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Some Properties of Isotone and Jointive Multiderivations on Lattices

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Abstract. In this paper, we introduce the notion of multiderivations on lattices and their fixed sets. By using the superjoinitivity and supermeetability properties of the multifunctions, we give some new results on properties of isotone and joinitive multiderivations on lattices. In this way, we show that under some conditions the fixed set of a multiderivation is an ideal.

1. Introduction

The Lattice algebra has a significant role in some branches, for example information theory ([2]), information retrieval ([7]), information access controls ([16]) and cryptanalysis ([10]). Formal models of secure computer systems use the algebraic concept of a lattice to describe certain components of the system (see [3], [15] and [9]). In 2008, the notion of derivation introduced in [18]. Analytic and algebraic properties of lattices have been studied by some researchers (see for example, [8], [11] and [12]). Also, derivations on rings, near-rings, *BCI*-algebras and lattices have been reviewed (see for example, [4], [5], [13] and [14]). There are some equivalent conditions under which a derivation is isotone on a lattices with a greatest element, modular lattices and distributive lattices ([18]). In fact, it has been characterized modular lattices and distributive lattices by isotone derivations and proved that the set of all fixed points of a derivation on a lattice is an ideal of the lattice ([18]). Also, it has been investigated some relations among derivations, ideals and fixed sets (see for example, [17]).

In this paper, we define superjoinitive and supermeetable correspondence from a lattice L to the power set $P(L)$. Then we introduce the concept of multiderivation on lattices and investigate some properties of the notion. In this way, we give some relations about multiderivations, ideals and fixed set of a multiderivation.

A lattice is a non-empty set L endowed with binary operations $\wedge$ and $\vee$ such that $x \wedge x = x$, $x \vee x = x$, $x \wedge y = y \wedge x$, $x \vee y = y \vee x$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$, $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ for all $x, y, z \in L$ ([6]). A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ or $x \vee y = y$ ([6]). A lattice L is called modular whenever $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$ with $x \leq z$ ([1]). A lattice L is said to be distributive whenever $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$ ([6]). A Boolean algebra is an algebra $(B, \wedge, \vee, ', 0, 1)$ with two binary operations $\wedge$ and $\vee$, one unary operation $'$

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and two nullary operations 0 and 1 such that $(B, \wedge, \vee)$ is a distributive lattice, $a \wedge 1 = a$ and $0 \vee a = a$ for all $a \in B$ and there is $a^{'} \in B$ such that $a \wedge a^{'} = 0$ and $a \vee a^{'} = 1$ for all $a \in B$ ([6]). Let L be a lattice. Finally, an ideal is a non-void subset I of a lattice L such that $x \leq y$ and $y \in I$ implies $x \in I$ and $x, y \in I$ implies $x \vee y \in I$ ([6]). We say that I is down-closed whenever satisfies the first property of the ideal.

Lemma 1.1. *[6] Let $(L, \wedge, \vee)$ be a lattice and $\leq$ the binary relation. Then $(L, \leq)$ is a poset, $x \wedge y$ is the greater lower bound (g.l.b) of $\{x, y\}$ and $x \vee y$ is the lower upper bound (l.u.b) of $\{x, y\}$ for all $x, y \in L$.*

It has been proved that $x \wedge y \leq x$, $x \wedge y \leq y$, $x \leq x \vee y$, $y \leq x \vee y$, $y \leq z$ implies $x \wedge y \leq x \wedge z$, $x \vee y \leq x \vee z$ for any $x, y, z \in L$ and $x \vee a \leq y \vee b$ and $x \wedge a \leq y \wedge b$ for all $x, y, a, b \in L$ with $x \leq y$ and $a \leq b$ ([6]).

2. The Preliminary

Let L be a lattice, 2^L the set of nonempty subsets of L and $M, N \in 2^L$. We define the operation $\wedge$ and $\vee$ on 2^L by $M \wedge N = \{x \in L :$ there exist $m \in M$ and $n \in N$ such that $x = m \wedge n\}$ and $M \vee N = \{x \in L :$ there exist $m \in M$ and $n \in N$ such that $x = m \vee n\}$. We abbreviate $M \wedge \{x\}$ and $M \vee \{x\}$ by $M \wedge x$ or $M \vee x$, respectively. We say $M \preceq N$ whenever for each $m \in M$ there exists $n \in N$ such that $m \leq n$. It is easy to see that $\preceq$ does not satisfy antisymmetry property, that is, the condition $M \preceq N$ and $N \preceq M$ does not imply $M = N$. Thus, $\preceq$ is not a partial order on 2^L. We say that the order $\preceq$ is a quasi-partial order on 2^L and $(2^L, \preceq)$ is a quasi-partial ordered set. One can see that $M \wedge N \preceq M$, $M \wedge N \preceq N$, $M \preceq M \vee N$, $N \preceq M \vee N$, $M \preceq N$ implies $P \wedge M \preceq P \wedge N$ and $P \vee M \preceq P \vee N$ for all $P, M, N \in 2^L$. If $M \preceq N$ and $P \preceq Q$ ($P, M, N \in 2^L$), then it is easy to check that $M \vee P \preceq N \vee Q$ and $M \wedge P \preceq N \wedge Q$. Now for record, we give next result which one can prove it easily.

Lemma 2.1. *Let $(2^L, \wedge, \vee, \preceq)$ be a quasi-partial ordered set and $M, N, C \in 2^L$. Then, we have $M \subseteq M \wedge M$ and $M \subseteq M \vee M$. If M be a sublattice of L, then $M = M \wedge M$ and $M = M \vee M$. Also, $M \wedge N = N \wedge M$, $M \vee N = N \vee M$, $(M \wedge N) \wedge C = M \wedge (N \wedge C)$, $(M \vee N) \vee C = M \vee (N \vee C)$, $M \subseteq (M \wedge N) \vee M$ and $M \subseteq (M \vee N) \wedge M$. If L is a distributive lattice, then $M \wedge (N \vee C) = (M \wedge N) \vee (M \wedge C)$ and $M \vee (N \wedge C) = (M \vee N) \wedge (M \vee C)$. Finally, $N \preceq M$ whenever $M \vee N \subseteq M$.*

Let L be a lattice and $\varphi : L \rightrightarrows 2^L$ a multifunction. We say that φ is supermeetable (and superjoinitive) whenever $\varphi(x \wedge y) \supseteq \varphi(x) \wedge \varphi(y)$ (and $\varphi(x \vee y) \supseteq \varphi(x) \vee \varphi(y)$) for all $x, y \in L$. We say that φ is super-homomorphism whenever φ is supermeetable and superjoinitive. Also, φ is called a meet-homomorphism (simply meetable) whenever $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and is called a join-homomorphism (simply joinitive) whenever $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \in L$. Finally, we say that φ is multi-homomorphism whenever it is both meet-homomorphism and join-homomorphism.

Theorem 2.2. *Let L be a lattice and $\varphi : L \rightrightarrows 2^L$ a super-homomorphism. Then $\varphi(L)$ is a lattice with binary operation $\wedge$ and $\vee$. If L is a distributive lattice, then $\varphi(L)$ so is.*

Proof. Since φ is supermeetable, $\varphi(x) = \varphi(x \wedge x) \supseteq \varphi(x) \wedge \varphi(x)$ for all $x \in L$ and so $\varphi(x) = \varphi(x) \wedge \varphi(x)$ for all $x \in L$. Similarly, we get $\varphi(x) = \varphi(x) \vee \varphi(x)$ for all $x \in L$. Hence, $\wedge$ and $\vee$ satisfy the reflexivity property on 2^L. Since φ is supermeetable and superjoinitie, $\varphi(x) = \varphi((x \wedge y) \vee x) \supseteq \varphi(x \wedge y) \vee \varphi(x) \supseteq (\varphi(x) \wedge \varphi(y)) \vee \varphi(x)$ and $(\varphi(x) \vee \varphi(y)) \wedge \varphi(x) = \varphi(x)$ for all $x, y \in L$. On the other hand, $\varphi(x) \subseteq (\varphi(x) \wedge \varphi(y)) \vee \varphi(x)$. Thus, $(\varphi(x) \wedge \varphi(y)) \vee \varphi(x) = \varphi(x)$ for all $x, y \in L$. Thus, $(\varphi(L), \wedge, \vee)$ is a lattice. $\square$

3. Multiderivations on Lattices

Let L be a lattice and $\mathfrak{d} : L \rightrightarrows 2^L$ a multifunction. We say that $\mathfrak{d}$ is a multiderivation whenever $\mathfrak{d}(x \wedge y) = (\mathfrak{d}(x) \wedge y) \vee (x \wedge \mathfrak{d}(y))$ for all $x, y \in L$. We abbreviate $\mathfrak{d}(x)$ by $\mathfrak{d}x$.

Example 3.1. *Consider the lattice $L = \{0, 1, a, b\}$ via the operations $0 \vee a = a$, $0 \vee b = b$, $0 \vee 1 = 1$, $a \vee b = a \vee 1 = 1$, $b \vee 1 = 1$, $0 \wedge a = 0 \wedge b = 0 \wedge 1 = 0$, $a \wedge b = 0$, $a \wedge 1 = a$ and $b \wedge 1 = b$. Define the multifunction $\mathfrak{d}$ on L by $\mathfrak{d}(0) = \{0\}$, $\mathfrak{d}(1) = \{0, 1, a, b\}$, $\mathfrak{d}(a) = \{0, a\}$ and $\mathfrak{d}(b) = \{0, b\}$. Then it is easy to check that $\mathfrak{d}$ is a multiderivation on L.*

Example 3.2. *Consider the lattice $L = \{0, 1, a, b\}$ via the operations $0 \vee a = a$, $0 \vee b = b$, $0 \vee 1 = 1$, $a \vee b = b$, $a \vee 1 = 1$, $b \vee 1 = 1$, $0 \wedge a = 0 \wedge b = 0 \wedge 1 = 0$, $a \wedge b = a$, $a \wedge 1 = a$ and $b \wedge 1 = b$. Define the multifunction $\mathfrak{d}$ on L by $\mathfrak{d}(0) = \{0\}$, $\mathfrak{d}(1) = \{0, 1\}$, $\mathfrak{d}(a) = \{0, b\}$ and $\mathfrak{d}(b) = \{0, a\}$. One can check that $\mathfrak{d}$ is not a multiderivation on L.*

The following example shows that we can consider infinite lattices.

Example 3.3. *Consider the lattice $L = [0, \infty)$ via binary operation infimum as $\wedge$ and supremum as $\vee$. Define multifunction $\mathfrak{d} : L \rightrightarrows 2^L$ by $\mathfrak{d}x = [0, x]$ for all $x \in L$. It is easy to check that $\mathfrak{d}$ is a joinitive multiderivation on L.*

Hereafter, we review some properties of the multiderivations on lattices.

Proposition 3.1. *Let L be a lattice and $\mathfrak{d} : L \rightrightarrows 2^L$ a multiderivation. Then, $\mathfrak{d}x \leq x$ and $\mathfrak{d}x \wedge \mathfrak{d}y \leq \mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \vee \mathfrak{d}y$ for all $x, y \in L$. If I is an ideal of L, then $\mathfrak{d}(I) \in 2^I$, where $\mathfrak{d}(I) = \bigcup_{x \in L} \mathfrak{d}x$. If L has a least element 0 and a greatest element 1, then $\mathfrak{d}0 = \{0\}$ and $\mathfrak{d}1 \leq 1$.*

Proof. Note that, $\mathfrak{d}x = \mathfrak{d}(x \wedge x) = (\mathfrak{d}x \wedge x) \vee (x \wedge \mathfrak{d}x) \leq x$ for all $x \in L$. Thus, $\mathfrak{d}x \wedge \mathfrak{d}y \leq \mathfrak{d}x \wedge y$ and $\mathfrak{d}x \wedge \mathfrak{d}y \leq x \wedge \mathfrak{d}y$ for all $x, y \in L$. Hence,

$$\mathfrak{d}x \wedge \mathfrak{d}y \subseteq (\mathfrak{d}x \wedge \mathfrak{d}y) \vee (\mathfrak{d}x \wedge \mathfrak{d}y) \leq (\mathfrak{d}x \wedge y) \vee (x \wedge \mathfrak{d}y) = \mathfrak{d}(x \wedge y)$$

for all $x, y \in L$. On the other hand, $\mathfrak{d}(x \wedge y) = (\mathfrak{d}x \wedge y) \vee (x \wedge \mathfrak{d}y) \leq \mathfrak{d}x \vee \mathfrak{d}y$ for all $x, y \in L$. Thus, $\mathfrak{d}x \wedge \mathfrak{d}y \leq \mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \vee \mathfrak{d}y$ for all $x, y \in L$. Now, let $y \in \mathfrak{d}(I)$. It follows that $y \in \mathfrak{d}(x)$ for some $x \in I$. But, $y \in \mathfrak{d}(x) \leq x$ and so $y \leq x$. Since I is an ideal of L, $y \in I$. Hence, $\mathfrak{d}(I) \subseteq I$. Finally, it is obvious that $\mathfrak{d}1 \leq 1$. Let x be an arbitrary element of L. Then, we have $\mathfrak{d}0 = \mathfrak{d}(x \wedge 0) = (\mathfrak{d}x \wedge 0) \vee (x \wedge \mathfrak{d}0) = 0 \vee (x \wedge \mathfrak{d}0) = x \wedge \mathfrak{d}0$. Hence, $\mathfrak{d}0 \leq x$ for all $x \in L$. If $x = 0$, then we get $\mathfrak{d}0 = \{0\}$. □

Proposition 3.2. *Let L be a lattice with a greatest element 1 and $\mathfrak{d}$ a multiderivation on L. If $\mathfrak{d}1 \leq x$, then $\mathfrak{d}1 \leq \mathfrak{d}x$. If $x \leq \mathfrak{d}1$, then $x \in \mathfrak{d}x$.*

Proof. Since 1 is the greatest element of L, $\mathfrak{d}x = \mathfrak{d}(1 \wedge x) = (\mathfrak{d}1 \wedge x) \vee (1 \wedge \mathfrak{d}x) = (\mathfrak{d}1 \wedge x) \vee \mathfrak{d}x$. Hence, $\mathfrak{d}x = \mathfrak{d}x \vee (x \wedge \mathfrak{d}1) = \mathfrak{d}x \vee \mathfrak{d}1$ and so $\mathfrak{d}1 \leq \mathfrak{d}x$. Now, let $x \leq \mathfrak{d}1$. Then, there exists $a \in \mathfrak{d}1$ such that $x \wedge a = x$. By using Proposition 3.1, we get

$$x = \mathfrak{d}x \vee x = \mathfrak{d}x \vee (x \wedge a) \in \mathfrak{d}x \vee (x \wedge \mathfrak{d}1) = \mathfrak{d}x.$$

□

Definition 3.4. *Let L be a lattice and $\mathfrak{d}$ a multiderivation on L. We say $\mathfrak{d}$ is an isotone multiderivation on L whenever $x \leq y$ implies $\mathfrak{d}x \leq \mathfrak{d}y$.*

Note that every superjoinitive multiderivation $\mathfrak{d}$ on a lattice L is isotone. Let $x, y \in L$ with $x \leq y$. Then, we have $\mathfrak{d}y = \mathfrak{d}(x \vee y) \supseteq \mathfrak{d}x \vee \mathfrak{d}y$. Thus, $\mathfrak{d}x \leq \mathfrak{d}y$.

Now, let L be a lattice and $A \subseteq L$ a sublattice of L. Define the multifunction $\mathfrak{d}_A$ on L by $\mathfrak{d}_A(x) = x \wedge A$ for all $x \in L$. Then, $\mathfrak{d}_A$ is a multiderivation on L. Note that,

$$\mathfrak{d}_A(x \wedge y) = x \wedge y \wedge A = (x \wedge y \wedge A) \vee (x \wedge y \wedge A) = (\mathfrak{d}_A(x) \wedge y) \vee (x \wedge \mathfrak{d}_A(y))$$

for all $x, y \in L$. We say that a multiderivation φ on L is principle whenever there exists a sublattice A of L such that $\varphi(x) = \mathfrak{d}_A(x)$ for all $x \in L$.

Lemma 3.3. *Every principle multiderivation on a lattice L is isotone.*

Proof. Let $\mathfrak{d}_A$ be a principle multiderivation on a lattice L and $x, y \in L$ with $x \leq y$. Then, we have $\mathfrak{d}_A(x) = x \wedge A \leq y \wedge A = \mathfrak{d}_A(y)$. Hence, $\mathfrak{d}_A$ is isotone. □

Definition 3.5. *Let L be a lattice and φ a multifunction on L. The set of all fixed points of the multifunction is defined by $Fix_\varphi(L) = \{x \in L : x \in \varphi x\}$.*

Let $\mathfrak{d}$ be a multiderivation on a lattice L. By using Proposition 3.1, we get $Fix_\mathfrak{d}(L) \neq \emptyset$ whenever L has a least element zero. By using Proposition 3.2, $\{x \in L : x \leq \mathfrak{d}1\} \subseteq Fix_\mathfrak{d}(L)$ whenever L has a greatest element 1.

Lemma 3.4. *Let L be a lattice and $\mathfrak{d}$ a multiderivation on L. If $x, y \in L$, $y \leq x$ and $x \in \mathfrak{d}x$, then $y \in \mathfrak{d}y$.*

Proof. Since $x \in \mathfrak{d}x$, by using Proposition 3.1 we get $\mathfrak{d}y \leq y$ which implies that

$$y = y \vee \mathfrak{d}y = (x \wedge y) \vee \mathfrak{d}y \in (\mathfrak{d}x \wedge y) \vee (x \wedge \mathfrak{d}y) = \mathfrak{d}(x \wedge y) = \mathfrak{d}y.$$

□

Theorem 3.5. *Let L be a lattice and $\mathfrak{d}$ a superjoinitive multiderivation on L. Then $Fix_\mathfrak{d}(L)$ is an ideal of L.*

Proof. By using Lemma 3.4, $Fix_\mathfrak{d}(L)$ is a down-closed set. Thus, it sufficient to show that $Fix_\mathfrak{d}(L)$ is closed under the operation $\vee$. Let $x, y \in Fix_\mathfrak{d}(L)$. Since $\mathfrak{d}$ is superjoinitive, $x \vee y \in \mathfrak{d}x \vee \mathfrak{d}y \subseteq \mathfrak{d}(x \vee y)$. This completes the proof. □

Let L be a lattice, $\mathfrak{d}$ a multiderivation on L and $x, y \in L$. Then, $\mathfrak{d}x = \mathfrak{d}x \vee (x \wedge \mathfrak{d}(x \vee y))$. In fact, $\mathfrak{d}x = \mathfrak{d}(x \wedge (x \vee y)) = (\mathfrak{d}x \wedge (x \vee y)) \vee (x \wedge \mathfrak{d}(x \vee y)) = \mathfrak{d}x \vee (x \wedge \mathfrak{d}(x \vee y))$. If L has a greatest element 1, then $1 \in \mathfrak{d}1$ if and only if $x \in \mathfrak{d}x$ for all $x \in L$. Finally, define the multifunction $\mathfrak{d}^2$ on L by $\mathfrak{d}^2 x = \mathfrak{d}(\mathfrak{d}(x)) = \bigcup_{t \in \mathfrak{d}x} \mathfrak{d}t$ for all $x \in L$. It is clear that $Fix_\mathfrak{d}(L) \subseteq Fix_\mathfrak{d}(L)$.

Proposition 3.6. *Let L be a lattice and $\mathfrak{d}$ a joinitive multiderivation on L. Then, $\mathfrak{d}x$ is a subset of $Fix_\mathfrak{d}(L)$ for all $x \in L$.*

Proof. Let $x \in L$ be given. Then, we have

$$\mathfrak{d}x = \mathfrak{d}(x \vee \mathfrak{d}x) = \bigcup_{t \in \mathfrak{d}x} \mathfrak{d}(x \vee t) = \bigcup_{t \in \mathfrak{d}x} \mathfrak{d}x \vee \mathfrak{d}t = \mathfrak{d}x \vee \bigcup_{t \in \mathfrak{d}x} \mathfrak{d}t = \mathfrak{d}x \vee \mathfrak{d}^2 x \subseteq$$

$$(\mathfrak{d}x \wedge \mathfrak{d}x) \vee (x \wedge \mathfrak{d}^2 x) = \bigcup_{t \in \mathfrak{d}x} (\mathfrak{d}x \wedge t) \vee (x \wedge \mathfrak{d}t) = \bigcup_{t \in \mathfrak{d}x} \mathfrak{d}(x \wedge t) = \mathfrak{d}(x \wedge \mathfrak{d}x) = \mathfrak{d}^2 x.$$

□

Theorem 3.7. *Let L be a lattice and $\mathfrak{d}$ a multiderivation on L. Then, $\mathfrak{d}$ is isotone if and only if $\mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge y$ and $\mathfrak{d}x \wedge y \leq \mathfrak{d}(x \wedge y)$ for all $x, y \in L$.*

Proof. If $\mathfrak{d}$ is isotone, then $\mathfrak{d}(x \wedge y) \subseteq \mathfrak{d}(x \wedge y) \wedge \mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge \mathfrak{d}y \leq \mathfrak{d}x \wedge y$. On the other hand, we have $\mathfrak{d}x \wedge y \leq (\mathfrak{d}x \wedge y) \vee (x \wedge \mathfrak{d}y) = \mathfrak{d}(x \wedge y)$. Now, assume that $\mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge y$ and $\mathfrak{d}x \wedge y \leq \mathfrak{d}(x \wedge y)$ for all $x, y \in L$. Let $x, y \in L$ with $x \leq y$. But, $\mathfrak{d}x = \mathfrak{d}(y \wedge x) \leq \mathfrak{d}y \wedge x$. Thus, for each $a \in \mathfrak{d}x$ there exists $b \in \mathfrak{d}y$ such that $a \leq b \wedge x$. Hence, $a \leq b$ and so $\mathfrak{d}x \leq \mathfrak{d}y$. □

Lemma 3.8. *Let L be a modular lattice and $\mathfrak{d}$ be a multiderivation on L. Then, $\mathfrak{d}$ is isotone if and only if $\mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge \mathfrak{d}y$ and $\mathfrak{d}x \wedge \mathfrak{d}y \leq \mathfrak{d}(x \wedge y)$ for all $x, y \in L$.*

Proof. First suppose that $\mathfrak{d}$ is isotone. Since L is modular, we have

$$\mathfrak{d}x \wedge \mathfrak{d}y = (\mathfrak{d}x \wedge \mathfrak{d}y) \wedge y \wedge x \leq (\mathfrak{d}x \vee \mathfrak{d}y) \wedge y \wedge x$$

$$= [(\mathfrak{d}x \wedge y) \vee \mathfrak{d}y] \wedge x = [(\mathfrak{d}x \wedge y) \vee \mathfrak{d}y] \wedge x = (\mathfrak{d}x \wedge y) \vee (x \wedge \mathfrak{d}y) = \mathfrak{d}(x \wedge y).$$

On the other hand, we have $\mathfrak{d}(x \wedge y) \subseteq \mathfrak{d}(x \wedge y) \wedge \mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge \mathfrak{d}y$. Now, suppose that $\mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge \mathfrak{d}y$ and $\mathfrak{d}x \wedge \mathfrak{d}y \leq \mathfrak{d}(x \wedge y)$ for all $x, y \in L$. Let $x, y \in L$ with $x \leq y$. Then, we get $\mathfrak{d}x = \mathfrak{d}(x \wedge y) \leq \mathfrak{d}x \wedge \mathfrak{d}y$ which implies that $\mathfrak{d}x \leq \mathfrak{d}y$. □

Example 3.6. *Consider the lattice $L = \{0, 1, a, b, c\}$ via the operations $0 \vee a = a$, $0 \vee b = b$, $0 \vee c = c$, $0 \vee 1 = 1$, $a \vee 1 = b \vee 1 = c \vee 1 = 1$, $a \vee b = a \vee c = b \vee c = 1$ and $a \wedge b = a \wedge c = b \wedge c = 0$. Now, consider the sublattice $A = \{0, 1, b\}$ of L and define the multiderivation $\mathfrak{d}_A$ on L by $\mathfrak{d}_A x = x \wedge A$. Then it is easy to see that $\mathfrak{d}_A$ is a principle multiderivation on L. Hence by Using Lemma 3.3, $\mathfrak{d}_A$ is isotone. Moreover by using Lemma 3.8, we have $\mathfrak{d}_A(x \wedge y) \preceq \mathfrak{d}_A x \wedge \mathfrak{d}_A y$ and $\mathfrak{d}_A x \wedge \mathfrak{d}_A y \preceq \mathfrak{d}_A(x \wedge y)$.*

Theorem 3.9. *Let L be a distributive lattice and $\mathfrak{d}$ a superjoinitive multiderivation on L. Then the followings are equivalent.*
a) $\mathfrak{d}$ is isotone,
b) $\mathfrak{d}(x \wedge y) \preceq \mathfrak{d}x \wedge \mathfrak{d}y$ and $\mathfrak{d}x \wedge \mathfrak{d}y \preceq \mathfrak{d}(x \wedge y)$ for all $x, y \in L$,
c) $\mathfrak{d}(x \vee y) \preceq \mathfrak{d}x \vee \mathfrak{d}y$ and $\mathfrak{d}x \vee \mathfrak{d}y \preceq \mathfrak{d}(x \vee y)$ for all $x, y \in L$.

Proof. Since every distributive lattice is modular, it is sufficient we prove that $\mathfrak{d}$ is isotone if and only if the condition (c) holds. First suppose that $\mathfrak{d}$ is isotone. We show that the condition (c) holds. Since $\mathfrak{d}$ is isotone and L is distributive, we have $\mathfrak{d}x \preceq \mathfrak{d}(x \vee y)$, $\mathfrak{d}y \preceq \mathfrak{d}(x \vee y)$ and

$$\mathfrak{d}x = \mathfrak{d}x \vee (x \wedge \mathfrak{d}(x \vee y)) = x \wedge (\mathfrak{d}x \vee \mathfrak{d}(x \vee y)).$$

Similarly, $\mathfrak{d}y = y \wedge (\mathfrak{d}y \vee \mathfrak{d}(x \vee y))$. Thus, we get

$$\begin{aligned}
\mathfrak{d}x \vee \mathfrak{d}y &= [x \wedge (\mathfrak{d}x \vee \mathfrak{d}(x \vee y))] \vee [y \wedge (\mathfrak{d}y \vee \mathfrak{d}(x \vee y))] \\
&= [(x \wedge (\mathfrak{d}x \vee \mathfrak{d}(x \vee y))) \vee y] \wedge [(x \wedge (\mathfrak{d}x \vee \mathfrak{d}(x \vee y))) \vee (\mathfrak{d}y \vee \mathfrak{d}(x \vee y))] \\
&= (x \vee y) \wedge (\mathfrak{d}x \vee \mathfrak{d}(x \vee y) \vee y) \wedge (x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y)) \wedge (\mathfrak{d}x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y) \vee \mathfrak{d}(x \vee y)) \\
&\supseteq (x \vee y) \wedge (y \vee \mathfrak{d}x \vee \mathfrak{d}(x \vee y)) \wedge (x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y)) \wedge (\mathfrak{d}x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y)) \\
&= (y \vee \mathfrak{d}x \vee \mathfrak{d}(x \vee y)) \wedge (x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y)) \wedge (\mathfrak{d}x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y)) \\
&\supseteq (y \vee \mathfrak{d}x \vee \mathfrak{d}(x \vee y)) \wedge (\mathfrak{d}x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y)) \supseteq \mathfrak{d}x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y).
\end{aligned}$$

Hence, $\mathfrak{d}x \vee \mathfrak{d}y \vee \mathfrak{d}(x \vee y) \subseteq \mathfrak{d}x \vee \mathfrak{d}y$ and so $\mathfrak{d}(x \vee y) \preceq \mathfrak{d}x \vee \mathfrak{d}y$. Since $\mathfrak{d}$ is superjoinitive, by using the relations $\mathfrak{d}x \preceq \mathfrak{d}(x \vee y)$ and $\mathfrak{d}y \preceq \mathfrak{d}(x \vee y)$, we get $\mathfrak{d}x \vee \mathfrak{d}y \preceq \mathfrak{d}(x \vee y) \vee \mathfrak{d}(x \vee y) \subseteq \mathfrak{d}(x \vee y)$ and so $\mathfrak{d}x \vee \mathfrak{d}y \preceq \mathfrak{d}(x \vee y)$ for all $x, y \in L$. Now, suppose that the condition (c) holds. Let $x, y \in L$ with $x \leq y$. Then, $\mathfrak{d}x \vee \mathfrak{d}y \preceq \mathfrak{d}(x \vee y) = \mathfrak{d}y$ and so $\mathfrak{d}x \preceq \mathfrak{d}y$. This completes the proof. □

Here, we provide an example to show that there are some infinite distributive lattices and isotone multiderivations which satisfy assumptions of our results.

Example 3.7. *Consider the distributive lattice $L = \mathbb{N} \cup \{0\}$ via binary operation infimum as $\wedge$ and supremum as $\vee$. Define multiderivation $\mathfrak{d} : L \to 2^L$ by $\mathfrak{d}x = \{0, 1, ..., x\}$ for all $x \in L$. One can check that $\mathfrak{d}$ is isotone and satisfies the conditions Theorems 3.7 and 3.9.*

Proposition 3.10. *Let L be a modular lattice and $\mathfrak{d}$ a multiderivation on L. Then, $\mathfrak{d}$ is isotone if and only if $x \in \mathfrak{d}x$ implies $\mathfrak{d}(x \vee y) \subseteq \mathfrak{d}x \vee \mathfrak{d}y$ for all $x, y \in L$.*

Proof. First suppose that $\mathfrak{d}$ is isotone and $x \in \mathfrak{d}x$. Then, we have $\mathfrak{d}y = \mathfrak{d}y \vee (y \wedge \mathfrak{d}(x \vee y))$. Since L is modular, $\mathfrak{d}y = (\mathfrak{d}y \vee y) \wedge \mathfrak{d}(x \vee y) = y \wedge \mathfrak{d}(x \vee y)$. Thus,

$$\mathfrak{d}x \vee \mathfrak{d}y = \mathfrak{d}x \vee (y \wedge \mathfrak{d}(x \vee y)) = (\mathfrak{d}x \vee y) \wedge \mathfrak{d}(x \vee y).$$

Since $x \in \mathfrak{d}x$, we get $\mathfrak{d}(x \vee y) = (x \vee y) \wedge \mathfrak{d}(x \vee y) \subseteq \mathfrak{d}x \vee \mathfrak{d}y$. Since $\mathfrak{d}$ is superjoinitive, we obtain $\mathfrak{d}(x \vee y) \subseteq \mathfrak{d}x \vee \mathfrak{d}y$. If $x \in \mathfrak{d}x$ implies $\mathfrak{d}(x \vee y) \subseteq \mathfrak{d}x \vee \mathfrak{d}y$ for all $x, y \in L$, then one can easily get that $\mathfrak{d}$ is isotone. □

Let L be a modular lattice and $\mathfrak{d}$ a superjoinitive multiderivation on L. Then, it is easy to conclude that $\mathfrak{d}$ is isotone if and only if $x \in \mathfrak{d}x$ implies $\mathfrak{d}(x \vee y) = \mathfrak{d}x \vee \mathfrak{d}y$ for all $x, y \in L$.

Proposition 3.11. *Let L be a lattice and $\mathfrak{d}_g$ and $\mathfrak{d}$ two joinitive multiderivations on L. If $Fix_{\mathfrak{d}_g}(L) = Fix_{\mathfrak{d}}(L)$, then $\mathfrak{d}_g \preceq \mathfrak{d}$ and $\mathfrak{d} \preceq \mathfrak{d}_g$.*

Proof. Let $Fix_{\mathfrak{d}_g}(L) = Fix_{\mathfrak{d}}(L)$ and $x \in L$. Then, $\mathfrak{d}_g x \subseteq \mathfrak{d}(\mathfrak{d}_g x)$ and $\mathfrak{d}x \subseteq \mathfrak{d}_g(\mathfrak{d}x)$. Hence, $\mathfrak{d}_g x \preceq \mathfrak{d}_g x \vee \mathfrak{d}x \subseteq \mathfrak{d}(\mathfrak{d}_g x) \vee \mathfrak{d}x = \mathfrak{d}(\mathfrak{d}_g x \vee x) = \mathfrak{d}x$ and so $\mathfrak{d}_g x \preceq \mathfrak{d}x$. Thus, $\mathfrak{d}_g \preceq \mathfrak{d}$. Similarly, one can get that $\mathfrak{d} \preceq \mathfrak{d}_g$. □

Theorem 3.12. *Let L be a distributive lattice, $A, B \in 2^L$ and*

$$\mathfrak{D}(L) = \{\mathfrak{d}_A : L \rightrightarrows 2^L \mid A \text{ be a sublattice of } L\},$$

where $\mathfrak{d}_A x = x \wedge A$ for all $x \in L$. Define the multifunctions $\mathfrak{d}_A \cdot \mathfrak{d}_B$ and $\mathfrak{d}_A + \mathfrak{d}_B$ by

$$(\mathfrak{d}_A \cdot \mathfrak{d}_B)x := (\mathfrak{d}_A x) \wedge (\mathfrak{d}_B x) \text{ and } (\mathfrak{d}_A + \mathfrak{d}_B)x := (\mathfrak{d}_A x) \vee (\mathfrak{d}_B x)$$

for all $A, B \in 2^L$ and $x \in L$ Then 2^L is isomorphic to $\mathfrak{D}(L)$.

Proof. Note that,

$$(\mathfrak{d}_A \cdot \mathfrak{d}_B)x = (\mathfrak{d}_A x) \wedge (\mathfrak{d}_B x) = (x \wedge A) \wedge (x \wedge B) = x \wedge (A \wedge B) = \mathfrak{d}_{A \wedge B} x$$

and

$$\begin{aligned}(\mathfrak{d}_A + \mathfrak{d}_B)x &= (\mathfrak{d}_A x) \vee (\mathfrak{d}_B x) = (x \wedge A) \vee (x \wedge B) = [(x \wedge A) \vee x] \wedge [(x \wedge A) \vee B] \\ &= x \wedge (x \vee A) \wedge (x \vee B) \wedge (A \vee B) = x \wedge (A \vee B) = \mathfrak{d}_{A \vee B} x.\end{aligned}$$

Thus, $\mathfrak{d}_A \cdot \mathfrak{d}_B = \mathfrak{d}_{A \wedge B}$ and $\mathfrak{d}_A + \mathfrak{d}_B = \mathfrak{d}_{A \vee B}$. Define the set function $\phi : 2^L \to \mathfrak{D}(L)$ by $\phi(A) = \mathfrak{d}_A$. One can easily check that ϕ is one-one, onto, $\phi(A \wedge B) = \mathfrak{d}_{A \wedge B} = \mathfrak{d}_A \cdot \mathfrak{d}_B$ and $\phi(A \vee B) = \mathfrak{d}_{A \vee B} = \mathfrak{d}_A + \mathfrak{d}_B$ for all $A, B \in 2^L$. □

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