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Continuity of Superposition Operators on the Double Sequence Spaces $\mathcal{L}_p$

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Abstract. In this paper, we define the superposition operator P_g where $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ by $P_g((x_{ks})) = g(k, s, x_{ks})$ for all real double sequence (x_{ks}). Chew & Lee [4] and Petranuarat & Kemprasit [7] have characterized $P_g : l_p \to l_1$ and $P_g : l_p \to l_q$ where $1 \le p, q < \infty$, respectively. The main goal of this paper is to construct the necessary and sufficient conditions for the continuity of $P_g : \mathcal{L}_p \to \mathcal{L}_1$ and $P_g : \mathcal{L}_p \to \mathcal{L}_q$ where $1 \le p, q < \infty$.

1. Introduction

Let $\mathbb{R}$ be set of all real numbers, $\mathbb{N}$ be the set of all natural numbers, $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ and Ω denotes the space of all real double sequences which is the vector space with coordinatewise addition and scalar multiplication. Let $x = (x_{ks}) \in \Omega$. If for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{ks} - l| < \varepsilon$ for all $k, s \ge N$, then we call that the double sequence $x = (x_{ks})$ is convergent in the Pringsheim's sense and denoted by $p - \lim x_{ks} = l$. The space of all convergent double sequences in the Pringsheim's sense is denoted by C_p. The space of all bounded double sequences is denoted by M_u, that is,

$$M_u := \left\{ x = (x_{ks}) \in \Omega : \|x\|_{M_u} = \sup_{k,s \in \mathbb{N}} |x_{ks}| < \infty \right\}$$

which is a Banach space with the norm $\|\cdot\|_{M_u}$. It's known that there are such sequences in the space C_p, but not in the space M_u. The space $\mathcal{L}_p$ is defined by

$$\mathcal{L}_p := \left\{ x = (x_{ks}) \in \Omega : \sum_{k,s=1}^{\infty} |x_{ks}|^p < \infty \right\}$$

where $1 \le p < \infty$ and $\sum\limits_{k,s=1}^{\infty} = \sum\limits_{k=1}^{\infty}\sum\limits_{s=1}^{\infty}$. $\mathcal{L}_p$ is a Banach space with the norm

$$\|x\|_p = \left(\sum_{k,s=1}^{\infty} |x_{ks}|^p \right)^{\frac{1}{p}} .$$

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It's know that $\mathcal{L}_p \subset M_u$ and $\mathcal{L}_p \subset \mathcal{L}_q$ where $1 \leq p < q < \infty$. If given the sequence $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ defined by $f(k,s) = x_{ks}$ and given the increasing functions $i : \mathbb{N} \to \mathbb{N}$ defined by $i(k) = i_k$, $j : \mathbb{N} \to \mathbb{N}$ defined by $j(s) = j_s$, then we define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with $h(k,s) = (i_k, j_s)$. In this case, the composite function such that $f \circ h(k,s) = x_{i_k j_s}$ is called subsequence of the sequence (x_{ks}). The sequence $e^{ks} = \left(e^{ks}_{ij}\right)$ defined by

$$e^{ks}_{ij} = \begin{cases} 1, & (k,s) = (i,j) \\ 0, & \text{otherwise} \end{cases} .$$

If we consider the sequence (s_{nm}) defined by $s_{nm} = \sum\limits_{k=1}^{n}\sum\limits_{s=1}^{m} x_{ks}$ $(n, m \in \mathbb{N})$, then the pair of $((x_{ks}), (s_{nm}))$ is called double series. Also (x_{ks}) is called general term of the series and (s_{nm}) is called the sequence of partial sum. If the sequence of partial sum (s_{nm}) is convergent to a real number s in the Pringsheim's sense, i.e.,

$$p - \lim_{n,m} \sum_{k=1}^{n}\sum_{s=1}^{m} x_{ks} = s$$

then the series $((x_{ks}), (s_{nm}))$ is called convergent in the Pringsheim's sense ,i.e., $p-$convergent and the sum of series equal to s, and is denoted by

$$\sum_{k,s=1}^{\infty} x_{ks} = s.$$

It's know that if the series is $p-$convergent, then the $p-$limit of the general term of the series is zero. The remaining term of the series $\sum\limits_{k=1}^{\infty}\sum\limits_{s=1}^{\infty} x_{ks}$ is defined by

$$R_{nm} = \sum_{k=1}^{n-1}\sum_{s=m}^{\infty} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=1}^{m-1} x_{ks} + \sum_{k=n}^{\infty}\sum_{s=m}^{\infty} x_{ks}. \tag{1}$$

We will demonstrate the formula (1) briefly with

$$\sum_{\max\{k,s\} \geq N} x_{ks}$$

for $n = m = N$. It's known that if the series is $p-$convergent, then the $p-$limit of the remaining term of the series is zero. Once find before mentioned and more details in [1], [2], [3], [10].

Superposition operators on sequence spaces are discussed by some authors. Chew and Lee [4] have given the necessary and sufficient conditions for the superposition operator acting from the sequence space l_p into l_1 with the continuity hypothesis. The characterization of the superposition operator acting from the sequence space l_p into l_q with $1 \leq p, q < \infty$ has given by Dedagich and Zabrejko [5]. Petranuarat and Kemprasit [7] have characterized the superposition operator acting from sequence space l_p into l_q with $1 \leq p, q < \infty$ by generalizing works in [4]. The reader may refer for relevant terminology on the superposition operators to [4], [5], [6], [7], [8], [9].

We extend the definition of superposition operators to double sequence spaces as follows. Let X, Y be two double sequence spaces. A superposition operator P_g on X is a mapping from X into Ω defined by $P_g(x) = (g(k,s,x_{ks}))_{k,s=1}^{\infty}$ where the function $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies
(1) $g(k,s,0) = 0$ for all $k,s \in \mathbb{N}$.
If $P_g(x) \in Y$ for all $x \in X$, we say that P_g acts from X into Y and write $P_g : X \to Y$. Moreover, we shall assume the additionally some of the following conditions:
(2) $g(k,s,.)$ is continuous for all $k,s \in \mathbb{N}$
(2') $g(k,s,.)$ is bounded on every bounded subset of $\mathbb{R}$ for all $k,s \in \mathbb{N}$.
It's obvious that if the function $g(k,s,.)$ satisfies (2), then g satisfies (2') from [9].

In this paper, we characterize the superposition operator acting from the double sequence space $\mathcal{L}_p$ into $\mathcal{L}_1$ where $1 \leq p < \infty$ under the hypothesis that the function $g(k,s,.)$ satisfies (2′) and its continuity by using the methods in [4], [7]. Then we generalize our works as the superposition operator acting from the space $\mathcal{L}_p$ into $\mathcal{L}_q$ where $1 \leq p,q < \infty$ without assuming that the function $g(k,s,.)$ is satisfies (2′) by using the methods in [7].

2. Superposition Operators of $\mathcal{L}_p$ into $\mathcal{L}_1$ $(1 \leq p < \infty)$

Theorem 2.1. *Let us suppose that $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies (2′). Then $P_g : \mathcal{L}_p \to \mathcal{L}_1$ if and only if there exist $\alpha, \beta > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that*

$$\left|g(k,s,t)\right| \leq c_{ks} + \alpha |t|^p$$

for each $k,s \in \mathbb{N}$ whenever $|t| \leq \beta$.

Proof. Assume that there exist $\alpha, \beta > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that $\left|g(k,s,t)\right| \leq c_{ks} + \alpha |t|^p$ for each $k,s \in \mathbb{N}$ whenever $|t| \leq \beta$. Let $x = (x_{ks}) \in \mathcal{L}_p$. Then, $\sum_{\max\{k,s\} \geq N} |x_{ks}|^p < \varepsilon < \beta^p$ for sufficiently large $N \in \mathbb{N}$. Hence it's obvious that $|x_{ks}| < \beta$ for all $k,s \in \mathbb{N}$ such that $\max\{k,s\} \geq N$. Thus,

$$\left|g(k,s,x_{ks})\right| \leq c_{ks} + \alpha |x_{ks}|^p$$

for all $k,s \in \mathbb{N}$ such that $\max\{k,s\} \geq N$. Then we get

$$\begin{aligned} \sum_{k,s=1}^{\infty} \left|g(k,s,x_{ks})\right| &= \sum_{k,s=1}^{N-1} \left|g(k,s,x_{ks})\right| + \sum_{\max\{k,s\} \geq N} \left|g(k,s,x_{ks})\right| \\ &\leq A + \sum_{\max\{k,s\} \geq N} c_{ks} + \alpha \sum_{\max\{k,s\} \geq N} |x_{ks}|^p \\ &\leq A + \sum_{k,s=1}^{\infty} c_{ks} + \alpha \sum_{k,s=1}^{\infty} |x_{ks}|^p < \infty. \end{aligned}$$

Since $P_g(x) = (g(k,s,x_{ks}))_{k,s=1}^{\infty}$, we obtain that $P_g(x) \in \mathcal{L}_1$. So, P_g acts from $\mathcal{L}_p$ to $\mathcal{L}_1$.

Conversely, suppose that P_g acts from $\mathcal{L}_p$ to $\mathcal{L}_1$. For all $\alpha, \beta > 0$ and $k,s \in \mathbb{N}$, we define

$$A(k,s,\alpha,\beta) = \left\{t \in \mathbb{R} : |t|^p \leq \min\left\{\beta, \alpha^{-1}\left|g(k,s,t)\right|\right\}\right\}$$

and

$$B(k,s,\alpha,\beta) = \sup\left\{\left|g(k,s,t)\right| : t \in A(k,s,\alpha,\beta)\right\}.$$

If $|t| \leq \beta$ and $t \in A(k,s,\alpha,\beta)$, then $\left|g(k,s,t)\right| \leq B(k,s,\alpha,\beta)$. If $|t| \leq \beta$ and $t \notin A(k,s,\alpha,\beta)$, then $\left|g(k,s,t)\right| < \alpha |t|^p$. Thus we have

$$\left|g(k,s,t)\right| \leq B(k,s,\alpha,\beta) + \alpha |t|^p$$

whenever $|t| \leq \beta$. Now, we shall show that $B(k,s,\alpha,\beta) \in \mathcal{L}_1$ for some $\alpha, \beta > 0$. Suppose that this does not hold, i.e., for all $\alpha, \beta > 0$, $\sum_{k,s=1}^{\infty} B(k,s,\alpha,\beta) = \infty$. Therefore for every $i \in \mathbb{N}$, $\sum_{k,s=1}^{\infty} B\left(k,s,2^i,2^{-i}\right) = \infty$. Then there exist the increasing sequences of positive integers (n_i) and (m_i) such that the pair of n_i, m_i is the least positive integers satisfying

$$\sum_{k=n_{i-1}+1}^{n_i} \sum_{s=m_{i-1}+1}^{m_i} B\left(k,s,2^i,2^{-i}\right) > 1.$$

So, we see that

$$\sum_{k=n_{i-1}+1}^{n_i-1}\sum_{s=m_{i-1}+1}^{m_i-1} B\left(k,s,2^i,2^{-i}\right) \leq 1. \tag{2}$$

For each $i \in \mathbb{N}$, there is $\varepsilon_i > 0$ such that

$$\sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i} B\left(k,s,2^i,2^{-i}\right) - \varepsilon_i\left(n_i - n_{i-1}\right)\left(m_i - m_{i-1}\right) > 1. \tag{3}$$

Let $i \in \mathbb{N}$ be fixed. Since g satisfies (2′), $0 \leq B\left(k,s,2^i,2^{-i}\right) < \infty$ for all $k,s \in \mathbb{N}$ such that $n_{i-1}+1 \leq k \leq n_i$ and $m_{i-1}+1 \leq s \leq m_i$. From the definition of $B\left(k,s,2^i,2^{-i}\right)$ for all $k,s \in \mathbb{N}$ with $n_{i-1}+1 \leq k \leq n_i$ and $m_{i-1}+1 \leq s \leq m_i$, there is $x_{ks} \in A\left(k,s,2^i,2^{-i}\right)$ such that

$$\left|g\left(k,s,x_{ks}\right)\right| > B\left(k,s,2^i,2^{-i}\right) - \varepsilon_i. \tag{4}$$

From (3) and (4), we have

$$\begin{aligned}
\sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i}\left|g\left(k,s,x_{ks}\right)\right| &> \sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i} B\left(k,s,2^i,2^{-i}\right) - \sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i}\varepsilon_i \\
&= \sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i} B\left(k,s,2^i,2^{-i}\right) - \varepsilon_i\left(n_i - n_{i-1}\right)\left(m_i - m_{i-1}\right) \\
&> 1.
\end{aligned}$$

Thus $\sum_{i=1}^{\infty}\left(\sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i}\left|g\left(k,s,x_{ks}\right)\right|\right) = \infty$, that is, $\left(g\left(k,s,x_{ks}\right)\right)_{k,s=1}^{\infty} \notin \mathcal{L}_1$. Since $x_{ks} \in A\left(k,s,2^i,2^{-i}\right)$,

$$\left|x_{ks}\right|^p \leq \frac{1}{2^i} \text{ and } \left|x_{ks}\right|^p \leq 2^{-i}\left|g\left(k,s,x_{ks}\right)\right| \tag{5}$$

for all $k,s \in \mathbb{N}$ with $n_{i-1}+1 \leq k \leq n_i$ and $m_{i-1}+1 \leq s \leq m_i$. Therefore, we obtain using (2) and (5) that

$$\begin{aligned}
\sum_{k=n_{i-1}+1}^{n_i}\sum_{s=m_{i-1}+1}^{m_i}\left|x_{ks}\right|^p &= \sum_{k=n_{i-1}+1}^{n_i-1}\sum_{s=m_{i-1}+1}^{m_i-1}\left|x_{ks}\right|^p + \left|x_{n_i m_i}\right|^p \\
&\leq \sum_{k=n_{i-1}+1}^{n_i-1}\sum_{s=m_{i-1}+1}^{m_i-1} 2^{-i}\left|g\left(k,s,x_{ks}\right)\right| + \frac{1}{2^i} \\
&\leq 2^{-i}\sum_{k=n_{i-1}+1}^{n_i-1}\sum_{s=m_{i-1}+1}^{m_i-1} B\left(k,s,2^i,2^{-i}\right) + \frac{1}{2^i} \\
&\leq \frac{1}{2^i} + \frac{1}{2^i} = \frac{2}{2^i}
\end{aligned}$$

which shows that $\left(x_{ks}\right) \in \mathcal{L}_p$. This contradicts the assumption that $P_g : \mathcal{L}_p \to \mathcal{L}_1$. □

Example 2.2. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ defined by*

$$g\left(k,s,t\right) = \frac{|t|}{3^{k+s}} + |t|^{p+1}$$

for all $k,s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Since $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s \in \mathbb{N}$, g satisfies (2′). Let $\beta = 2$ and $|t| \le 2$. Then for all $k,s \in \mathbb{N}$,

$$\begin{aligned} \left|g(k,s,t)\right| &= \frac{|t|}{3^{k+s}} + |t|^{p+1} \\ &= \frac{|t|}{3^{k+s}} + |t|^{p}\,|t| \\ &\le \frac{2}{3^{k+s}} + 2\,|t|^{p}. \end{aligned}$$

Since $\sum_{k,s=1}^{\infty} \frac{2}{3^{k+s}} < \infty$, we put $c_{ks} = \frac{2}{3^{k+s}}$ for all $k,s \in \mathbb{N}$. If we take $\alpha = 2$, then we have $\left|g(k,s,t)\right| \le c_{ks} + \alpha\,|t|^p$ whenever $|t| \le \beta$. By Theorem 2.1, we find that P_g acts from $\mathcal{L}_p$ to $\mathcal{L}_1$.

Theorem 2.3. *If $P_g : \mathcal{L}_p \to \mathcal{L}_1$, then P_g is continuous on $\mathcal{L}_p$ if and only if the function $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s \in \mathbb{N}$.*

Proof. Assume that P_g is continuous on $\mathcal{L}_p$. Let $\varepsilon > 0$ be given. Also, let $m,n \in \mathbb{N}$ and $t \in \mathbb{R}$. Since P_g is continuous at $te^{mn} \in \mathcal{L}_p$, there exists $\delta > 0$ such that $\|z - te^{mn}\|_p < \delta$ implies $\left\|P_g(z) - P_g(te^{mn})\right\|_1 < \varepsilon$ for all $z = (z_{ks}) \in \mathcal{L}_p$. Let $u \in \mathbb{R}$ such that $|u - t| < \delta$ and define y_{ks} by

$$y_{ks} = \begin{cases} u, & k = m \text{ and } s = n \\ 0, & \text{otherwise} \end{cases}.$$

Hence $y = (y_{ks}) \in \mathcal{L}_p$ and $|u - t| = \left\|y - te^{mn}\right\|_p < \delta$. Therefore, we get $\left|g(k,s,u) - g(k,s,t)\right| = \left\|P_g(y) - P_g(te^{mn})\right\|_1 < \varepsilon$.

Conversely, suppose that $g(k,s,.)$ is continuous on $\mathbb{R}$ for all $k,s \in \mathbb{N}$. So, g satisfies (2′). Since $P_g : \mathcal{L}_p \to \mathcal{L}_1$, there exist $\alpha,\beta > 0$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that for each $k,s \in \mathbb{N}$,

$$\left|g(k,s,t)\right| \le c_{ks} + \alpha\,|t|^p \text{ whenever } |t| \le \beta \tag{6}$$

by Theorem 2.1. Since $x = (x_{ks}) \in \mathcal{L}_p$ and $(c_{ks}) \in \mathcal{L}_1$, there exists sufficiently large $N \in \mathbb{N}$ such that

$$\sum_{\max\{k,s\} \ge N} |x_{ks}|^p < \min\left\{\frac{\varepsilon}{6\alpha}, \frac{1}{2^p}\left(\frac{\varepsilon}{6\alpha}\right)\right\} \tag{7}$$

$$|x_{ks}| < \frac{\beta}{2} \text{ for all } k,s \in \mathbb{N} \text{ such that } \max\{k,s\} \ge N \tag{8}$$

and

$$\sum_{\max\{k,s\} \ge N} |c_{ks}| < \frac{\varepsilon}{6}.$$

From (6) and (8), we find $\left|g(k,s,x_{ks})\right| \le c_{ks} + \alpha\,|x_{ks}|^p$ for all $k,s \in \mathbb{N}$ such that $\max\{k,s\} \ge N$. Hence, we find

$$\sum_{\max\{k,s\} \ge N} \left|g(k,s,x_{ks})\right| \le \sum_{\max\{k,s\} \ge N} c_{ks} + \alpha \sum_{\max\{k,s\} \ge N} |x_{ks}|^p < \frac{\varepsilon}{3}. \tag{9}$$

There exists $\delta > 0$ with $\delta < \min\left\{\frac{\beta}{2}, \frac{1}{2}\left(\frac{\varepsilon}{6\alpha}\right)^{\frac{1}{p}}\right\}$ such that for all $k,s \in \{1,2,...,N-1\}$ and $t \in \mathbb{R}$,

$$|t - x_{ks}| < \delta \text{ implies } \left|g(k,s,t) - g(k,s,x_{ks})\right| < \frac{\varepsilon}{3(N-1)^2} \tag{10}$$

because $g(k,s,.)$ is continuous at x_{ks} for all $k,s \in \{1,2,...,N-1\}$. Let $z \in \mathcal{L}_p$ such that $\|z-x\|_p < \delta$. Then

$$|z_{ks} - x_{ks}| < \delta \tag{11}$$

for all $k,s \in \mathbb{N}$. For all $k,s \in \{1,2,...,N-1\}$, we find

$$\left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right| < \frac{\varepsilon}{3(N-1)^2}$$

by (10). Therefore

$$\sum_{k,s=1}^{N-1} \left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right| < \frac{\varepsilon}{3}. \tag{12}$$

We see that $\left\|(z_{ks})_{\max\{k,s\}\geq N} - (x_{ks})_{\max\{k,s\}\geq N}\right\|_p < \frac{1}{2}\left(\frac{\varepsilon}{6\alpha}\right)^{\frac{1}{p}}$. So,

$$\begin{aligned}\left(\sum_{\max\{k,s\}\geq N} |z_{ks}|^p\right)^{\frac{1}{p}} &= \left\|(z_{ks})_{\max\{k,s\}\geq N}\right\|_p \\ &\leq \left\|(z_{ks})_{\max\{k,s\}\geq N} - (x_{ks})_{\max\{k,s\}\geq N}\right\|_p + \left\|(x_{ks})_{\max\{k,s\}\geq N}\right\| \\ &< \left(\frac{\varepsilon}{6\alpha}\right)^{\frac{1}{p}}\end{aligned}$$

from (7). For all $k,s \in \mathbb{N}$ such that $\max\{k,s\} \geq N$, we find

$$|z_{ks}| \leq |z_{ks} - x_{ks}| + |x_{ks}| < \delta + \frac{\beta}{2} < \beta$$

by (11). It's follows that,

$$\left|g(k,s,z_{ks})\right| \leq c_{ks} + \alpha |z_{ks}|^p$$

for all $k,s \in \mathbb{N}$ such that $\max\{k,s\} \geq N$ from (6). Therefore,

$$\sum_{\max\{k,s\}\geq N} \left|g(k,s,z_{ks})\right| \leq \sum_{\max\{k,s\}\geq N} c_{ks} + \alpha \sum_{\max\{k,s\}\geq N} |z_{ks}|^p < \frac{\varepsilon}{3}.$$

Then, we obtain

$$\begin{aligned}\left\|P_g(z) - P_g(x)\right\|_1 &= \sum_{k,s=1}^{N-1} \left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right| + \sum_{\max\{k,s\}\geq N} \left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right| \\ &\leq \sum_{k,s=1}^{N-1} \left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right| + \sum_{\max\{k,s\}\geq N} \left|g(k,s,z_{ks})\right| + \sum_{\max\{k,s\}\geq N} \left|g(k,s,x_{ks})\right| \\ &< \varepsilon\end{aligned}$$

by (9) and (12). □

3. Superposition Operators of $\mathcal{L}_p$ into $\mathcal{L}_q$ $(1 \leq p, q < \infty)$

Proposition 3.1. *Let X be a double sequence space. If $\mathcal{L}_1 \subseteq X$ and $P_g : X \to M_u$, then there exist $N \in \mathbb{N}$ and $\alpha > 0$ such that $(g(k,s,.))_{k,s=N}^{\infty}$ is uniformly bounded on $[-\alpha, \alpha]$.*

Proof. Suppose that the converse of this holds. Then there is a subsequence $(i_k, j_s)_{k,s=1}^{\infty}$ of $(i,j)_{i,j=1}^{\infty}$ and a sequence $\left(x_{i_k j_s}\right)_{k,s=1}^{\infty}$ such that

$$x_{i_k j_s} \in \left[-2^{-(k+s)}, 2^{-(k+s)}\right] \text{ and } \left|g\left(i_k, j_s, x_{i_k j_s}\right)\right| > k+s$$

for all $k,s \in \mathbb{N}$. Then we find $\left(x_{i_k j_s}\right)_{k,s=1}^{\infty} \in \mathcal{L}_1$ and $\left(g\left(i_k, j_s, x_{i_k j_s}\right)\right)_{k,s=1}^{\infty} \notin M_u$. Let $\left(y_{ij}\right)_{i,j=1}^{\infty}$ defined by

$$y_{ij} = \begin{cases} x_{i_k j_s}, & i_k = i \text{ and} j_s = j \\ 0, & \text{otherwise} \end{cases}.$$

Hence, we obtain $\left(y_{ij}\right)_{i,j=1}^{\infty} \in \mathcal{L}_1 \subseteq X$ and $\left(g\left(i,j,y_{ij}\right)\right)_{i,j=1}^{\infty} \notin M_u$. Therefore, $P_g : X \nrightarrow M_u$. □

Theorem 3.2. *$P_g : \mathcal{L}_p \to \mathcal{L}_q$ if and only if there exist $\alpha > 0$, $\beta > 0$, $N \in \mathbb{N}$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that*

$$\left|g(k,s,t)\right|^q \leq c_{ks} + \alpha |t|^p \text{ whenever } |t| \leq \beta \tag{13}$$

for all $k,s \in \mathbb{N}$ with $\max\{k,s\} \geq N$.

Proof. Suppose that P_g acts from $\mathcal{L}_p$ to $\mathcal{L}_q$. Since $\mathcal{L}_q \subset M_u$, $P_g : \mathcal{L}_p \to M_u$. Also since $\mathcal{L}_1 \subseteq \mathcal{L}_p$, we see that there exist α_0 and $N \in \mathbb{N}$ such that $(g(k,s,.))_{k,s=N}^{\infty}$ is uniformly bounded on $[-\alpha_0, \alpha_0]$ by Proposition 3.1. Therefore,

$$\sup_{t \in [-\alpha_0, \alpha_0]} \left|g(k,s,t)\right|^q < \infty$$

for all $k,s \geq N$. We define $A(k,s,\alpha,\beta) \subseteq [-\alpha_0, \alpha_0]$ by

$$A(k,s,\alpha,\beta) = \left\{t \in [-\alpha_0, \alpha_0] : |t|^p \leq \min\left\{\beta, \alpha^{-1}\left|g(k,s,t)\right|^q\right\}\right\} \tag{14}$$

and

$$B(k,s,\alpha,\beta) = \sup\left\{\left|g(k,s,t)\right|^q : t \in A(k,s,\alpha,\beta)\right\} \tag{15}$$

for all $\alpha > 0$, $\beta > 0$ and $k,s \geq N$. We assert that $\sum_{k,s=N}^{\infty} B(k,s,\alpha,\beta) < \infty$ for some $\alpha, \beta > 0$. To show the validity of this fact, we assume the converse, that is, $\sum_{k,s=N}^{\infty} B\left(k,s,2^j,2^{-j}\right) = \infty$ for each $j \in \mathbb{N} \cup \{0\}$. Therefore, we see that for all $j \in \mathbb{N} \cup \{0\}$ and $n \geq N$ there exist $n' > n$ and $m' > n$ such that

$$\sum_{k=n}^{n'} \sum_{s=n}^{m'} B\left(k,s,2^j,2^{-j}\right) > 1. \tag{16}$$

Then there exist $n'_1 > N$ and $m'_1 > N$ such that

$$\sum_{k=N+1}^{n'_1} \sum_{s=N+1}^{m'_1} B\left(k,s,2^0,2^{-0}\right) > 1.$$

Let

$$\begin{aligned} n_1 &= \min\left\{ n' \in \mathbb{N} | \, m' \in \mathbb{N}, \; n', m' > N, \text{ and } \sum_{k=N+1}^{n'} \sum_{s=N+1}^{m'} B\left(k, s, 2^0, 2^{-0}\right) > 1 \right\} \\ m_1 &= \min\left\{ m' \in \mathbb{N} | \, n' \in \mathbb{N}, \; n', m' > N, \text{ and } \sum_{k=N+1}^{n'} \sum_{s=N+1}^{m'} B\left(k, s, 2^0, 2^{-0}\right) > 1 \right\}. \end{aligned}$$

Also there exist $n_2' > n_1$ and $m_2' > m_1$ such that

$$\sum_{k=n_1+1}^{n_2'} \sum_{s=m_1+1}^{m_2'} B\left(k, s, 2^1, 2^{-1}\right) > 1$$

by using (16) . We write

$$\begin{aligned} n_2 &= \min\left\{ n' \in \mathbb{N} | \, m' \in \mathbb{N}, n' > n_1, m' > m_1, \text{ and } \sum_{k=n_1+1}^{n'} \sum_{s=m_1+1}^{m'} B\left(k, s, 2^1, 2^{-1}\right) > 1 \right\} \\ m_2 &= \min\left\{ m' \in \mathbb{N} | \, n' \in \mathbb{N}, n' > n_1, m' > m_1, \text{ and } \sum_{k=n_1+1}^{n'} \sum_{s=m_1+1}^{m'} B\left(k, s, 2^1, 2^{-1}\right) > 1 \right\}. \end{aligned}$$

Hence by induction, there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a subsequence $(m_k)_{k=1}^{\infty}$ of $(m)_{m=1}^{\infty}$ such that $n_1, m_1 > N$ and

$$\begin{aligned} n_{j+1} &= \min\left\{ n' \in \mathbb{N} | \, m' \in \mathbb{N}, n' > n_1, m' > m_1 \text{ and } \sum_{k=n_j+1}^{n'} \sum_{s=m_j+1}^{m'} B\left(k, s, 2^j, 2^{-j}\right) > 1 \right\} \\ m_{j+1} &= \min\left\{ m' \in \mathbb{N} | \, n' \in \mathbb{N}, m' > m_1, n' > n_1 \text{ and } \sum_{k=n_j+1}^{n'} \sum_{s=m_j+1}^{m'} B\left(k, s, 2^j, 2^{-j}\right) > 1 \right\}. \end{aligned}$$

Therefore, we see

$$\sum_{k=n_j+1}^{n_{j+1}-1} \sum_{s=m_j+1}^{m_{j+1}-1} B\left(k, s, 2^j, 2^{-j}\right) \leq 1. \tag{17}$$

We set $\mathcal{F} = \{(k,s) : k \leq n_1 \;\vee\; s \leq m_1\}$. If $(k,s) \in \mathcal{F}$, let $x_{ks} = 0$. If $k > n_1$ and $s > m_1$, then there exists $j \in \mathbb{N}$ such that $n_j < k \leq n_{j+1}$ and $m_j < s \leq m_{j+1}$. Thus there is $(x_{ks}) \in A\left(k, s, 2^j, 2^{-j}\right)$ and

$$0 \leq B\left(k, s, 2^j, 2^{-j}\right) < \left|g\left(k, s, x_{ks}\right)\right|^q + 2^{-(k+s)} \tag{18}$$

by (15). Also from (14), we have

$$|x_{ks}| \leq \min\left\{2^{-j}, 2^{-j}\left|g\left(k, s, x_{ks}\right)\right|^q\right\}. \tag{19}$$

Therefore for each $r \in \mathbb{N}$, we find

$$
\begin{aligned}
r &< \sum_{j=1}^{r}\left(\sum_{k=n_j+1}^{n_{j+1}}\sum_{s=m_j+1}^{m_{j+1}} B\left(k,s,2^j,2^{-j}\right)\right)\\
&< \sum_{k=n_1+1}^{n_{r+1}}\sum_{s=m_1+1}^{m_{r+1}}\left\{\left|g\left(k,s,x_{ks}\right)\right|^q+2^{-(k+s)}\right\}\\
&< \sum_{k=1}^{n_{r+1}}\sum_{s=1}^{m_{r+1}}\left|g\left(k,s,x_{ks}\right)\right|^q+\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}2^{-(k+s)}
\end{aligned}
$$

by using (18). Since $\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}2^{-(k+s)}<\infty$, we find $P_g(x)=\left(g\left(k,s,x_{ks}\right)\right)_{k,s=1}^{\infty}\notin\mathcal{L}_q$. We see by (17) and (19) that

$$
\begin{aligned}
\sum_{k,s=1}^{\infty}\left|x_{ks}\right|^p &= \sum_{j=1}^{\infty}\left(\sum_{k=n_j+1}^{n_{j+1}}\sum_{s=m_j+1}^{m_{j+1}}\left|x_{ks}\right|^p\right)\\
&\le \sum_{j=1}^{\infty}\left(\sum_{k=n_j+1}^{n_{j+1}-1}\sum_{s=m_j+1}^{m_{j+1}-1}\left|x_{ks}\right|^p+\left|x_{n_{j+1}m_{j+1}}\right|^p\right)\\
&\le \sum_{j=1}^{\infty}\left(2^{-j}\sum_{k=n_j+1}^{n_{j+1}-1}\sum_{s=m_j+1}^{m_{j+1}-1}\left|g\left(k,s,x_{ks}\right)\right|^q+2^{-j}\right)\\
&\le \sum_{j=1}^{\infty}\left(2^{-j}\sum_{k=n_j+1}^{n_{j+1}-1}\sum_{s=m_j+1}^{m_{j+1}-1}B\left(k,s,2^j,2^{-j}\right)+2^{-j}\right)\\
&\le \sum_{j=1}^{\infty}2.2^{-j}
\end{aligned}
$$

which means that $(x_{ks})\in\mathcal{L}_p$. But it contradicts that $P_g:\mathcal{L}_p\to\mathcal{L}_q$. So, we see that there exist $\alpha>0$ and $\beta>0$ such that $\sum_{k,s=N}^{\infty}B\left(k,s,\alpha,\beta\right)<\infty$.
Let $\gamma=\min\left\{\alpha_0,\beta^{\frac{1}{p}}\right\}$ and define (c_{ks}) by

$$
c_{ks}=\begin{cases} B\left(k,s,\alpha,\beta\right), & k,s\ge N\\ 0, & \text{otherwise}\end{cases}.
$$

It's obvious that $(c_{ks})_{k,s=1}^{\infty}\in\mathcal{L}_1$. Also $[-\gamma,\gamma]\subseteq[-\alpha_0,\alpha_0]$ and $|t|^p\le\beta$ for each $t\in[-\gamma,\gamma]$. Let $k,s\ge N$ and $t\in[-\gamma,\gamma]$. If $t\in A\left(k,s,\alpha,\beta\right)$, then $\left|g\left(k,s,t\right)\right|^q\le B\left(k,s,\alpha,\beta\right)=c_{ks}\le c_{ks}+\alpha\left|t\right|^p$. If $t\notin A\left(k,s,\alpha,\beta\right)$, then $|t|^p>\alpha^{-1}\left|g\left(k,s,t\right)\right|^q$ and so we find $\left|g\left(k,s,t\right)\right|^q<\alpha\left|t\right|^p\le c_{ks}+\alpha\left|t\right|^p$. Hence the inequality (13) holds.

Conversely, suppose that there is $\alpha>0$, $\beta>0$, $N\in\mathbb{N}$ and $(c_{ks})_{k,s=1}^{\infty}\in\mathcal{L}_1$ such that

$$
\left|g\left(k,s,t\right)\right|^q\le c_{ks}+\alpha\left|t\right|^p \text{ whenever } |t|\le\beta
$$

for all k,s with $\max\{k,s\}\ge N$. Let $(x_{ks})\in\mathcal{L}_p$. Then there is $N'>N$ such that $\sum_{\max\{k,s\}\ge N'}\left|x_{ks}\right|^p<\varepsilon<\beta^p$. Hence for all $k,s\in\mathbb{N}$ such that $\max\{k,s\}\ge N'$, it's obvious that $|x_{ks}|<\beta$. Therefore,

$$
\left|g\left(k,s,x_{ks}\right)\right|^q\le c_{ks}+\alpha\left|x_{ks}\right|^p
$$

for all $k, s \in \mathbb{N}$ such that $\max\{k, s\} \geq N'$. So, we have

$$\sum_{\max\{k,s\}\geq N'} \left|g(k,s,x_{ks})\right|^q \leq \sum_{\max\{k,s\}\geq N'} c_{ks} + \alpha \sum_{\max\{k,s\}\geq N'} |x_{ks}|^p .$$

Then we obtain that $\sum_{k,s=1}^{\infty} \left|g(k,s,x_{ks})\right|^q < \infty$. This completes the proof. □

Example 3.3. *Let $g : \mathbb{N}^2 \times \mathbb{R} \to \mathbb{R}$ defined by*

$$g(k,s,t) = \left(\frac{1}{2^{\frac{k+s}{q}}} + |t|^{\frac{p}{q}}\right)|t|$$

for all $k, s \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Let $N = 1$, $\beta = 2$ and $|t| \leq 2$. Then for all $k, s \in \mathbb{N}$,

$$\begin{aligned} \left|g(k,s,t)\right|^q &= \left(\frac{1}{2^{\frac{k+s}{q}}} + |t|^{\frac{p}{q}}\right)^q |t|^q \\ &\leq 2^q \max\left\{\frac{1}{2^{\frac{k+s}{q}\cdot q}}, |t|^{\frac{p}{q}\cdot q}\right\} \cdot |t|^q \\ &\leq 2^{2q}\left(\frac{1}{2^{k+s}} + |t|^p\right) \\ &\leq \frac{4^q}{2^{k+s}} + 4^q |t|^p . \end{aligned}$$

Since $\sum_{k,s=1}^{\infty} \frac{4^q}{2^{k+s}} < \infty$, we put $c_{ks} = \frac{4^q}{2^{k+s}}$ for all $k, s \in \mathbb{N}$. If we take $\alpha = 4^q$, then we have $\left|g(k,s,t)\right| \leq c_{ks} + \alpha |t|^p$ whenever $|t| \leq \beta$. By Theorem 3.2, we find that P_g acts from $\mathcal{L}_p$ to $\mathcal{L}_q$.

Proposition 3.4. *Let X be a normed double sequence space containing all finite double sequences and Y be a normed double sequence space such that $Y \subseteq M_u$. Suppose that*
(i) $P_g : X \to Y$,
(ii) there exists $\alpha > 0$ such that $\|e^{mn}\|_X \leq \alpha$ for all $m, n \in \mathbb{N}$,
(iii) $\|.\|_{M_u} \leq \beta \|.\|_Y$ on Y for some $\beta > 0$.
If P_g is continuous at x, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t - x_{ks}| < \delta \text{ implies } \left|g(k,s,t) - g(k,s,x_{ks})\right| < \varepsilon$$

for all $k, s \in \mathbb{N}$ and $t \in \mathbb{R}$.

Proof. Let any $\varepsilon > 0$. Since P_g is continuous, there exists $\delta > 0$ such that

$$\|z - x\|_X < \delta \text{ implies } \left\|P_g(z) - P_g(x)\right\|_Y < \varepsilon \tag{20}$$

for all $z \in X$. Let $k, s \in \mathbb{N}$ and $t \in \mathbb{R}$ with $|t - x_{ks}| < \frac{\delta}{\alpha}$. Let $u = (t - x_{ks}) e^{mn} + x$, hence $u \in X$, $u_{ks} = t$ and from (ii)

$$\|u - x\|_X = |t - x_{ks}| \, \|e^{mn}\|_X < \delta.$$

Thus, we find $\left\|P_g(u) - P_g(x)\right\|_Y < \frac{\varepsilon}{\beta}$ by (20). Therefore, we obtain

$$\left|g(k,s,t) - g(k,s,x_{ks})\right| \leq \left\|P_g(u) - P_g(x)\right\|_{M_u} \leq \beta \left\|P_g(u) - P_g(x)\right\|_Y < \varepsilon$$

by (iii). □

Theorem 3.5. *Let the superposition operator $P_g : \mathcal{L}_p \to \mathcal{L}_q$. If P_g is continuous at $x \in \mathcal{L}_p$ if and only if $g(k,s,.)$ is continuous at x_{ks} for all $k,s \in \mathbb{N}$.*

Proof. Since the conditions in Proposition 3.4 provided, we see that the necessary condition can be showed easily.

Conversely, suppose that $g(k,s,.)$ is continuous at x_{ks} for all $k,s \in \mathbb{N}$. We need to show that P_g is continuous at $x \in \mathcal{L}_p$. Since $P_g : \mathcal{L}_p \to \mathcal{L}_q$, there exist $\alpha > 0$, $\beta > 0$, $N_1 \in \mathbb{N}$ and $(c_{ks})_{k,s=1}^{\infty} \in \mathcal{L}_1$ such that

$$\left|g(k,s,t)\right|^q \leq c_{ks} + \alpha |t|^p \tag{21}$$

whenever $|t| \leq \beta$ for all k,s with $\max\{k,s\} \geq N_1$. Let any $\varepsilon > 0$. Since $\mathcal{L}_p \subseteq M_u$,

$$p - \lim_{n,m\to\infty}\left[\sum_{k=0}^{n-1}\sum_{s=m}^{\infty}|x_{ks}|^p + \sum_{k=n}^{\infty}\sum_{s=1}^{m-1}|x_{ks}|^p + \sum_{k=n}^{\infty}\sum_{s=m}^{\infty}|x_{ks}|^p\right] = 0$$

and

$$p - \lim_{n,m\to\infty}\left[\sum_{k=0}^{n-1}\sum_{s=m}^{\infty}c_{ks} + \sum_{k=n}^{\infty}\sum_{s=1}^{m-1}c_{ks} + \sum_{k=n}^{\infty}\sum_{s=m}^{\infty}c_{ks}\right] = 0$$

respectively, there exists $N \in \mathbb{N}$ with $N \geq N_1$ such that

$$|x_{ks}| \leq \beta \text{ for all } k,s \text{ with} \max\{k,s\} \geq N, \tag{22}$$

$$\sum_{\max\{k,s\}\geq N}|x_{ks}|^p \leq \min\left\{\left(\frac{\beta}{2}\right)^p, \frac{\varepsilon^q}{\alpha 2^{q+3}}, \frac{1}{2^p}\left(\frac{\varepsilon^q}{\alpha 2^{q+3}}\right)\right\} \tag{23}$$

and

$$\sum_{\max\{k,s\}\geq N} c_{ks} \leq \frac{\varepsilon^q}{2^{q+3}}. \tag{24}$$

We write $\left|g(k,s,x_{ks})\right|^q \leq c_{ks} + \alpha |x_{ks}|^p$ for all k,s with $\max\{k,s\} \geq N$ by using (21) and (22). Since $g(k,s,.)$ is continuous at x_{ks} for all $k,s \in \{1,2,...,N-1\}$, there is $\delta \in \mathbb{R}$ satisfying $0 < \delta \leq \min\left\{\frac{\beta}{2}, \frac{1}{2}\left(\frac{\varepsilon^q}{\alpha 2^{q+3}}\right)^{\frac{1}{p}}\right\}$ such that

$$|t - x_{ks}| < \delta \text{ implies } \left|g(k,s,t) - g(k,s,x_{ks})\right| < \left(\frac{\varepsilon^q}{2(N-1)^2}\right)^{\frac{1}{q}} \tag{25}$$

for all $t \in \mathbb{R}$ and $k,s \in \{1,2,...,N-1\}$.
Let $z \in \mathcal{L}_p$ with $\|z - x\|_p < \delta$. Then, we see that

$$\left\|(z_{ks})_{\max\{k,s\}\geq N} - (x_{ks})_{\max\{k,s\}\geq N}\right\|_p < \delta.$$

Hence we find

$$\begin{aligned} |z_{ks}| &\leq \left\|(z_{ks})_{\max\{k,s\}\geq N}\right\|_p \\ &\leq \left\|(z_{ks})_{\max\{k,s\}\geq N} - (x_{ks})_{\max\{k,s\}\geq N}\right\|_p + \left\|(x_{ks})_{\max\{k,s\}\geq N}\right\|_p \\ &< \delta + \frac{\beta}{2} \leq \beta \end{aligned} \tag{26}$$

by using (23), for all k,s with $\max\{k,s\} \geq N$. Also, we find

$$\begin{aligned}\left(\sum_{\max\{k,s\}\geq N} |z_{ks}|^p\right)^{\frac{1}{p}} &= \left\|(z_{ks})_{\max\{k,s\}\geq N}\right\|_p \\ &\leq \left\|(z_{ks})_{\max\{k,s\}\geq N} - (x_{ks})_{\max\{k,s\}\geq N}\right\|_p + \left\|(x_{ks})_{\max\{k,s\}\geq N}\right\|_p \\ &\leq \left(\frac{\varepsilon^q}{\alpha 2^{q+3}}\right)^{\frac{1}{p}}. \end{aligned} \tag{27}$$

We write $\left|g(k,s,z_{ks})\right|^q \leq c_{ks} + \alpha |z_{ks}|^p$ for all k,s with $\max\{k,s\} \geq N$ by using (21) and (26). Therefore, we have

$$\begin{aligned}\left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right|^q &\leq 2^q \max\left\{\left|g(k,s,z_{ks})\right|^q, \left|g(k,s,x_{ks})\right|^q\right\} \\ &\leq 2^q\left(\left|g(k,s,z_{ks})\right|^q + \left|g(k,s,x_{ks})\right|^q\right) \\ &\leq 2^q\left(2c_{ks} + \alpha|z_{ks}|^p + \alpha|x_{ks}|^p\right).\end{aligned}$$

Then we find

$$\begin{aligned}\sum_{\max\{k,s\}\geq N} \left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right|^q &\leq 2^{q+1}\sum_{\max\{k,s\}\geq N} c_{ks} + 2^q\alpha \sum_{\max\{k,s\}\geq N}|z_{ks}|^p + 2^q\alpha\sum_{\max\{k,s\}\geq N}|x_{ks}|^p \\ &< \frac{\varepsilon^q}{2}\end{aligned}$$

by using (23), (24) and (27). We know that $|z_{ks} - x_{ks}| \leq \|z - x\|_p < \delta$ for all $k,s \in \{1,2,...,N-1\}$ and so $\left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right|^q \leq \frac{\varepsilon^q}{2(N-1)^2}$ from (25). Therefore, we obtain

$$\begin{aligned}\sum_{k,s=1}^{\infty}\left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right|^q &= \sum_{k,s=1}^{N-1}\left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right|^q + \sum_{\max\{k,s\}\geq N}\left|g(k,s,z_{ks}) - g(k,s,x_{ks})\right|^q \\ &< (N-1)^2\frac{\varepsilon^q}{2(N-1)^2} + \frac{\varepsilon^q}{2} < \varepsilon^q.\end{aligned}$$

This completes the proof. □

4. Concluding Remarks

The necessary and sufficient conditions for the continuity of the superposition operator P_g have been formulated, as stated in Theorem 2.3 and Theorem 3.5 . For the future, we will formulate the necessary and sufficient conditions for the boundedness of the superposition operator P_g.

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