Filomat 30:10 (2016), 2803–2807
DOI 10.2298/FIL1610803B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Absolute Weighted Mean Summability of Infinite Series and Fourier Series

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Abstract. In this paper, firstly we proved a known theorem dealing with absolute weighted mean summability of infinite series under weaker conditions and then we obtained an application of it to the Fourier series. Some new results are also deduced.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [3])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [5], [7])

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k = \sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^\alpha \right|^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

2010 *Mathematics Subject Classification*. Primary 26D15; Secondary 40D15, 40F05, 40G99, 42A24, 46A45
Keywords. Weighted mean; summability factors; infinite series; Fourier series; Hölder inequality; Minkowski inequality; sequence space
Received: 04 August 2014; Accepted: 27 December 2014
Communicated by Dragan S. Djordjević
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The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (w_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$, (resp. $|\bar{N}, p_n|$) summability. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} | \lambda_n - \lambda_{n+1} | < \infty$.

2. Known Result

The following theorem is known dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 2.1 ([2]) Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{6}$$

Let (X_n) be a positive monotonic nondecreasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{7}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1), \tag{8}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. The Main Result

The aim of this paper is to prove Theorem 2.1 under weaker conditions. Now we shall prove the following theorem.

Theorem 3.1 Let (X_n) be a positive monotonic nondecreasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions (6)-(8) and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n{}^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Remark 3. 2 It should be noted that condition (10) is reduced to the condition (9), when k=1. When $k > 1$, condition (10) is weaker than condition (9) but the converse is not true. As in [9] we can show that if (9) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n{}^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m).$$

If (10) is satisfied, then for $k>1$ we obtain that

$$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k=\sum_{n=1}^{m}X_n^{k-1}\frac{p_n}{P_n}\frac{|t_n|^k}{X_n{}^{k-1}}=O(X_m^{k-1})\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{|t_n|^k}{X_n{}^{k-1}}=O(X_m^k)\neq O(X_m).$$

We need the following lemma for the proof of our theorem.
Lemma 3. 3 ([2]) Under the conditions of Theorem 3. 1, we have that

$$\sum_{n=1}^{\infty}X_n|\Delta\lambda_n|<\infty, \tag{11}$$

$$nX_n|\Delta\lambda_n|=O(1)\quad as\quad n\to\infty. \tag{12}$$

4. Proof of Theorem 3.1 Let (T_n) be the sequence of $(\bar{N},p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{r=0}^{v}a_r\lambda_r=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v. \tag{13}$$

Then, for $n\geq1$, we get

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v. \tag{14}$$

Applying Abel's transformation to the right-hand side of (14), we have

$$\begin{aligned}
T_n-T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta\left(\frac{P_{v-1}\lambda_v}{v}\right)\sum_{r=1}^{v}ra_r+\frac{p_n\lambda_n}{nP_n}\sum_{r=1}^{n}va_v\\
&= \frac{(n+1)p_nt_n\lambda_n}{nP_n}-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v}\\
&\quad+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\Delta\lambda_vt_v\frac{v+1}{v}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_{v+1}t_v\frac{1}{v}\\
&= T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,r}\right|^k<\infty,\quad for\quad r=1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|\frac{p_n}{P_n}|t_n|^k=O(1)\sum_{n=1}^{m}|\lambda_n|\frac{p_n}{P_n}\frac{|t_n|^k}{X_n{}^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}}+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{|t_n|^k}{X_n{}^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m=O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3. 1 and Lemma 3. 3. Also, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}} = O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Again, by using (6), we get that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,3}\right|^k &= \sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left(\sum_{v=1}^{n-1}vp_v|\Delta\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^kp_v|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}v|\Delta\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v}\frac{p_r}{P_r}\frac{|t_r|^k}{X_r{}^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta\left(v|\Delta\lambda_v|\right)\right|X_v + O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v| + O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3. 1 and and Lemma 3. 3. Finally, by using (6), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,4}\right|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left(\sum_{v=1}^{n-1}|\lambda_{v+1}|p_v|t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}|\lambda_{v+1}|^kp_v|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}} = O(1) \quad as \quad m\to\infty.
\end{aligned}$$

This completes the proof of Theorem 3. 1. It should be noted that if we take $p_n = 1$ for all values of n, then we get the known result of Mazhar dealing with $|C, 1|_k$ summability factors of infinite series under weaker conditions (see [8]).

5. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} C_n(x)$$

$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$, and $\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du$, $(\alpha > 0)$.

It is well know that if $\phi_1(t) \in \mathcal{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [4]). Using this fact, we get the following main result dealing with Fourier series.

Theorem 5. 1 If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3. 1, then the series series $\sum C_n(x)\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

If we take $p_n = 1$ for all values of n, then we obtain a new result dealing with $|C, 1|_k$ summability factors of Fourier series.

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