cases.

Case 1: Suppose that (i) holds true. Again, by (2) and (3), we get

$$p(x_{n+1}, Tx_{n+1}) \le \frac{r}{\alpha} p(x_n, Tx_n) \qquad \text{for all } n \in \mathbb{N} \cup \{0\},$$

which implies

$$p(x_n, Tx_n) \leq k^n p(x_0, Tx_0) \qquad \text{for all } n \in \mathbb{N} \cup \{0\},$$

where $k = \dfrac{r}{\alpha} < 1$. Consequently,

$$\liminf_{n\to+\infty} f_T x_n = \lim_{n\to+\infty} f_T x_n = \lim_{n\to+\infty} p(x_n, Tx_n) = 0.$$

Since $x_{n+1} \in Tx_n$, f_T is T-0-lower semi-continuous, $x_n \to x$ and $p(x, x) = 0$, we have

$$f_T x = p(x, Tx) = 0.$$

By Lemma 2.5, from $p(x, Tx) = p(x, x)$ and the closedness of Tx, we get that $x \in Tx$. Thus $Fix(T) \neq \emptyset$.
Case 2: If (ii) holds true, then from $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ and $p^*((x_n, x_{n+1}), (x, x)) \to p^*((x, x), (x, x))$, we get that $(x, x) \in Gr(T)$ and hence $x \in Tx$. Thus x is a fixed point of T. This completes the proof. □

The following theorem is a local version of Theorem 3.1.

Theorem 3.2. *Let (X, p) be a 0-complete partial metric space, $x_0 \in X$, $R > 0$ and let $T : X \to C^p(X)$ be a multi-valued mapping. Suppose that there exists $r \in (0, \alpha)$, with $\alpha \in (0, 1)$, such that for any $x \in \overline{B}_p(x_0, R)$ there is $y \in I_\alpha^x$ satisfying*

$$p(y, Ty) \leq r\, p(x, y).$$

If $p(x_0, Tx_0) \leq \alpha(1 - r/\alpha)R$, then T has a fixed point in $\overline{B}_p(x_0, R)$ provided one of the following conditions holds:

(i) f_T is T-0-lower semi-continuous,

(ii) T is closed.

Proof. Proceeding as in the proof of Theorem 3.1, we construct an iterative sequence $\{x_n\}$ with initial point x_0, with $x_{n+1} \in I_\alpha^{x_n}$ and satisfying the conditions (2)-(4) for all $n \in \mathbb{N} \cup \{0\}$. From (4) and $p(x_0, Tx_0) \leq \alpha(1 - r/\alpha)R$, we obtain

$$p(x_n, x_{n+1}) \leq \frac{r^n}{\alpha^n} p(x_0, x_1) \leq \frac{r^n}{\alpha^{n+1}} p(x_0, Tx_0) \leq \frac{r^n}{\alpha^n}\left(1 - \frac{r}{\alpha}\right)R$$

for all $n \in \mathbb{N} \cup \{0\}$.

This implies $x_n \in \overline{B}_p(x_0, R)$. In fact,

$$p(x_0, x_n) \leq \sum_{k=1}^{n} p(x_{k-1}, x_k) \leq \sum_{k=1}^{n} \frac{r^{k-1}}{\alpha^{k-1}}\left(1 - \frac{r}{\alpha}\right)R < R$$

and so $x_n \in \overline{B}_p(x_0, R)$. Again, proceeding as in the proof of Theorem 3.1, we deduce that T has a fixed point in $\overline{B}_p(x_0, R)$. □

Now, we show that Theorem 3.1 is a generalization of the following version of Nadler's fixed point theorem.

Theorem 3.3. *Let (X, p) be a 0-complete partial metric space. If $T : X \to C^p(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have $H_p(Tx, Ty) \leq rp(x, y)$, where $r \in (0, 1)$, then T has a fixed point in X.*

Proof. First, notice that if (X, p) is complete, then it is 0-complete. Then, we prove that the function $f_T : X \to \mathbb{R}$ defined by $f_T x = p(x, Tx)$ is T-0-lower semi-continuous. Let $x \in X$ be such that $p(x, x) = 0$ and $\{x_n\} \subset X$, with $x_{n+1} \in Tx_n$, be a sequence convergent to x. From

$$p(x, Tx) \leq p(x, x_{n+1}) + H_p(Tx_n, Tx) \leq p(x, x_{n+1}) + rp(x_n, x),$$

letting $n \to +\infty$ we obtain $f_T x = 0$ and hence

$$f_T x \leq \liminf_{n\to+\infty} f_T x_n.$$

Now, we show that T satisfies condition (1) of Theorem 3.1. Indeed, for any $x \in X$ and $y \in Tx$, we have

$$p(y, Ty) \leq H_p(Tx, Ty) \leq rp(x, y)$$

and hence T satisfies all the conditions of Theorem 3.1. Consequently, we deduce the existence of a fixed point of T. □

The following example illustrates our Theorem 3.1.

Example 3.4. Let $X = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0, 1\}$ be endowed with the partial metric $p(x, y) = \frac{1}{2}|x - y| + \frac{1}{4}\max\{x, y\}$, for all $x, y \in X$. Clearly, (X, p) is a 0-complete partial metric space. Define $T : X \to C^p(X)$ as

$$Tx = \begin{cases} \{\frac{1}{2^{n+1}}, 1\} & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}, \\ \{0, \frac{1}{2}\} & \text{if } x = 0, \\ \{\frac{1}{2}\} & \text{if } x = 1. \end{cases}$$

Now, for all $n \in \mathbb{N}$, we have

$$p\left(\frac{1}{2^n}, T\frac{1}{2^n}\right) = \min\left\{\frac{1}{2^{n+1}}, \frac{2^n - 1}{2^{n+1}} + \frac{1}{4}\right\} = \frac{1}{2^{n+1}} = p\left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right).$$

Also, we have

$$p(1, T1) = p\left(1, \frac{1}{2}\right) = \frac{1}{2}, \quad p(1, 1) = \frac{1}{4} \quad \text{and} \quad p(0, T0) = \min\left\{0, \frac{3}{8}\right\} = 0 = p(0, 0).$$

Furthermore, there exists $y \in I^x_\alpha$, with $\alpha \in (1/2, 1)$, for any $x \in X$ such that

$$p(y, Ty) \leq \frac{1}{2}p(x, y).$$

Then, since f_T is T-0-lower semi-continuous, the existence of a fixed point follows from Theorem 3.1.

On the other hand, Theorem 2.6 (Aydi et al. [8]) is not applicable in this case; in fact, we have

$$\begin{aligned} H_p\left(T\frac{1}{2^n}, T0\right) &= \max\left\{\delta_p\left(T\frac{1}{2^n}, T0\right), \delta_p\left(T0, T\frac{1}{2^n}\right)\right\} \\ &= \max\left\{\frac{2^n - 1}{2^{n+2}} + \frac{1}{8}, \frac{1}{2}\right\} \\ &= \frac{1}{2} > \frac{3}{8} \geq \frac{3}{2^{n+2}} = p\left(\frac{1}{2^n}, 0\right) \end{aligned}$$

for all $n \in \mathbb{N}$.

Therefore, we conclude that our Theorem 3.1 is a proper extension of Theorem 2.6.

Next, we study data dependence of fixed points for multi-valued mappings, by using the technique presented in this section. First, we prove the following fixed point result.

Theorem 3.5. *Let (X, p) be a 0-complete partial metric space and let $T : X \to C^p(X)$ be a multi-valued mapping. Suppose that for some $\alpha \in (0,1)$ there exist $r, s \in (0,1)$, with $r\alpha^{-1} + s < 1$, such that for any $x \in X$ there is $y \in I_\alpha^x$ satisfying*

$$p(y, Ty) \leq r\, p(x, y) + sp(x, Tx).$$

Then T has a fixed point in X provided that one of the following conditions holds:

(i) f_T is T-0-lower semi-continuous,

(ii) T is closed.

Proof. Since Tx is a non-empty closed set for any $x \in X$, then I_α^x is non-empty. Now, for a fixed point $x_0 \in X$, there exists $x_1 \in I_\alpha^{x_0}$ such that

$$p(x_1, Tx_1) \leq rp(x_0, x_1) + s\,p(x_0, Tx_0) \leq r\alpha^{-1}p(x_0, Tx_0) + s\,p(x_0, Tx_0) = (r\alpha^{-1} + s)p(x_0, Tx_0).$$

If x_1 is not a fixed point of T, we choose $x_2 \in I_\alpha^{x_1}$ such that

$$\begin{aligned} p(x_2, Tx_2) &\leq rp(x_1, x_2) + s\,p(x_1, Tx_1) \\ &\leq r\alpha^{-1}p(x_1, Tx_1) + s\,p(x_1, Tx_1) \\ &= (r\alpha^{-1} + s)p(x_1, Tx_1) \\ &\leq (r\alpha^{-1} + s)^2 p(x_0, Tx_0). \end{aligned}$$

Again, if x_2 is not a fixed point of T (and so on), by iterating this procedure, we can get an iterative sequence $\{x_n\}$, where $x_{n+1} \in I_\alpha^{x_n}$ and

$$p(x_n, Tx_n) \leq (r\alpha^{-1} + s)^n p(x_0, Tx_0) \qquad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

On the other hand, $x_{n+1} \in I_\alpha^{x_n}$ implies

$$p(x_n, x_{n+1}) \leq \alpha^{-1}(r\alpha^{-1} + s)^n p(x_0, Tx_0) \qquad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Again, proceeding as in the proof of Theorem 3.1, the reader can conclude that T has a fixed point in X. □

In view of Theorem 3.5, we prove the following data dependence theorem for two multi-valued mappings.

Theorem 3.6. *Let (X, p) be a 0-complete partial metric space and let $S, T : X \to C^p(X)$ be two multi-valued mappings such that $\sup_{x \in X} H_p(Sx, Tx) < +\infty$. Suppose that for some $\alpha \in (0,1)$ there exist $r, s \in (0,1)$ with $r\alpha^{-1} + s < 1$ such that for any $x \in X$ there is $y \in I_\alpha^x$ satisfying*

$$p(y, Sy) \leq r\, p(x, y) + sp(x, Sx) \quad \text{and} \quad p(y, Ty) \leq r\, p(x, y) + sp(x, Tx).$$

Then

$$H_p(Fix(S), Fix(T)) \leq \frac{\alpha^{-1}}{1 - (r\alpha^{-1} + s)} \sup_{x \in X} H_p(Sx, Tx),$$

provided that one of the following conditions holds:

(i) f_T and f_S are 0-lower semi-continuous,

(ii) S and T are closed.

Proof. By Theorem 3.5 we deduce that $Fix(S), Fix(T) \neq \emptyset$. Moreover, $Fix(S)$ and $Fix(T)$ are closed. Indeed, for instance, let $\{u_n\} \subset Fix(S)$, such that $u_n \to u$, as $n \to +\infty$. Then, if S is closed, the conclusion follows easily. On the other hand, if $f_S x := p(x, Sx)$ is 0-lower semi-continuous we have

$$\begin{aligned} 0 &\leq f_S u \leq \liminf_{n\to+\infty} f_S u_n = \liminf_{n\to+\infty} p(u_n, Su_n) \\ &= \liminf_{n\to+\infty} p(u_n, u_n) \leq \liminf_{n\to+\infty} p(u_n, u) = p(u, u). \end{aligned}$$

Since $f_S u = p(u, Su) \geq p(u, u)$, it follows $p(u, Su) = p(u, u)$ and so by Lemma 2.5 $u \in Su$, that is, $u \in Fix(S)$. The same holds for $Fix(T)$.
Now, let $x_0 \in Fix(S)$. Then there exists $x_1 \in I_\alpha^{x_0}$ with $p(x_1, Tx_1) \leq rp(x_0, x_1) + sp(x_0, Tx_0)$. Since $\alpha p(x_0, x_1) \leq p(x_0, Tx_0)$, we obtain

$$p(x_0, x_1) \leq \alpha^{-1} p(x_0, Tx_0) \quad \text{and} \quad p(x_1, Tx_1) \leq rp(x_0, x_1) + sp(x_0, Tx_0) \leq (r\alpha^{-1} + s)p(x_0, Tx_0).$$

Then, by iterating this procedure, we can get an iterative sequence $\{x_n\}$ such that

(i) $x_0 \in Fix(S)$,

(ii) $p(x_n, Tx_n) \leq (r\alpha^{-1} + s)^n p(x_0, Tx_0)$ for all $n \in \mathbb{N} \cup \{0\}$,

(iii) $p(x_n, x_{n+1}) \leq \alpha^{-1}(r\alpha^{-1} + s)^n p(x_0, Tx_0)$ for all $n \in \mathbb{N} \cup \{0\}$.

From (iii), we deduce that $\{x_n\}$ is a 0-Cauchy sequence and hence it converges to an element $u \in X$ with $p(u, u) = 0$. As in the proof of Theorem 3.1, from (ii) we immediately get that $u \in Fix(T)$. Again, if $m > n$ from

$$p(x_n, x_m) \leq \frac{\alpha^{-1}(r\alpha^{-1} + s)^n}{1 - (r\alpha^{-1} + s)} p(x_0, Tx_0),$$

letting $m \to +\infty$, we deduce

$$p(x_n, u) \leq \frac{\alpha^{-1}(r\alpha^{-1} + s)^n}{1 - (r\alpha^{-1} + s)} p(x_0, Tx_0),$$

for each $n \in \mathbb{N} \cup \{0\}$. Then, for $n = 0$, we get

$$\begin{aligned} p(x_0, u) &\leq \frac{\alpha^{-1}}{1 - (r\alpha^{-1} + s)} p(x_0, Tx_0) \\ &\leq \frac{\alpha^{-1}}{1 - (r\alpha^{-1} + s)} H_p(Sx_0, Tx_0) \\ &\leq \frac{\alpha^{-1}}{1 - (r\alpha^{-1} + s)} \sup_{x\in X} H_p(Sx, Tx). \end{aligned}$$

In a similar way we can prove that, for each $y_0 \in Fix(T)$, there exists $v \in Fix(S)$ such that

$$p(y_0, v) \leq \frac{\alpha^{-1}}{1 - (r\alpha^{-1} + s)} \sup_{x\in X} H_p(Sx, Tx).$$

Thus, the proof is complete. □

4. Homotopy Result in 0-Complete Partial Metric Space

In this section, inspired by [22] and following a similar argument, we apply our Theorem 3.2 to get a homotopy result. Before establishing our theorem, we need the following proposition.

Proposition 4.1. *Let (X, p) be a partial metric space and let $T\colon X \to C^p(X)$ be a multi-valued mapping. Suppose that there exists $r \in (0, \alpha)$, with $\alpha \in (0, 1)$, such that for any $x \in X$*

$$p(y, Ty) \leq r\, p(x, y), \tag{5}$$

for all $y \in I_\alpha^x$. Then, if $z \in Tz$ for some $z \in X$, we deduce that $p(z, z) = 0$.

Proof. Let $z \in Tz \in C^p(X)$ so that, by Lemma 2.5, $p(z, Tz) = p(z, z)$. Clearly, $z \in I_\alpha^z$, for any $\alpha \in (0, 1)$. Consequently, assuming $p(z, z) > 0$, by using (5), we get

$$p(z, Tz) \leq rp(z, z),$$

which yields to contradiction since $r < \alpha < 1$. Thus $p(z, z) = 0$. □

Let (X, p) be a partial metric space and $T\colon X \times [0, 1] \to C^p(X)$ be a multi-valued operator, referring to Section 3, for $\alpha \in (0, 1)$ we introduce the set

$$I_\alpha^{(x,t)} := \{y \in T(x, t) : \alpha\, p(x, y) \leq p(x, T(x, t))\}.$$

Moreover, on $X \times [0, 1] \times X$, we consider the partial metric p^* defined by

$$p^*((x, t, y), (u, \tau, v)) = p(x, u) + |t - \tau| + p(y, v) \qquad \text{for all } (x, t, y), (u, \tau, v) \in X \times [0, 1] \times X.$$

Then, we adapt the definitions of closed graph and closed multi-valued mapping as follows.

Definition 4.2. *Let (X, p) be a partial metric space and let $T : X \times [0, 1] \to C^p(X)$ be a multi-valued operator. The graph of T is the subset $\{(x, t, y) : x \in X, t \in [0, 1], y \in T(x, t)\}$ of $X \times [0, 1] \times X$; we denote the graph of T by $G(T)$. Then, T is a closed multi-valued operator if the graph $G(T)$ is a closed subset of $(X \times [0, 1] \times X, p^*)$.*

Now, we are ready to prove the following result.

Theorem 4.3. *Let (X, p) be a 0-complete partial metric space, F be a closed subset of X and U be an open subset of X with $U \subset F$. Suppose that $T\colon F \times [0, 1] \to C^p(X)$ is a closed multi-valued operator satisfying the following conditions:*

(i) *$x \notin T(x, t)$ for each $x \in F \setminus U$ and each $t \in [0, 1]$;*

(ii) *there exists $r \in (0, \alpha)$, with $\alpha \in (0, 1)$ such that, for all $x \in F$ and each $t \in [0, 1]$, there is $y \in I_\alpha^{(x,t)}$ satisfying*

$$p(y, T(y, t)) \leq rp(x, y)$$

and $p(x, x) = 0$ if $x \in I_\alpha^{(x,t)}$;

(iii) *there exists a continuous increasing function $\phi : [0, 1] \to \mathbb{R}$ such that*

$$H_p(T(x, t_1), T(x, t_2)) \leq |\phi(t_1) - \phi(t_2)|, \text{ for all } t_1, t_2 \in [0, 1] \text{ and each } x \in F.$$

If $T(\cdot, 0)$ has a fixed point in F, then $T(\cdot, 1)$ has a fixed point in U.

Proof. Define the set

$$Q\colon = \{(t, x) \in [0, 1] \times U : x \in T(x, t)\}.$$

Obviously Q is nonempty. Then, consider on Q the partial order defined as follows:

$$(t, x) \preceq (s, y) \quad \text{iff} \quad t \leq s \text{ and } p(x, y) \leq \frac{2\alpha^{-1}}{1 - r\alpha^{-1}}[\phi(s) - \phi(t)].$$

Let K be a totally ordered subset of Q and consider $t^* = \sup\{t : (t,x) \in K\}$. Consider a sequence $\{(t_n, x_n)\} \subset K$ such that $(t_n, x_n) \preceq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$, as $n \to +\infty$. Then, we get

$$p(x_m, x_n) \leq \frac{2\alpha^{-1}}{1 - r\alpha^{-1}}[\phi(t_m) - \phi(t_n)],$$

for all $m, n \in \mathbb{N}$, with $m > n$. Letting $m, n \to +\infty$, we obtain $p(x_m, x_n) \to 0$ and hence $\{x_n\}$ is a 0-Cauchy sequence and it converges to an element $x^* \in X$. Since T is a closed multi-valued operator, then $(x_n, t_n, x_n) \in G(T)$ and $(x_n, t_n, x_n) \to (x^*, t^*, x^*)$, as $n \to +\infty$, ensure that $x^* \in T(x^*, t^*)$. Moreover, from (i) we deduce that $x^* \in U$, and hence $(t^*, x^*) \in Q$. Since K is totally ordered, we get $(t, x) \preceq (t^*, x^*)$, for each $(t, x) \in K$. Thus (t^*, x^*) is an upper bound of K. Consequently, the lemma of Zorn applies to this case and so Q has a maximal element, say $(t_0, x_0) \in Q$. Now, to complete the proof, we need to show that $t_0 = 1$.

Suppose the contrary, that is, assume $t_0 < 1$. Then, choose $R > 0$ and $t \in (t_0, 1]$ such that $B_p(x_0, R) \subset U$ and $R := (2\alpha^{-1}/(1 - r\alpha^{-1}))[\phi(t) - \phi(t_0)]$. Therefore, we have

$$\begin{aligned} p(x_0, T(x_0, t)) &\leq H_p(T(x_0, t_0), T(x_0, t)) \\ &\leq [\phi(t) - \phi(t_0)] \\ &= (1 - r\alpha^{-1})R/2\alpha^{-1} \\ &< (1 - r\alpha^{-1})R/\alpha^{-1}. \end{aligned}$$

It follows that the multi-valued operator $T(\cdot, t) : \overline{B}_p(x_0, R) \to C^p(X)$ satisfies all the hypotheses of Theorem 3.2. This implies that there exists a fixed point $x \in \overline{B}_p(x_0, R)$ of $T(\cdot, t)$ and so $(t, x) \in Q$. Since

$$p(x_0, x) \leq R = \frac{2\alpha^{-1}}{1 - r\alpha^{-1}}[\phi(t) - \phi(t_0)], \tag{6}$$

then we get $(t_0, x_0) \prec (t, x)$, which contradicts the maximality of (t_0, x_0). □

Remark 4.4. *Notice that $x \in \overline{B}_p(x_0, R) \cap T(x, t)$ ensures that $p(x, x) = 0$ and there exists $x_n \in B_p(x_0, R) \cap B_p(x, \frac{1}{n})$ such that $p(x, x_n) < \frac{1}{n}$, for all $n \in \mathbb{N}$. It follows that*

$$p(x_0, x) \leq p(x_0, x_n) + p(x_n, x) \leq \frac{2\alpha^{-1}}{1 - r\alpha^{-1}}[\phi(t) - \phi(t_0)] + \frac{1}{n},$$

for all $n \in \mathbb{N}$ and hence (6) *holds true.*

It is obvious that each metric space (X, d) is a partial metric space (in which each point has zero self-distance, see [16]). Then, as a consequence of Theorem 4.3 we prove a nonlinear alternative of Leray-Schauder type, see also Agarwal et al. [3] and Frigon and O'Regan [13].

Theorem 4.5. *Let $(X, \|\cdot\|)$ be a Banach space, U be an open subset of X and $0 \in U$. Suppose that $S : \overline{U} \to X$ is a closed multi-valued mapping satisfying the following conditions:*

(i) $S(\overline{U}) = \cup_{x \in \overline{U}} Sx$ is bounded;

(ii) there exists $r \in (0, \alpha)$, with $\alpha \in (0, 1)$ such that, for all $x \in \overline{U}$ and each $t \in [0, 1]$, there is $y \in I_\alpha^{(x,t)} := \{y \in tSx : \alpha\, p(x, y) \leq p(x, tSx)\}$ satisfying

$$p(y, tSy) \leq r p(x, y)$$

and $p(x, x) = 0$ if $x \in I_\alpha^{(x,t)}$, where p is the (partial) metric induced by the norm.

Then, one of the following assertions holds:

(c1) S has a fixed point in $\overline{U}$,

(c2) there exist $t \in (0, 1)$ and u in the boundary of U, say ∂U, such that $u \in tSu$.

Proof. By negation, we assume that both assertions ($c1$) and ($c2$) do not hold; otherwise the proof is finished. Then, $u \notin tSu$ for all $u \in \partial U$ and $t \in [0,1]$.
Now, let $T : \overline{U} \times [0,1] \to X$ be defined as $T(x,t) = tSx$, for all $x \in \overline{U}$ and $t \in [0,1]$. Since S is closed, then also T is closed. Next, let G be the zero mapping on $\overline{U}$. Since $G(0) = \{0\}$, obviously G has a fixed point in U. Also, S and G are homotopic mappings. Then, we put $T(x,0) = Gx$ and $T(x,1) = Sx$. Also, notice that condition (iii) of Theorem 4.3 holds true with $\phi(t) = Mt$ for all $t \in [0,1]$ and some constant $M > 0$, since $S(\overline{U})$ is bounded. Thus, we can apply Theorem 4.3 to deduce that there exists $x \in U$ with $x \in Sx$, that is, ($c1$) holds true. □

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