Filomat 30:10 (2016), 2731–2742
DOI 10.2298/FIL1610731R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On p-Cauchy Numbers

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Abstract. In this paper we define a new family of p-Cauchy numbers by means of the confluent hypergeometric function. We establish some basic properties. As consequence, a number of algorithms based on three-term recurrence relation for computing Cauchy numbers of both kinds, Bernoulli numbers of the second kind are derived.

1. Introduction

Following the usual notations, the falling factorial $\{x\}_n$ (for $x \in \mathbb{C}$) is defined by $\{x\}_0 = 1, \{x\}_n = x(x-1)\cdots(x-n+1)$ for $n > 0$ and, the rising factorial or Pochhammer symbol often denoted by $(x)_n$, is defined by $(x)_n = x(x+1)\cdots(x+n-1)$ with $(x)_0 = 1$. The Cauchy numbers of both kinds are introduced by Comtet [6] as integral of the falling and rising factorials

$$c_n^{(1)} := \int_0^1 \{x\}_n \, dx, \quad \text{and} \quad c_n^{(2)} := \int_0^1 (x)_n dx.$$

These numbers appears in diverse contexts, like number theory [1, 24], special functions and approximation theory [9]. Some analogues of the Cauchy numbers were recently studied by Liu *et al.* [15] in order to evaluate some integrals. In the sequel we follow the notation and terminology of [17]. The Cauchy numbers of the first kind $C_n := c_n^{(1)}$, are defined by the generating function

$$\sum_{n\geq 0} C_n \frac{z^n}{n!} = \frac{z}{\ln(1+z)},$$

and can be written explicitly as

$$C_n = \sum_{k=0}^{n} \frac{s(n,k)}{k+1}, \tag{1}$$

where $s(n,k)$ denotes the (signed) Stirling numbers of the first kind defined by means of the following generating function (see [22, p. 76])

$$\frac{1}{k!} (\ln(1+x))^k = \sum_{n\geq k} s(n,k) \frac{x^n}{n!}, \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 11B37; Secondary 11B83, 33C15
Keywords. Cauchy numbers, confluent hypergeometric function, Stirling numbers, three-term recurrence relation
Received: 08 July 2014; Accepted: 01 December 2014.
Communicated by Hari M. Srivastava
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and satisfy the recurrence relation (see [6])

$$s(n+1,k) = s(n,k-1) - ns(n,k).$$

The Bernoulli numbers of the second kind b_n are defined by

$$b_n := \frac{c_n^{(1)}}{n!}.$$

The generating function of b_n is given by

$$\sum_{n\geq 0} b_n z^n = \frac{z}{\ln(1+z)},$$

The Cauchy numbers of the second kind $\widehat{C}_n$ or Nörlund numbers $B_n^{(n)}$ are defined by

$$\widehat{C}_n = B_n^{(n)} := (-1)^n c_n^{(2)}$$

The generating function of $\widehat{C}_n$ is given by

$$\sum_{n\geq 0} \widehat{C}_n \frac{z^n}{n!} = \frac{z}{(1+z)\ln(1+z)}.$$

An explicit representation of $\widehat{C}_n$ is given by

$$\widehat{C}_n = \sum_{k=0}^{n} (-1)^k \frac{s(n,k)}{k+1}.$$

We recall that the Nörlund polynomials $B_n^{(x)}$ are defined by

$$\sum_{n\geq 0} B_n^{(x)} \frac{z^n}{n!} = \left(\frac{z}{e^z-1}\right)^x.$$

The numbers $B_n^{(n)}$ are called Nörlund numbers (see [5, 8, 14, 22, 23]).

The following relationship [17, p.1910, Eq. (2.2)]

$$C_n = \widehat{C}_n + n\widehat{C}_{n-1},$$

hold true between the Cauchy numbers of the first kind C_n and the Cauchy numbers of the second kind $\widehat{C}_n$.

Recently, many research articles have been devoted to Cauchy numbers of both kinds [2, 8, 13] and many generalization are introduced (poly-Cauchy numbers [10, 11], hypergeometric Cauchy numbers [12], generalized Cauchy numbers [25, 26]). As an example of a recent application of the Cauchy numbers of both kinds see [16], where Masjed-Jamei *et al.* have derived the explicit forms of the weighted Adams-Bashforth rules [16, proposition 4] and Adams-Moulton rules [16, proposition 5] in terms of the Cauchy numbers of the first and second kind. In [18, 19] Qi gave a new explicit formula for Bernoulli numbers of the second kind involving Stirling numbers of the first kind.

Theorem 1.1 (Qi [19]). *For $n \geq 2$, Bernoulli numbers b_n of the second kind can be computed by*

$$b_n = (-1)^n \frac{1}{n!}\left(\frac{1}{n+1} + \sum_{k=2}^{n} \frac{s(n,k-1) - ns(n-1,k-1)}{k}\right).$$

In this paper, a number of algorithms based on three-term recurrence relation for calculating Cauchy numbers of both kinds are derived. In particular, we prove the following recurrence relation for computing Bernoulli numbers of the second kind. First, we construct an infinite matrix $\left(b_{n,p}\right)_{n,p\geq 0}$ as follow: the first row of the matrix is $b_{0,p} := 1$ and each entry is given by

$$b_{n+1,p} = \frac{1-n}{1+n} b_{n,p} - \frac{p+1}{(p+2)(1+n)} b_{n,p+1}.$$

Finally, the first column of the matrix are Bernoulli numbers of the second kind $b_{n,0} := b_n$.

2. The p-Cauchy Numbers of the First Kind

For every integer $p \geq -1$, we define a sequence of rational numbers $C_{n,p}$ $(n \geq 0)$ by

$$f_p(z) := \sum_{n\geq 0} C_{n,p} \frac{z^n}{n!} = {}_1F_1\left(\begin{matrix} 1 \\ p+2 \end{matrix} ; z \right)\Bigg|_{z=\ln(1+z)}, \tag{3}$$

where ${}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; z \right)$ denotes the Kummer confluent hypergeometric function [3] defined by

$$\sum_{n\geq 0} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{4}$$

Remark 2.1. *For $p = -1$, we have*

$$\sum_{n\geq 0} C_{n,-1} \frac{z^n}{n!} = 1 + z,$$

and for $p = 0, C_{n,0} := C_n$ denotes the classical Cauchy numbers of the first kind.

The first few exponential generating function for $C_{n,p}$ $(p = 1, 2, 3)$ are

$$\sum_{n\geq 0} C_{n,1} \frac{z^n}{n!} = -\frac{2(\ln(1+z) - z)}{(\ln(1+z))^2},$$
$$\sum_{n\geq 0} C_{n,2} \frac{z^n}{n!} = -\frac{3((\ln(1+z))^2 + \ln(1+z) - 2z)}{(\ln(1+z))^3},$$
$$\sum_{n\geq 0} C_{n,3} \frac{z^n}{n!} = -\frac{4((\ln(1+z))^3 + 3(\ln(1+z))^2 + 6\ln(1+z) - 6z)}{(\ln(1+z))^4}.$$

We start with an explicit formula for p-Cauchy numbers of the first kind $C_{n,p}$.

Theorem 2.2. *For $p \geq -1$, we have*

$$C_{n,p} = \sum_{k=0}^{n} \binom{k+p+1}{k}^{-1} s(n,k). \tag{5}$$

Proof. By using (2), we obtain

$$
\begin{aligned}
\sum_{n\geq 0}\sum_{k=0}^{n}\binom{k+p+1}{k}^{-1}s(n,k)\frac{z^n}{n!} &= \sum_{k\geq 0}\binom{k+p+1}{k}^{-1}\sum_{n\geq k}s(n,k)\frac{z^n}{n!}.\\
&= \sum_{k\geq 0}\frac{(p+1)!k!}{(k+p+1)!}\frac{(\ln(1+z))^k}{k!}.\\
&= \sum_{k\geq 0}\frac{(1)_k}{(p+2)_k}\frac{(\ln(1+z))^k}{k!}\\
&= {}_1F_1\left(\begin{matrix}1\\ p+2\end{matrix};z\right)\Bigg|_{z=\ln(1+z)}\\
&= \sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{z^n}{n!}$, on both sides, we arrive at the result. □

In the special case of (5), when $p = 0$, we obtain the explicit formula of the classical Cauchy numbers of the first kind (1).

Recall that the r−Stirling numbers of the second kind $\left\{\begin{matrix}n\\ k\end{matrix}\right\}_r$ is the number of partitions of a set of cardinality n into exactly k nonempty, disjoint subsets, such that the first r elements are in distinct subsets. They may be defined recursively as follows [4]

$$
\begin{aligned}
&\left\{\begin{matrix}n\\ k\end{matrix}\right\}_r = 0, && n < r,\\
&\left\{\begin{matrix}n\\ k\end{matrix}\right\}_r = \delta_{k,r}, && n = r,\\
&\left\{\begin{matrix}n\\ k\end{matrix}\right\}_r = k\left\{\begin{matrix}n-1\\ k\end{matrix}\right\}_r + \left\{\begin{matrix}n-1\\ k-1\end{matrix}\right\}_r, && n > r,
\end{aligned}
\tag{6}
$$

where $\delta_{k,r}$ is the Kronecker symbol. The exponential generating function is given by

$$
\sum_{n\geq k}\left\{\begin{matrix}n+r\\ k+r\end{matrix}\right\}_r\frac{x^n}{n!} = \frac{1}{k!}e^{rx}\left(e^x-1\right)^k. \tag{7}
$$

From the result in [20] for the Stirling transform, we have

Corollary 2.3. *For $m \geq 0$, we have*

$$
\sum_{k=0}^{n}\left\{\begin{matrix}n+m\\ k+m\end{matrix}\right\}_m C_{m+k,p} = \sum_{k=0}^{m}s(m,k)\binom{n+k+p+1}{n+k}^{-1}. \tag{8}
$$

In particular for $m = 0$, we get

$$
\sum_{k=0}^{n}\left\{\begin{matrix}n\\ k\end{matrix}\right\}C_{k,p} = \frac{n!\,(p+1)!}{(n+p+1)!}.
$$

It follows from the general theory of hypergeometric functions [3] that the confluent hypergeometric function has an integral representation

$$ {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1}\, dt, \tag{9} $$

where Γ denotes the gamma function.

Theorem 2.4. *We have*

$$ f_p(z) = (p+1) \int_0^1 (1+z)^t (1-t)^p\, dt. \tag{10} $$

The next result gives the recurrence relation for the p-Cauchy numbers of the first kind $C_{n,p}$. The proof is based on (10).

Theorem 2.5. *The p-Cauchy numbers of the first kind satisfies the following three-term recurrence relation*

$$ C_{n+1,p} = (1-n) C_{n,p} - \frac{p+1}{p+2} C_{n,p+1}, \tag{11} $$

with the initial sequence $C_{0,p} = 1$.

Proof. By differentiation (10) with respect to z, we obtain

$$ \begin{aligned} (1+z)\frac{d}{dz} f_p(z) &= (p+1) \int_0^1 t(1+z)^t (1-t)^p\, dt \\ &= -(p+1) \int_0^1 (1-t-1)(1+z)^t (1-t)^p\, dt \\ &= -(p+1) \int_0^1 (1+z)^t (1-t)^{p+1}\, dt + (p+1) \int_0^1 (1+z)^t (1-t)^p\, dt \\ &= -\frac{p+1}{p+2} f_{p+1}(z) + f_p(z), \end{aligned} $$

or equivalently

$$ (1+z) \sum_{n\geq 0} C_{n+1,p} \frac{z^n}{n!} = -\frac{p+1}{p+2} \sum_{n\geq 0} C_{n,p+1} \frac{z^n}{n!} + \sum_{n\geq 0} C_{n,p} \frac{z^n}{n!}. $$

After some rearrangement, we get

$$ \sum_{n\geq 0} \left(C_{n+1,p} + (n-1) C_{n,p} \right) \frac{z^n}{n!} = -\frac{p+1}{p+2} \sum_{n\geq 0} C_{n,p+1} \frac{z^n}{n!}. $$

The conclusion follows by comparing coefficients of $\frac{z^n}{n!}$. □

By using (11), we list the p-Cauchy numbers of the first kind $C_{n,p}$ for $0 \leq n \leq 6$ and $0 \leq p \leq 3$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & C_{0,p} \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots & C_{1,p} \\ -1/6 & -1/6 & -3/20 & -2/15 & \cdots & C_{2,p} \\ 1/4 & 4/15 & 1/4 & 8/35 & \cdots & C_{3,p} \\ -19/30 & -7/10 & -47/70 & -131/210 & \cdots & C_{4,p} \\ 9/4 & 107/42 & 139/56 & 1496/630 & \cdots & C_{5,p} \\ -863/84 & -995/84 & -9809/840 & -2323/210 & \cdots & C_{6,p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ C_{n,0} & C_{n,1} & C_{n,2} & C_{n,3} & & C_{n,p} \end{pmatrix}$$

Corollary 2.6. *The p-Bernoulli numbers of the second kind satisfies the following three-term recurrence relation*

$$b_{n+1,p} = \frac{1-n}{1+n} b_{n,p} - \frac{p+1}{(p+2)(1+n)} b_{n,p+1}, \tag{12}$$

with the initial sequence $b_{0,p} = 1$.

The next result presents a different types of recurrence for p-Cauchy numbers of the first kind $C_{n,p}$. We need the following lemma in order to prove the Theorem 2.8.

Lemma 2.7. *We have*

$${}_1F_1\left(\begin{matrix} 1 \\ p+2 \end{matrix}; z\right) = \left(1 + \frac{z}{p+2}\right) {}_1F_1\left(\begin{matrix} 1 \\ p+3 \end{matrix}; z\right) - \frac{z}{p+3}\, {}_1F_1\left(\begin{matrix} 1 \\ p+4 \end{matrix}; z\right). \tag{13}$$

Proof. This comes directly from (4) . □

Theorem 2.8. *The p-Cauchy numbers of the first kind satisfies the following recurrence relation*

$$C_{n+1,p+1} = (1-n) C_{n,p} - \left(\frac{n}{p+2} + 1\right) C_{n,p+1} + \frac{n+1}{p+3} C_{n,p+2},$$

with the initial sequence $C_{0,p} = 1$ *and the final sequence* $C_{n,0} = C_n$.

Proof. By (13) with $z := \ln(1+z)$, we have

$$f_p(z) = \left(1 + \frac{z}{p+2}\right) f_{p+1}(z) - \frac{z}{p+3} f_{p+2}(z)$$
$$\sum_{n \geq 0} C_{n,p} \frac{z^n}{n!} = \sum_{n \geq 0} \left(C_{n,p+1} + \frac{1}{p+2} n C_{n-1,p+1} - \frac{1}{p+3} n C_{n-1,p+2}\right) \frac{z^n}{n!}.$$

Equating the coefficients of $\frac{z^n}{n!}$ and using (11), we get the result. □

3. The *p*-Cauchy Numbers of the Second Kind

Similarly to p-Cauchy numbers of the first kind $C_{n,p}$, we define p-Cauchy numbers of the second kind

$\widehat{C}_{n,p}$ by the exponential generating function

$$\widehat{f_p}(z) := \sum_{n\geq 0} \widehat{C}_{n,p} \frac{z^n}{n!} = {}_1F_1\left(\begin{matrix} 1 \\ p+2 \end{matrix} ; z \right)\Bigg|_{z=-\ln(1+z)}.$$

The first few exponential generating function for $\widehat{C}_{n,p}$ $(p = 1, 2)$ are

$$\sum_{n\geq 0} \widehat{C}_{n,1} \frac{z^n}{n!} = -\frac{2((1+z)\ln(1+z) - z)}{(1+z)(\ln(1+z))^2},$$
$$\sum_{n\geq 0} \widehat{C}_{n,2} \frac{z^n}{n!} = -\frac{3((1+z)(\ln(1+z))^2 - 2(1+z)\ln(1+z) + 2z)}{(1+z)(\ln(1+z))^3}.$$

Theorem 3.1. *The p-Cauchy numbers of the first kind satisfies the following*

1. *explicit formula, for $p \geq -1$*

$$\widehat{C}_{n,p} = \sum_{k=0}^{n} (-1)^k \binom{k+p+1}{k}^{-1} s(n,k).$$

2. *integral representation*

$$\widehat{f_p}(z) = (p+1) \int_0^1 (1+z)^{-t} (1-t)^p \, dt.$$

3. *three-term recurrence relation*

$$\widehat{C}_{n+1,p} = \frac{p+1}{p+2} \widehat{C}_{n,p+1} - (n+1) \widehat{C}_{n,p}, \tag{14}$$

with the initial sequence $\widehat{C}_{0,p} = 1$.

By using (14), we list the p-Cauchy numbers of the second kind $\widehat{C}_{n,p}$ for $0 \leq n \leq 6$ and $0 \leq p \leq 3$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & \widehat{C}_{0,p} \\ -1/2 & -1/3 & -1/4 & -1/5 & \cdots & \widehat{C}_{1,p} \\ 5/6 & 1/2 & 7/20 & 4/15 & \cdots & \widehat{C}_{2,p} \\ -9/4 & -19/15 & -17/20 & -32/35 & \cdots & \widehat{C}_{3,p} \\ 251/30 & 9/2 & 41/14 & 89/42 & \cdots & \widehat{C}_{4,p} \\ -475/12 & -863/42 & -731/56 & -5849/630 & \cdots & \widehat{C}_{5,p} \\ 19087/84 & 1375/12 & 8563/120 & 1501/30 & \cdots & \widehat{C}_{6,p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \widehat{C}_{n,0} & \widehat{C}_{n,1} & \widehat{C}_{n,2} & \widehat{C}_{n,3} & & \widehat{C}_{n,p} \end{pmatrix}$$

4. Some Properties of p-Cauchy Numbers

In the present section, we derive expressions for the generating function of the p-Cauchy numbers which allow to generalize the relationship between the Cauchy numbers of the first kind C_n, and the Cauchy numbers of the second kind $\widehat{C}_n$. In order to establish the Rodrigues-type formula for the generating function of p-Cauchy numbers of the first kind, we need the following lemma.

Lemma 4.1. *We have*

$$ {}_1F_1\left(\begin{matrix} p+1 \\ p+2 \end{matrix} ; \pm \ln(1+z) \right) = (\pm 1)^p (p+1) \left((1+z) \frac{d}{dz} \right)^p {}_1F_1\left(\begin{matrix} 1 \\ 2 \end{matrix} ; \pm \ln(1+z) \right). \tag{15} $$

Proof. The verification of (15) follows by induction on p . It follows from induction hypothesis that

$$ \pm \frac{p+1}{p+2} {}_1F_1\left(\begin{matrix} p+2 \\ p+3 \end{matrix} ; \pm \ln(1+z) \right) = (1+z) \frac{d}{dz} \left[{}_1F_1\left(\begin{matrix} p+1 \\ p+2 \end{matrix} ; \pm \ln(1+z) \right) \right] $$

$$ = (\pm 1)^p (p+1) \left((1+z) \frac{d}{dz} \right)^{p+1} {}_1F_1\left(\begin{matrix} 1 \\ 2 \end{matrix} ; \pm \ln(1+z) \right). $$

Lemma 4.1 is proved. □

The next result establishe the Rodrigues-type formula for the generating function of the p-Cauchy numbers of the first kind.

Theorem 4.2. *The exponential generating function for p-Cauchy numbers of the first kind is given by*

$$ \sum_{n\geq 0} C_{n,p} \frac{z^n}{n!} = (-1)^p (p+1) (1+z) \left((1+z) \frac{d}{dz} \right)^p \left(\frac{z}{(1+z) \ln(1+z)} \right) $$

Proof. According to the Kummer transformation [3], we can write

$$ {}_1F_1\left(\begin{matrix} 1 \\ p+2 \end{matrix} ; \ln(1+z) \right) = (1+z) \; {}_1F_1\left(\begin{matrix} p+1 \\ p+2 \end{matrix} ; -\ln(1+z) \right), $$

by (15) we arrive at the desired result. □

Theorem 4.3. *The exponential generating function for p-Cauchy numbers of the first kind is given by*

$$ {}_1F_1\left(\begin{matrix} 1 \\ p+2 \end{matrix} ; \ln(1+z) \right) = \frac{(p+1)!z}{(\ln(1+z))^{p+1}} - \sum_{k=1}^{p} \frac{\{p+1\}_k}{(\ln(1+z))^k}. $$

Proof. After an integration by parts, we can rewrite (10) as

$$ f_p(z) = \frac{p+1}{\ln(1+z)} f_{p-1}(z) - \frac{p+1}{\ln(1+z)}. \tag{16} $$

Now, applying (16) inductively we get the desired result. □

Similarly, we have

Theorem 4.4.

$$ \sum_{n\geq 0} \widehat{C}_{n,p} \frac{z^n}{n!} = \frac{p+1}{1+z} \left((1+z) \frac{d}{dz} \right)^p \left(\frac{z}{\ln(1+z)} \right) $$

$$ = \frac{(-1)^p (p+1)!z}{(1+z) (\ln(1+z))^{p+1}} - \sum_{k=1}^{p} \frac{(-1)^k \{p+1\}_k}{(\ln(1+z))^k}. $$

It is known that the Cauchy numbers of both kinds satisfies the following relation

$$ C_n = \widehat{C}_n + n\widehat{C}_{n-1}. $$

The next results generalize the above identity for p-Cauchy numbers.

Theorem 4.5. *For nonnegative integers n and p, we have*

$$C_{n,p} = (-1)^p (p+1) \sum_{k=0}^{p} \left\{ {p \atop k} \right\} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n}{j} j! \widehat{C}_{n-j+k}.$$

Proof. Since

$$\left((1+z) \frac{d}{dz} \right)^p = \sum_{k=0}^{p} \left\{ {p \atop k} \right\} (1+z)^k \frac{d^k}{dx^k},$$

we have

$$\begin{aligned} \sum_{n\geq 0} C_{n,p} \frac{z^n}{n!} &= (-1)^p (p+1) \sum_{k=0}^{p} \left\{ {p \atop k} \right\} (1+z)^{k+1} \frac{d^k}{dx^k} \left(\sum_{n\geq 0} \widehat{C}_n \frac{z^n}{n!} \right) \\ &= (-1)^p (p+1) \sum_{k=0}^{p} \left\{ {p \atop k} \right\} \sum_{j=o}^{k+1} \binom{k+1}{j} z^j \left(\sum_{n\geq 0} \widehat{C}_{n+k} \frac{z^n}{n!} \right). \end{aligned}$$

After some rearrangement, we arrive at the result by comparing the coefficients of $\frac{z^n}{n!}$, on both sides. □

The next identity is proven similarly and omitted.

Theorem 4.6. *For nonnegative integers n and p, we have*

$$\widehat{C}_{n,p} + n\widehat{C}_{n-1,p} = (p+1) \sum_{k=0}^{p} \left\{ {p \atop k} \right\} \sum_{j=0}^{k} \binom{k}{j} \binom{n}{j} j! C_{n-j+k}.$$

5. The *p*-Cauchy Polynomials

In this section, we define the p-Cauchy polynomials of both kinds as Appell polynomials which have been studied from multiple points of view [7, 21]. Recall that a polynomial set $(A_n(x))_{n\geq 0}$ is said to be of Appell type if $(A_n(x))_{n\geq 0}$ satisfying the identity

$$\frac{d}{dx} A_n(x) = n A_{n-1}(x),\ n \geq 1.$$

It is worth to note that some interesting works dealing with various families of numbers and associated polynomials have been developed (see, for example [22, 23]).

For $p \geq 0$, let us consider the p-Cauchy polynomials of the first kind $C_{n,p}(x)$ defined by means of the following generating function

$$\sum_{n\geq 0} C_{n,p}(x) \frac{z^n}{n!} = {}_1F_1 \left({1 \atop p+2} ; \ln(1+z) \right) e^{xz}. \tag{17}$$

In the same manner we can define the p-Cauchy number of the second kind. The following properties of the p-Cauchy polynomials of the first kind $C_{n,p}(x)$ are readily derived from (17).

Theorem 5.1. *For $n \geq 0$ and $p \geq 0$; we have*

$$\begin{aligned}
&C_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} C_{k,p} x^{n-k}, \\
&\frac{d}{dx} C_{n,p}(x) = n C_{n-1,p}(x), \\
&\int_0^x C_{n,p}(t)\, dt = \frac{C_{n+1,p}(x) - C_{n+1,p}}{n+1}, \\
&\int_0^1 C_{n,p}(t)\, dt = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} C_{k,p}, \\
&C_{n,p}(x+y) = \sum_{k=0}^{n} \binom{n}{k} C_{k,p}(x)\, y^{n-k}, \\
&C_{n,p}(x+1) - C_{n,p}(x) = \sum_{k=0}^{n-1} \binom{n}{k} C_{k,p}(x), \\
&C_{k,p}(mx) = \sum_{k=0}^{n} \binom{n}{k} C_{k,p}(x) (m-1)^{n-k} x^{n-k}.
\end{aligned}$$

The first few p-Cauchy polynomials of the first kind are

$$\begin{aligned}
&C_{0,p}(x) = 1, \\
&C_{1,p}(x) = x + \frac{1}{p+2}, \\
&C_{2,p}(x) = x^2 + \frac{2x}{p+2} - \frac{p+1}{(p+2)(p+3)}, \\
&C_{3,p}(x) = x^3 + \frac{3x^2}{p+2} - \frac{3(p+1)x}{(p+2)(p+3)} + \frac{2(p+1)}{(p+2)(p+4)}, \\
&C_{4,p}(x) = x^4 + \frac{4x^3}{p+2} - \frac{6(p+1)x^2}{(p+2)(p+3)} + \frac{8(p+1)x}{(p+2)(p+4)} - \frac{2(p+1)\left(3p^2 + 22p + 38\right)}{(p+2)(p+3)(p+4)(p+5)}.
\end{aligned}$$

The next result establishes the recurrence formula for computing p-Cauchy polynomials of the first kind. We shall need the following Lemma.

Lemma 5.2.

$$\sum_{k=0}^{n} k \binom{n}{k} C_{k,p} x^{n-k} = \sum_{k=0}^{n-1} \frac{n!}{k!} (-1)^{n-k} \left(\frac{p+1}{p+2} C_{k,p+1}(x) - C_{k,p}(x) \right).$$

Proof. Let us denote $\sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k}$ by $p_n(x)$, since $k\binom{n}{k} = n\binom{n-1}{k-1}$, and using (14), we have

$$\begin{aligned}
p_n(x) &= \sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k} \\
&= n\sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1,p} x^{n-k-1} \\
&= n\sum_{t=0}^{n-1} \binom{n-1}{k} \left((1-k) C_{k,p} - \frac{p+1}{p+2} C_{k,p+1}\right) x^{n-k-1} \\
&= nC_{n-1,p}(x) - \frac{p+1}{p+2} nC_{n-1,p+1}(x) - np_{n-1}(x).
\end{aligned}$$

Applying this recursively we get the desired result. □

Theorem 5.3. *The p-Cauchy polynomials of the first kind $C_{n,p}(x)$ satisfies the following recurrence relation*

$$C_{n+1,p}(x) = xC_{n,p}(x) - n!\sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!} \left(\frac{(p+1)}{(p+2)} C_{j,p+1}(x) - C_{j,p}(x)\right), \tag{18}$$

with the initial sequence $C_{0,p}(x) = 1$.

Proof. We have

$$x\frac{d}{dx} C_{n,p}(x) = nC_{n,p}(x) - \sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k}.$$

By applying Lemma 5.2 and after some rearrangement, we get (18). □

Acknowledgements

The author would like to thank an anonymous reviewer for their careful reading of this manuscript and their constructive remarks.

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