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Generalized Riesz Potential Spaces and their Characterization via Wavelet-Type Transform

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Abstract. We introduce a wavelet-type transform generated by the so-called beta-semigroup, which is a natural generalization of the Gauss-Weierstrass and Poisson semigroups associated to the Laplace-Bessel convolution. By making use of this wavelet-type transform we obtain new explicit inversion formulas for the generalized Riesz potentials and a new characterization of the generalized Riesz potential spaces. We show that the usage of the concept beta-semigroup gives rise to minimize the number of conditions on wavelet measure, no matter how big the order of the generalized Riesz potentials is.

1. Introduction

Let $\mathbb{R}^n_+ = \{x = (x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n : x_n > 0\}$ and $S(\mathbb{R}^n_+)$ be the space of functions, which are restrictions to $\mathbb{R}^n_+$ of the Schwartz test functions on $\mathbb{R}^n$ that are even in the last variable x_n. The closure of the space $S(\mathbb{R}^n_+)$ in the norm

$$\left\|f\right\|_{p,\nu} = \left(\int\limits_{\mathbb{R}^n_+} \left|f(x)\right|^p x_n^{2\nu} dx\right)^{\frac{1}{p}} \tag{1}$$

is denoted by $L_{p,\nu} \equiv L_{p,\nu}(\mathbb{R}^n_+)$. Here $\nu > 0$ is a fixed parameter, $1 \leq p < \infty$ and $dx = dx_1...dx_{n-1}dx_n$. The notation $C_0 \equiv C_0(\mathbb{R}^n_+)$ stands for the closure of the spaces $S(\mathbb{R}^n_+)$ in the sup-norm.

The Fourier-Bessel transform and its inverse are defined as

$$(F_\nu\varphi)(x) = \int\limits_{\mathbb{R}^n_+} \varphi(y)e^{-ix'\cdot y'} j_{\nu-\frac{1}{2}}(x_n y_n) y_n^{2\nu} dy, \qquad \left(F_\nu^{-1}\varphi\right)(x) = c_\nu(n)\left(F_\nu\varphi\right)(-x', x_n) \tag{2}$$

where $x' \cdot y' = x_1y_1 + ... + x_{n-1}y_{n-1}$, $\varphi \in L_{1,\nu}(\mathbb{R}^n_+)$,

$$c_\nu(n) = \left[(2\pi)^{n-1}2^{2\nu-1}\Gamma^2(\nu + \frac{1}{2})\right]^{-1} \tag{3}$$

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and $j_s(t)$ $(t > 0,\ s > -\frac{1}{2})$ is the normalized Bessel function: $j_s(t) = \frac{2^s\Gamma(p+1)J_s(t)}{t^s}$ ($J_s(t)$ is the first kind Bessel function).

The Fourier-Bessel transform is an automorphism of the space $S(\mathbb{R}^n_+)$ and if the function $\varphi \in L_{1,\nu}(\mathbb{R}^n_+)$ is radial, then $F_\nu\varphi$ is also radial (see for details, [16],[30]).

Denote by T^y the generalized translation (shift) operator, acting as

$$(T^y\varphi)(x) = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma(\frac{1}{2})}\int_0^\pi \varphi\left(x'-y';\ \sqrt{x_n^2 - 2x_ny_n\cos\theta + y_n^2}\right)\sin^{2\nu-1}\theta d\theta. \tag{4}$$

The convolution (Bessel convolution) generated by the translation T^y is defined as

$$(\varphi \circledast \psi)(x) = \int_{\mathbb{R}^n_+} \varphi(\xi)T^\xi\psi(x)\xi_n^{2\nu}d\xi,\ (d\xi = d\xi_1...d\xi_n), \tag{5}$$

for which $\varphi \circledast \psi = \psi \circledast \varphi$. The following Young inequality for convolution (5) is well known:

$$\left\|\varphi \circledast \psi\right\|_{r,\nu} \le \left\|\varphi\right\|_{p,\nu}\left\|\psi\right\|_{q,\nu},\quad 1 \le p, q, r \le \infty \text{ and } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} - 1. \tag{6}$$

The action of the Fourier-Bessel transform to Bessel convolution is as follows:

$$F_\nu(\varphi \circledast \psi) = F_\nu\varphi.F_\nu\psi\ . \tag{7}$$

The generalized Riesz potentials generated by the generalized translation (4) are defined in terms of Fourier-Bessel transforms as follows

$$I^\alpha_\nu f = F_\nu^{-1}\left(|\xi|^{-\alpha}F_\nu f\right); f \in S(\mathbb{R}^n_+), 0 < \alpha < n+2\nu\ . \tag{8}$$

These potentials admit the following integral representation as the Bessel convolution (see [9],[1],[2]):

$$(I^\alpha_\nu f)(x) = \frac{1}{\gamma_{n,\nu}(\alpha)}\int_{\mathbb{R}^n_+}\left|y\right|^{\alpha-n-2\nu}T^y f(x)y_n^{2\nu}dy\ , \tag{9}$$

where

$$\gamma_{n,\nu}(\alpha) = \frac{2^{\alpha-1}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{n+2\nu-\alpha}{2}\right)}, 0 < \alpha < n+2\nu. \tag{10}$$

Many known results for the classical Riesz potentials are also valid for the potentials $I^\alpha_\nu f$. For instance, the analog of Hardy-Littlewood-Sobolev theorem in this case is formulated as (see [9]):

$$\left\|I^\alpha_\nu f\right\|_{q,\nu} \le c.\left\|f\right\|_{p,\nu},\ (1 < p < \frac{n+2\nu}{\alpha} \text{ and } \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2\nu});$$

If $p = 1$ then

$$meas\left\{x \in \mathbb{R}^n_+ : \left|(I^\alpha_\nu f)(x)\right| > \lambda\right\} \le \left(\frac{c_q\left\|f\right\|_{1,\nu}}{\lambda}\right)^q,$$

where $q = \frac{n+2\nu}{n+2\nu-\alpha}$ and for measurable $E \subset \mathbb{R}^n_+$, $measE = \int_E x_n^{2\nu}dx$.

The potentials $I_\nu^\alpha f$ have remarkable one-dimensional integral representations in terms of the Poisson and Gauss-Weierstrass semigroups, generated by the generalized translation T^y. Namely,

$$(I_\nu^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \left(P_t^{(\nu)} f\right)(x) dt; \tag{11}$$

$$(I_\nu^\alpha f)(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} \left(G_t^{(\nu)} f\right)(x) dt. \tag{12}$$

Here the Poisson semigroup $P_t^{(\nu)} f$ and the Gauss-Weierstrass semigroup $G_t^{(\nu)} f$ generated by the generalized translation are defined as follows (see [9], [10], [1]):

$$\left(P_t^{(\nu)} f\right)(x) = \int_{\mathbb{R}_+^n} p_\nu(y;t) T^y f(x) y_n^{2\nu} dy \,, \,(t>0), \tag{13}$$

$$p_\nu(y;t) \equiv F_\nu^{-1}(e^{-t|x|})(y) = \frac{2}{\pi^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n+2\nu+1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} \frac{t}{\left(\left|y\right|^2+t^2\right)^{\frac{n+2\nu+1}{2}}}; \tag{14}$$

$$\left(G_t^{(\nu)} f\right)(x) = \int_{\mathbb{R}_+^n} g_\nu(y;t) T^y f(x) y_n^{2\nu} dy \,, \,(t>0), \tag{15}$$

$$g_\nu(y;t) \equiv F_\nu^{-1}(e^{-t|x|^2})(y) = \frac{2\pi^{\nu+\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} (4\pi t)^{-\frac{n+2\nu}{2}} e^{-\frac{|y|^2}{4t}} \,. \tag{16}$$

The one-dimensional integral representations (11), (12) of the generalized Riesz potentials $I_\nu^\alpha f$ have proved to be extremely useful for explicit inversion of these potentials (see for details [9], [1], [3], [4]).

In [4] and [27], it has been introduced the so-called beta-semigroup

$$\left(B_t^{(\beta)} f\right)(x) = \int_{\mathbb{R}^n} \omega^{(\beta)}\left(\left|y\right|, t\right) f(x-y) dy, \ (t>0), \tag{17}$$

generated by the radial kernel

$$\omega^{(\beta)}\left(\left|y\right|, t\right) = F^{-1}(e^{-t|x|^\beta})(y) \equiv (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-t|x|^\beta} e^{ix\cdot y} dx,$$

and using this beta-semigroup it has been obtained integral representation of the classical Riesz and Bessel potentials and a new characterization for the Riesz potential spaces. Here F^{-1} is the inverse Fourier transform, $x \cdot y = x_1 y_1 + ... + x_n y_n$, $|x| = \sqrt{x \cdot x}$ and $\beta \in (0, \infty)$. The another application of the beta-semigroup (17) to Bessel potentials spaces and Radon transform is given in [4] and [5].

In this work we define a semigroup, generated by the radial kernel

$$\omega^{(\beta)}\left(\left|y\right|, t\right) = F_\nu^{-1}(e^{-t|x|^\beta})(y) \equiv c_\nu(n) \int_{\mathbb{R}_+^n} e^{-t|x|^\beta} e^{ix'\cdot y'} j_{\nu-\frac{1}{2}}(x_n y_n) x_n^{2\nu} dx$$

and by making use of this semigroup, we obtain one-dimensional integral representation for the generalized Riesz potentials $I_\nu^\alpha f$. Further, we define a wavelet-type transform generated by this semigroup and by some "wavelet-measure", then using this wavelet-type transform we obtain new explicit inversion formulas for the generalized Riesz potentials (9). Finally, we give a new characterization of generalized Riesz potential spaces. We show that the usage of the concept beta-semigroup gives rise to minimize the number of conditions on wavelet measure μ, no matter how big the order α of the generalized Riesz potentials is.

2. Beta-Semigroup Generated by the $F_\nu^{-1}\left(\exp(-t\,|x|^\beta)\right)$ and Application to Generalized Riesz Potentials

Given $\beta > 0$, consider $F_\nu^{-1}\left(\exp(-t\,|x|^\beta)\right)(y)$, $(t > 0;\, x, y \in \mathbb{R}^n_+)$. It is known that, if $\varphi \in L_{1,\nu}$ is radial, then $F_\nu\varphi$ also is radial ([16], [30]). Therefore, $F_\nu^{-1}\left(\exp(-t\,|x|^\beta)\right)(y)$ is radial. Denote

$$\omega_\nu^{(\beta)}\left(\left|y\right|, t\right) = F_\nu^{-1}\left(\exp(-t\,|x|^\beta)\right)(y) = c_\nu(n)\int_{\mathbb{R}^n_+} e^{-t|x|^\beta} e^{ix'\cdot y'} j_{\nu-\frac{1}{2}}(x_n y_n) x_n^{2\nu} dx \tag{18}$$

The Beta-semigroup, generated by the kernel (18) is defined (formally now) as convolution-type operator:

$$\left(W_t^{(\beta)} f\right)(x) = \left(\omega_\nu^{(\beta)}(|.|, t) \circledast f\right)(x) \equiv \int_{\mathbb{R}^n_+} \omega_\nu^{(\beta)}\left(\left|y\right|, t\right) T^y f(x) y_n^{2\nu} dy. \tag{19}$$

In case of $\beta = 1$ and $\beta = 2$, (19) coincides with the generalized Poisson semigroup (13) and generalized Gauss-Weierstrass semigroup (15), respectively. Unlike (14) and (16), the kernel function $\omega_\nu^{(\beta)}\left(\left|y\right|, t\right)$ cannot be computed explicitly, however, some important properties of $\omega_\nu^{(\beta)}\left(\left|y\right|, t\right)$ are well determined by the following lemma.

Lemma 2.1. *(cf. [4], [27]) Let $x, y \in \mathbb{R}^n_+$, $0 < t < \infty$ and $0 < \beta < \infty$. Then,*

(a) $\omega_\nu^{(\beta)}\left(\lambda^{\frac{1}{\beta}}\left|y\right|, \lambda t\right) = \lambda^{-\frac{n+2\nu}{\beta}}\omega_\nu^{(\beta)}\left(\left|y\right|, t\right)$, $(\lambda > 0)$.
In particular, for $\lambda = \frac{1}{t}$ we have

$$\omega_\nu^{(\beta)}\left(\left|y\right|, t\right) = t^{-\frac{n+2\nu}{\beta}}\omega_\nu^{(\beta)}\left(t^{-\frac{1}{\beta}}\left|y\right|, 1\right); \tag{20}$$

(b) *If $0 < \beta \le 2$, then $\omega_\nu^{(\beta)}\left(\left|y\right|, t\right) > 0$ for all $y \in \mathbb{R}^n_+$ and $t > 0$;*

(c) *If $\beta = 2k$, $(k \in \mathbb{N})$, then $\omega_\nu^{(\beta)}\left(\left|y\right|, t\right) \in S(\mathbb{R}^n_+)$, $\forall t > 0$;*

(d) $\int_{\mathbb{R}^n_+} \omega_\nu^{(\beta)}\left(\left|y\right|, t\right) y_n^{2\nu} dy = 1$, $\forall t > 0$; *provided that $0 < \beta \le 2$ or $\beta = 2k$, $(k \in \mathbb{N})$;*

(e) *Let $f \in L_{p,\nu}$, $1 \le p \le \infty$. If $0 < \beta \le 2$ or $\beta = 2k$, $(k \in \mathbb{N})$, then*

$$\left\|W_t^{(\beta)} f\right\|_{p,\nu} \le c(\beta)\left\|f\right\|_{p,\nu}.$$

Here, $c(\beta) = \int_{\mathbb{R}^n_+}\left|\omega_\nu^{(\beta)}\left(\left|y\right|, 1\right)\right| y_n^{2\nu} dy < \infty$ *and $c(\beta) = 1$ provided $0 < \beta \le 2$;*

(f) *Let $f \in L_{p,\nu}$, $1 \le p \le \infty$. If $0 < \beta \le 2$ or $\beta = 2k$, $(k \in \mathbb{N})$, then*

$$\sup_{t>0}\left|\left(W_t^{(\beta)} f\right)(x)\right| \le c\left(M_\nu f\right)(x),$$

where $M_\nu f$ is the generalized Hardy-Littlewood maximal function ([1],[13],[14]).

$$\left(M_\nu f\right)(x) = \sup_{r>0}\frac{1}{r^{n+2\nu}\omega(n;\nu)}\int_{B_r^+}\left|T^x f(y)\right| y_n^{2\nu} dy, \tag{21}$$

$B_r^+ = \{x : x \in \mathbb{R}^n_+, |x| \le r\}$ *and* $\omega(n;\nu) = \int_{B_1^+} x_n^{2\nu} dx$;

(g) $\sup_{x\in\mathbb{R}^n_+}\left|\left(W_t^{(\beta)} f\right)(x)\right| \le ct^{-\frac{n+2\nu}{p\beta}}\left\|f\right\|_{p,\nu}$, $1 \le p < \infty$, *where $0 < \beta \le 2$ or $\beta = 2k$, $(k \in \mathbb{N})$;*

(h) *Let* $0 < \beta \le 2$ *or* $\beta = 2k$, $(k \in \mathbb{N})$. *Then for any* $f \in L_{p,\nu}$ *and any* $t, \tau \in (0, \infty)$

$$W_t^{(\beta)}\left(W_\tau^{(\beta)} f\right) = W_{t+\tau}^{(\beta)} f, \text{(the semigroup property)};$$

(i) *Let* $f \in L_{p,\nu}$, $1 \le p \le \infty$ $(L_{\infty,\nu} \equiv C_0$, *the closure of the space of* $S(\mathbb{R}_+^n)$ *in the sup-norm). Then for* $0 < \beta \le 2$ *or* $\beta = 2k$, $(k \in \mathbb{N})$, *we have*

$$\lim_{t\to 0^+}\left(W_t^{(\beta)} f\right)(x) = f(x),$$

where the limit is understood in the $L_{p,\nu}$*-norm as well as pointwise for almost all* $x \in \mathbb{R}_+^n$. *In case of* $f \in L_{\infty,\nu} \equiv C_0$, *the convergence is uniform.*

Remark 2.2. *In accordance with (i) it will be assumed that* $W_0^{(\beta)} f = f$.

Remark 2.3. *In our opinion, the statements of this Lemma except of (b) and (c), are valid also for any* $\beta > 2$. *In order to proof this, it is sufficient to show the following asymptotic formula for any positive* $\beta \ne 2k$, $(k \in \mathbb{N})$.

$$\omega_\nu^{(\beta)}\left(\left|y\right|, 1\right) = c_\beta \left|y\right|^{-n-2\nu-\beta} \;(1 + o(1)) \text{ as } \left|y\right| \to \infty. \tag{22}$$

We believe that, the formula (22) is valid true but we don't know its proof and we suggest it, as an open problem.

Proof. **(a)** We have

$$\begin{aligned}\omega_\nu^{(\beta)}\left(\left|y\right|, t\right) &= c_\nu(n) \int_{\mathbb{R}_+^n} e^{-t|x|^\beta} e^{ix'\cdot y'} j_{\nu-\frac{1}{2}}(x_n y_n) x_n^{2\nu} dx \quad \left(\text{set } x = \lambda^{\frac{1}{\beta}} z,\ dx = \lambda^{\frac{n}{\beta}} dz\right)\\ &= c_\nu(n) \lambda^{\frac{2\nu}{\beta}} \lambda^{\frac{n}{\beta}} \int_{\mathbb{R}_+^n} e^{-\lambda t|z|^\beta} e^{iz'\cdot\lambda^{\frac{1}{\beta}} y'} j_{\nu-\frac{1}{2}}(z_n \lambda^{\frac{1}{\beta}} y_n) z_n^{2\nu} dz\\ &= \lambda^{\frac{n+2\nu}{\beta}} \omega_\nu^{(\beta)}\left(\lambda^{\frac{1}{\beta}}\left|y\right|, \lambda t\right).\end{aligned}$$

(b) For the classical Fourier transform F, the positivity of $F^{-1}(e^{-t|x|^\beta})$, $(0 < \beta \le 2)$ can be found in [17], p.44-45 (the case of $n = 1$) and in [19] (the case of $n > 1$); see also, [4], p.11-13. For the cases $\beta = 1$ and $\beta = 2$, the positivity of $\omega_\nu^{(\beta)}\left(\left|y\right|, t\right) \equiv F_\nu^{-1}(e^{-t|x|^\beta})(y)$ follows immediately from (14) and (16). Let now $0 < \beta < 2$. By Bernstein's theorem ([8], chapter 18, sec.4; see also [11], p.223) there is a non-negative finite measure μ_β on $[0, \infty)$, so that, $\mu_\beta([0, \infty)) = 1$ and $e^{-z^{\beta/2}} = \int_0^\infty e^{-\tau z} d\mu_\beta(\tau)$, $z \in [0, \infty)$. Replace z by $|x|^2$ to get

$$e^{-|x|^\beta} = \int_0^\infty e^{-\tau|x|^2} d\mu_\beta(\tau). \tag{23}$$

From (23) we have

$$\omega_\nu^{(\beta)}\left(\left|y\right|, 1\right) \equiv F_\nu^{-1}(e^{-|x|^\beta})(y) = \int_0^\infty F_\nu^{-1}(e^{-\tau|x|^2})(y) d\mu_\beta(\tau)$$

$$\overset{(16)}{=} \frac{2\pi^{\nu+\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty (4\pi\tau)^{-\frac{n+2\nu}{2}} e^{-\frac{|y|^2}{4\tau}} d\mu_\beta(\tau) > 0. \tag{24}$$

(c) Since the transform F_ν is an automorphizm of the space $S(\mathbb{R}^n_+)$ and $e^{-|x|^{2k}} \in S(\mathbb{R}^n_+)$, it follows that $\omega_\nu^{(2k)}\left(\left|y\right|,t\right) \in S(\mathbb{R}^n_+)$ and therefore, it is infinitely smooth and rapidly decreasing on $\mathbb{R}^n_+$.

(d) For $k \in \mathbb{N}$, $\omega_\nu^{(2k)}\left(\left|y\right|,t\right) \in S(\mathbb{R}^n_+)$, $(\forall t > 0)$ and therefore, $\omega_\nu^{(2k)}\left(\left|y\right|,t\right) \in L_{1,\nu}$, $(\forall t > 0)$. Then

$$F_\nu\left(\omega_\nu^{(2k)}\left(\left|y\right|,t\right)\right) = e^{-t|x|^2}.$$

Setting $x = (0,...,0)$, we have

$$\int\limits_{\mathbb{R}^n_+} \omega_\nu^{(2k)}\left(\left|y\right|,t\right) y_n^{2\nu} dy = 1.$$

Let now $0 < \beta < 2$. By making use of (24) and the formula

$$\int\limits_{\mathbb{R}^n_+} e^{-\frac{|y|^2}{4\tau}} y_n^{2\nu} dy = \frac{1}{2}\pi^{\frac{n-1}{2}}\Gamma(\nu + \frac{1}{2})\,(4\tau)^{\frac{n+2\nu}{2}} \quad \text{(see [6])},$$

we have

$$\begin{aligned}\int\limits_{\mathbb{R}^n_+} \omega_\nu^{(2k)}\left(\left|y\right|,1\right) y_n^{2\nu} dy &= \frac{2\pi^{\nu+\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\int\limits_0^\infty (4\pi\tau)^{-\frac{n+2\nu}{2}}\left(\int\limits_{\mathbb{R}^n_+} e^{-\frac{|y|^2}{4\tau}} y_n^{2\nu} dy\right) d\mu_\beta(\tau) \\ &= \pi^{\frac{n+2\nu}{2}}\pi^{-\frac{n+2\nu}{2}}\int\limits_0^\infty d\mu_\beta(\tau) = 1.\end{aligned}$$

Now, from homogeneity property (20), it follows immediately that

$$\int\limits_{\mathbb{R}^n_+} \omega_\nu^{(\beta)}\left(\left|y\right|,t\right) y_n^{2\nu} dy = \int\limits_{\mathbb{R}^n_+} \omega_\nu^{(\beta)}\left(\left|y\right|,1\right) y_n^{2\nu} dy = 1.$$

(e) follows by the Minkowski inequality.

(f) Theorem 2.1 from [1] states that if the function $\varphi \in L_{1,\nu}$ has a decreasing and positive radial majorant $\psi(|x|)$ with $\int\limits_{\mathbb{R}^n_+} \psi(|x|)x_n^{2\nu}dx < \infty$, then for any $f \in L_{p,\nu}$ $(1 \le p \le \infty)$

$$\sup_{\varepsilon>0}\left|\left(\varphi_\varepsilon \circledast f\right)(x)\right| \le \left\|\psi\right\|_{1,\nu}\left(M_\nu f\right)(x);\ (\varphi_\varepsilon(x) = \varepsilon^{-n-2\nu}\varphi(\frac{1}{\varepsilon}x)). \tag{25}$$

By setting $\psi(|x|) = \omega_\nu^{(\beta)}(|x|,1)$, $\varepsilon = t^{\frac{1}{\beta}}$ and taking into account (20) and (25) we have for $0 < \beta \le 2$ and $\beta = 2k$

$$\sup_{t>0}\left|\left(W_t^{(\beta)}f\right)(x)\right| \le c\left(M_\nu f\right)(x);\ c = \int\limits_{\mathbb{R}^n_+} |\omega_\nu^{(\beta)}\left(\left|y\right|,1\right)| y_n^{2\nu} dy < \infty.$$

It is clear from (24) that, the function $\omega_\nu^{(\beta)}\left(\left|y\right|,1\right)$ decreases monotonically. In case of $\beta = 2k$, $\omega_\nu^{(\beta)}\left(\left|y\right|,1\right) \in S(\mathbb{R}^n_+)$ and therefore, it has a decreasing, radial and integrable majorant.

(g) The application of the Hölder inequality (i.e. the case of $r = \infty$ in (6)) yields

$$\left|\left(W_t^{(\beta)}f\right)(x)\right| \le \left\|f\right\|_{p,\nu}\left(\int\limits_{\mathbb{R}^n_+}\left|\omega_\nu^{(\beta)}\left(\left|y\right|,t\right)\right|^{p'} y_n^{2\nu} dy\right)^{\frac{1}{p'}}$$

$$\overset{(20)}{=} \left\|f\right\|_{p,\nu} t^{-\frac{n+2\nu}{\beta}} \left(\int\limits_{\mathbb{R}^n_+} \left|\omega^{(\beta)}_\nu \left(t^{-\frac{1}{\beta}} \left|y\right| ; 1\right)\right|^{p'} y_n^{2\nu} dy \right)^{\frac{1}{p'}} \quad \left(\text{we set } y = t^{\frac{1}{\beta}} x,\ dy = t^{\frac{n}{\beta}} dx\right)$$

$$= \left\|f\right\|_{p,\nu} t^{-\frac{n+2\nu}{\beta}} t^{\frac{n+2\nu}{\beta}\frac{1}{p'}} \left(\int\limits_{\mathbb{R}^n_+} \left|\omega^{(\beta)}_\nu (|x| ; 1)\right|^{p'} y_n^{2\nu} dy \right)^{\frac{1}{p'}} = ct^{-\frac{n+2\nu}{p\beta}} \left\|f\right\|_{p,\nu},$$

where c does not depend of f.

(h) If $f \in S(\mathbb{R}^n_+)$, then the statement is obvious in terms of Fourier-Bessel transform. For arbitrary $f \in L_{p,\nu}$ the result follows by density of $S(\mathbb{R}^n_+)$ in $L_{p,\nu}$ ($L_{\infty,\nu} \equiv C_0$), by taking into account the statement (e).

(i) Using the equality $\int\limits_{\mathbb{R}^n_+} \omega^{(\beta)}_\nu \left(\left|y\right|, t\right) y_n^{2\nu} dy = 1$, $(\forall t > 0)$ and Minkowski inequality, we have for $f \in L_{p,\nu}$ ($L_{\infty,\nu} \equiv C_0$) that

$$\left\|W^{(\beta)}_t f - f\right\|_{p,\nu} \leq \int\limits_{\mathbb{R}^n_+} \left|\omega^{(\beta)}_\nu \left(\left|y\right| ; t\right)\right| \left\|T^y f(.) - f(.)\right\|_{p,\nu} y_n^{2\nu} dy$$

$$\overset{(20)}{=} t^{-\frac{n+2\nu}{\beta}} \int\limits_{\mathbb{R}^n_+} \left|\omega^{(\beta)}_\nu \left(t^{-\frac{1}{\beta}} \left|y\right| ; 1\right)\right| \left\|T^y f(.) - f(.)\right\|_{p,\nu} y_n^{2\nu} dy \quad \left(\text{set } y = t^{\frac{1}{\beta}} z,\ dy = t^{\frac{n}{\beta}} dz\right)$$

$$= \int\limits_{\mathbb{R}^n_+} \left|\omega^{(\beta)}_\nu (|z| ; 1)\right| \left\|T^{t^{\frac{1}{\beta}} z} f(.) - f(.)\right\|_{p,\nu} z_n^{2\nu} dz.$$

Since $\left\|T^{t^{\frac{1}{\beta}} z} f(.) - f(.)\right\|_{p,\nu} \leq 2 \left\|f\right\|_{p,\nu}$ and

$$\lim_{t\to 0^+} \left\|T^{t^{\frac{1}{\beta}} z} f(.) - f(.)\right\|_{p,\nu} = 0 \ ([18]),$$

it follows from Lebesgue dominated convergence theorem that

$$\lim_{t\to 0^+} \left\|W^{(\beta)}_t f - f\right\|_{p,\nu} = 0,\ 1 \leq p \leq \infty; \quad (L_{0,\infty} \equiv C_0 \ \text{ and in this case convergence is uniform.})$$

Since $W^{(\beta)}_t f \to f$ pointwise (in fact, uniformly) as $t \to 0$ for any $f \in L_{p,\nu} \cap C_0$ and this class is dense in $L_{p,\nu}$, $(1 \leq p < \infty)$, then owing to (f) (of the Lemma 2.1) and famous theorem on pointwise (a.e.) convergence [28], p.60, it follows that $\lim\limits_{t\to 0^+} W^{(\beta)}_t f(x) = f(x)$ for almost all $x \in \mathbb{R}^n_+$. The proof of Lemma 2.1 is complete. □

By making use of the generalized beta-semigroup $W^{(\beta)}_t f$, it is possible to obtain the following one-dimensional integral representation of the generalized Riesz potentials $I^\alpha_\nu f$.

Lemma 2.4. *Let* $0 < \alpha < n + 2\nu$ *and* $f \in L_{p,\nu}(\mathbb{R}^n_+)$, $1 \leq p < \frac{n+2\nu}{\alpha}$. *Then the generalized Riesz potentials* $I^\alpha_\nu f$ *admit the following one-dimensional representation:*

$$\left(I^\alpha_\nu f\right)(x) = \frac{1}{\Gamma(\frac{\alpha}{\beta})} \int\limits_0^\infty t^{\frac{\alpha}{\beta}-1} \left(W^{(\beta)}_t f\right)(x) dt, \tag{26}$$

where $0 < \beta \leq 2$ *or* $\beta = 2k$, $k \in \mathbb{N}$.

The formula (26) has exactly the same form as formula (17) in our paper [27] and resembles the classical Balakrishnan formulas for fractional powers of operators (see [23], p.121). It is clear that the formulas (11) and (12) are special cases of (26) (put $\beta = 1$ and $\beta = 2$). Note that this formula is given in [15] and proved in complete analogy with Theorem 2 from [27].

3. A Wavelet-Type Transform Generated by the β-Semigroup $W_t^{(\beta)} f$ and Inversion of Generalized Riesz Potentials

In this section it will be assumed that the parameter β is even natural number. By making use of the β-semigroup (19) we define the following integral transform (cf. [3], p. 339):

$$(Ag)(x,t) \equiv \left(A_{\beta,\nu,\mu} g\right)(x,t) = \int_0^\infty \left(W_{t\eta}^{(\beta)} g\right)(x) d\mu(\eta). \tag{27}$$

Here $x \in \mathbb{R}_+^n$, $t > 0$, $g \in L_{p,\nu}$ and μ is a finite Borel measure on $[0,\infty)$ with $\mu([0,\infty)) = 0$. From now on such a signed Borel measure μ will be called a wavelet measure and the relevant integral transform $(Ag)(x,t)$ will be called a wavelet-type transform.

The integral operator (27) is bounded in $L_{p,\nu}$−spaces. Indeed, by the Lemma 2.1-(e) and the Minkowski inequality, we have for $1 \le p \le \infty$

$$\left\|(Ag)(.,t)\right\|_{p,\nu} \le \int_0^\infty \left\|W_{t\eta}^{(\beta)} g\right\|_{p,\nu} d\left|\mu\right|(\eta) \le c(\beta) \left\|\mu\right\| \left\|g\right\|_{p,\nu},$$

where $\left\|\mu\right\| = \int_{[0,\infty)} d\left|\mu\right|(\eta) < \infty$.

The transform (27) enables one to get a new explicit inversion formula for the generalized Riesz potentials $I_\nu^\alpha f$, $(f \in L_{p,\nu}, 1 \le p < \frac{n+2\nu}{\alpha})$. For this, we need some lemmas.

Lemma 3.1. *(see [12], formula 3.238(3).)*

$$\int_1^s t^{-\frac{\alpha}{\beta}-1}(s-t)^{\frac{\alpha}{\beta}-1} dt = \frac{\Gamma\left(\frac{\alpha}{\beta}\right)}{\Gamma\left(1+\frac{\alpha}{\beta}\right)} \frac{1}{s}(s-1)^{\frac{\alpha}{\beta}}, (s > 1, \alpha > 0, \beta > 0).$$

Lemma 3.2. *(cf. Lemma 1.3 from [20]) Let*

$$K_\theta(\tau) = \frac{1}{\tau}\left(I_+^{\theta+1}\mu\right)(\tau), (\theta > 0, \tau > 0),$$

where

$$\left(I_+^{\theta+1}\mu\right)(\tau) = \frac{1}{\Gamma(1+\theta)} \int_0^\tau (\tau-\eta)^\theta d\mu(\eta), (\theta > 0)$$

is the Riemann-Liouville fractional integral of order $(\theta+1)$ of the measure μ. Suppose that μ satisfies the following conditions:

$$\int_1^\infty \eta^\gamma d\left|\mu\right|(\eta) < \infty \text{ for some } \gamma > \theta; \tag{28}$$

$$\int_0^\infty \eta^j d\mu(\eta) = 0, \ \forall j = 0, 1, ..., [\theta] \text{ (the integer part of } \theta). \tag{29}$$

Then $K_\theta(\tau)$ has a decreasing integrable majorant and

$$\int_0^\infty K_\theta(\tau)d\tau \equiv c_{\theta,\mu} = \begin{Bmatrix} \Gamma(-\theta)\int_0^\infty \eta^\theta d\mu(\eta),\ if\ \theta \neq 1,2,... \\ \frac{(-1)^{\theta+1}}{\theta!}\int_0^\infty \eta^\theta \ln\eta d\mu(\eta),\ if\ \theta = 1,2,... \end{Bmatrix} \tag{30}$$

In particular case, when $0 < \theta < 1$, the conditions (28)-(29) and relation (30) have the simpler form:

$$\int_1^\infty \eta d\left|\mu\right|(\eta) < \infty; \tag{31}$$

$$\int_0^\infty d\mu(\eta) = 0; \tag{32}$$

$$\int_0^\infty K_\theta(\tau)d\tau \equiv c_{\theta,\mu} = \Gamma(-\theta)\int_0^\infty \eta^\theta d\mu(\eta). \tag{33}$$

Lemma 3.3. *Let $f \in L_{p,\nu}$, $1 \le p < \frac{n+2\nu}{\alpha}$, and the integral transform Ag be defined as in (27). Then for $\varphi = I_\nu^\alpha f$,*

$$(A\varphi)(x,t) = \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty\left(\int_0^\infty (\tau-\eta t)_+^{\frac{\alpha}{\beta}-1}\left(W_\tau^{(\beta)}f\right)(x)d\tau\right)d\mu(\eta), \tag{34}$$

where $a_+^\lambda = \begin{Bmatrix} a^\lambda, & if\ a>0 \\ 0, & if\ a \le 0\end{Bmatrix}$ with $\lambda = \frac{\alpha}{\beta}-1$ and $a = \tau - \eta t$.

Proof. Since the operators $I_\nu^\alpha f$ and $W_t^{(\beta)}$ have a convolution structure, they are commutative and therefore,

$$\begin{aligned}(A\varphi)(x,t) &= \int_0^\infty \left(W_{t\eta}^{(\beta)} I_\nu^\alpha f\right)(x)d\mu(\eta) = \int_0^\infty \left(I_\nu^\alpha W_{t\eta}^{(\beta)} f\right)(x)d\mu(\eta) \quad \text{(we use (26))} \\ &= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty\left(\int_0^\infty \tau^{\frac{\alpha}{\beta}-1}\left(W_\tau^{(\beta)}W_{t\eta}^{(\beta)}f\right)(x)d\tau\right)d\mu(\eta) \\ &= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty\left(\int_0^\infty \tau^{\frac{\alpha}{\beta}-1}\left(W_{\tau+t\eta}^{(\beta)}f\right)(x)d\tau\right)d\mu(\eta) \\ &= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)}\int_0^\infty\left(\int_0^\infty (\tau-\eta t)_+^{\frac{\alpha}{\beta}-1}\left(W_\tau^{(\beta)}f\right)(x)d\tau\right)d\mu(\eta).\end{aligned}$$

□

Lemma 3.4. *Denote*

$$\left(D_\varepsilon^\alpha\varphi\right)(x) \equiv \left(D_{\varepsilon,\beta}^\alpha\varphi\right)(x) = \int_\varepsilon^\infty t^{-\frac{\alpha}{\beta}-1}(A\varphi)(x,t)dt,\ (\varepsilon > 0). \tag{35}$$

Then for $\varphi = I_\nu^\alpha f$, $(f \in L_{p,\nu}, 1 \le p < \frac{n+2\nu}{\alpha})$ *we have*

$$\left(D_\varepsilon^\alpha \varphi\right)(x) = \int_0^\infty \left(W_{\varepsilon\tau}^{(\beta)} f\right)(x) K_{\frac{\alpha}{\beta}}(\tau) d\tau, \tag{36}$$

where $K_\theta(\tau)$ *is defined as in Lemma 3.2.*

Proof. Using (34) and Fubini's theorem, we have

$$\begin{aligned}
\left(D_\varepsilon^\alpha \varphi\right)(x) &= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_\varepsilon^\infty t^{-\frac{\alpha}{\beta}-1} \left(\int_0^\infty d\mu(\eta) \int_0^\infty (\tau - \eta t)_+^{\frac{\alpha}{\beta}-1} \left(W_\tau^{(\beta)} f\right)(x) d\tau \right) dt \\
&= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty \left(W_\tau^{(\beta)} f\right)(x) \left(\int_0^{\frac{\tau}{\varepsilon}} \eta^{\frac{\alpha}{\beta}-1} d\mu(\eta) \int_\varepsilon^{\frac{\tau}{\eta}} t^{-\frac{\alpha}{\beta}-1} \left(\frac{\tau}{\eta} - t\right)^{\frac{\alpha}{\beta}-1} dt \right) d\tau \\
&= \frac{1}{\Gamma\left(\frac{\alpha}{\beta}\right)} \int_0^\infty \left(W_{\varepsilon\tau}^{(\beta)} f\right)(x) \left(\int_0^{\tau} \eta^{\frac{\alpha}{\beta}-1} d\mu(\eta) \int_1^{\frac{\tau}{\eta}} t^{-\frac{\alpha}{\beta}-1} \left(\frac{\tau}{\eta} - t\right)^{\frac{\alpha}{\beta}-1} dt \right) d\tau \\
&\text{(we use Lemma 3.1)} \\
&= \int_0^\infty \left(W_{\varepsilon\tau}^{(\beta)} f\right)(x) \left(\frac{1}{\tau} \frac{1}{\Gamma\left(1 + \frac{\alpha}{\beta}\right)} \int_0^\tau (\tau - \eta)^{\frac{\alpha}{\beta}} d\mu(\eta) \right) d\tau \\
&= \int_0^\infty \left(W_{\varepsilon\tau}^{(\beta)} f\right)(x)\, K_{\frac{\alpha}{\beta}}(\tau) d\tau.
\end{aligned}$$

□

Lemma 3.5. *Let the family of operators* $D_\varepsilon^\alpha \equiv D_{\varepsilon,\beta}^\alpha$, $(\varepsilon > 0)$ *be defined as in (35) and let* $\beta > \alpha$. *Suppose that the wavelet measure* μ *satisfies the conditions* $\int_0^\infty d\mu(\eta) = 0$ *and* $\int_0^\infty \eta\, d|\mu|(\eta) < \infty$. *Then the maximal operator*

$$f(x) \longmapsto \sup_{\varepsilon > 0} \left| \left(D_\varepsilon^\alpha I_\nu^\alpha f\right)(x) \right| \tag{37}$$

is weak (p, p) *type for* $1 \le p < \frac{n+2\nu}{\alpha}$.

Proof. The condition $\beta > \alpha$ yields $0 < \alpha/\beta < 1$. From Lemma 3.2 it follows that the function $K_{\frac{\alpha}{\beta}}(\tau)$ has a decreasing integrable majorant and therefore, $\int_0^\infty \left|K_{\frac{\alpha}{\beta}}(\tau)\right| d\tau < \infty$. Then by making use of (36) and Lemma 2.1-(f), we have

$$\left| \left(D_\varepsilon^\alpha I_\nu^\alpha f\right)(x) \right| \le \sup_{t > 0} \left| \left(W_t^{(\beta)} f\right)(x) \right| \int_0^\infty \left|K_{\frac{\alpha}{\beta}}(\tau)\right| d\tau \le C \left(M_\nu f\right)(x).$$

Since the Hardy-Littlewood type maximal operator $M_\nu f$ is weak (p, p) type (see e.g. [13, 14]), then the maximal operator (37) also is weak (p, p) type for $1 \le p < \frac{n+2\nu}{\alpha}$. □

Now, we can formulate the main theorem of this section.

Theorem 3.6. *Let $\alpha > 0$, $1 \le p < \frac{n+2\nu}{\alpha}$, $f \in L_{p,\nu}$ and the parameter $\beta > \alpha$ is of the form $\beta = 2k$, $k \in \mathbb{N}$. Suppose that μ is a finite Borel measure on $[0, \infty)$ satisfying the following conditions:*

$$\textbf{(a)} \int_0^\infty d\mu(\eta) = 0 \quad and \quad \textbf{(b)} \int_1^\infty \eta d\left|\mu\right|(\eta) < \infty. \tag{38}$$

Then

$$\int_0^\infty (AI_\nu^\alpha f)(x,t) t^{-\frac{\alpha}{\beta}-1} dt \equiv \lim_{\varepsilon \to 0} \int_\varepsilon^\infty (AI_\nu^\alpha f)(x,t) t^{-\frac{\alpha}{\beta}-1} dt = c_{\frac{\alpha}{\beta},\mu} f(x), \tag{39}$$

where the operator A and the coefficient $c_{\theta,\mu}$ (with $\theta = \frac{\alpha}{\beta}$) are defined as in (27) and (33) respectively. The limit in (39) exists in the $L_{p,\nu}$-norm and pointwise for almost all $x \in \mathbb{R}_+^n$. If $f \in C_0 \cap L_{p,\nu}$, the convergence in (39) is uniform.

Proof. By making use of (35) and (36) we have

$$\int_\varepsilon^\infty (AI_\nu^\alpha f)(x,t) t^{-\frac{\alpha}{\beta}-1} dt \equiv (D_\varepsilon^\alpha I_\nu^\alpha f)(x) = \int_0^\infty \left(W_{\varepsilon\tau}^{(\beta)} f\right)(x) K_{\frac{\alpha}{\beta}}(\tau) d\tau. \tag{40}$$

Since $\beta > \alpha$, then $\theta = \frac{\alpha}{\beta} < 1$ and therefore $[\theta] = 0$. Thus, the conditions (28)-(29) of the Lemma 3.2 become in the form (31)-(32), that are coincides with the conditions (38). These conditions guarantee that the function $K_{\frac{\alpha}{\beta}}(\tau)$ in Lemma 3.2 has a decreasing integrable majorant and satisfied the equality (33). Hence, we have for $\beta = 2k > \alpha$ and $f \in L_{p,\nu}$, $1 \le p < \frac{n+2\nu}{\alpha}$,

$$\int_\varepsilon^\infty (AI_\nu^\alpha f)(x,t) t^{-\frac{\alpha}{\beta}-1} dt - c_{\frac{\alpha}{\beta},\mu} f(x) \overset{(33)}{=} (D_\varepsilon^\alpha I_\nu^\alpha f)(x) - f(x) \int_0^\infty K_{\frac{\alpha}{\beta}}(\tau) d\tau$$

$$\overset{(40)}{=} \int_0^\infty \left[\left(W_{\varepsilon\tau}^{(\beta)} f\right)(x) - f(x)\right] K_{\frac{\alpha}{\beta}}(\tau) d\tau,$$

and therefore,

$$\left\|D_\varepsilon^\alpha I_\nu^\alpha f - c_{\frac{\alpha}{\beta},\mu} f\right\|_{p,\nu} \le \int_0^\infty \left\|W_{\varepsilon\tau}^{(\beta)} f - f\right\|_{p,\nu} \left|K_{\frac{\alpha}{\beta}}(\tau)\right| d\tau. \tag{41}$$

The application of Lemma 2.1-(i) and Lebesgue convergence theorem gives

$$\lim_{\varepsilon \to 0} \left\|D_\varepsilon^\alpha I_\nu^\alpha f - c_{\frac{\alpha}{\beta},\mu} f\right\|_{p,\nu} = 0. \tag{42}$$

For $f \in C_0 \cap L_{p,\nu}$ we have

$$\lim_{\varepsilon \to 0} \sup_x \left|D_\varepsilon^\alpha I_\nu^\alpha f(x) - c_{\frac{\alpha}{\beta},\mu} f(x)\right| = 0.$$

The proof of pointwise convergence, as expected, is based on the maximal function technique. Since the maximal operator $f(x) \mapsto \sup_{\varepsilon>0} \left|D_\varepsilon^\alpha I_\nu^\alpha f(x)\right|$ is weak (p,p) type for $1 \le p < \frac{n+2\nu}{\alpha}$ (see Lemma 3.5) and the family

$(D^\alpha_\varepsilon I^\alpha_\nu f)(x)$ converges to $c_{\frac{\alpha}{\beta},\mu} f(x)$ pointwise (in fact, uniformly) as $\varepsilon \to 0$ for any $f \in C_0 \cap L_{p,\nu}$ (this class is dense in $L_{p,\nu}$), then owing to Theorem 3.12 from [29], p.60, it follows that

$$(D^\alpha_\varepsilon I^\alpha_\nu f)(x) \to c_{\frac{\alpha}{\beta},\mu} f(x) \text{ a.e., as } \varepsilon \to 0^+.$$

The proof is complete. □

Example 3.7. *As easily to see that the measures*

(a) $d\mu(\eta) = (1-\eta)e^{-\eta}d\eta$ *and (b)* $d\mu(\eta) = h(\eta)d\eta$, *where* $h(\eta) = \left\{\begin{array}{c} 1,\ 0 \le \eta < 1 \\ -1,\ 1 \le \eta < 2 \\ 0,\ 2 \le \eta < \infty \end{array}\right\}$ *are satisfy the conditions (31)-(32), and with accordance to (33),* $c_{\frac{\alpha}{\beta},\mu} \neq 0$ *for these measures. It is easy to construct many another examples of wavelet measure* μ *on* $[0,\infty)$ *which are satisfy the conditions (31)-(32) with* $c_{\frac{\alpha}{\beta},\mu} \neq 0$.

4. A Characterization of the Generalized Riesz Potential Spaces

Generalized Riesz potential space is defined as follows:

$$I^\alpha_\nu\left(L_{p,\nu}\right) = \left\{\varphi : \varphi = I^\alpha_\nu f,\ f \in L_{p,\nu}(\mathbb{R}^n_+)\right\},\ 1 \le p < \frac{n+2\nu}{\alpha}. \tag{43}$$

The norm in the space $I^\alpha_\nu\left(L_{p,\nu}\right)$ is defined by the relation (cf. [23], p.553) $\left\|\varphi\right\|_{L_{p,\nu}} = \left\|f\right\|_{p,\nu}$, which makes $I^\alpha_\nu\left(L_{p,\nu}\right)$ a Banach space. We are going to give a new (wavelet) characterization of the space $I^\alpha_\nu\left(L_{p,\nu}\right)$. Note that most of the known characterizations of the classical Riesz potential spaces $I^\alpha(L_p)$ and its generalizations $L^\alpha_{p,\nu}(\mathbb{R}^n)$ (Samko's spaces) are given in terms of finite differences, the order of which increases with parameter α (see [23], [24], [21], [22]). A wavelet approach to characterization of classical Riesz's potentials is given by B. Rubin [21], p.235-237. As seen from Rubin's theorem in [21], p.235, the number of vanishing moments of the wavelet measure μ increases with α. In [5, 27] it has been shown that the usage of the concept "beta-semigroup" (which is a natural generalization of the well-known Gauss-Weierstrass and Poisson semigroups) enables one to minimize the number of conditions on wavelet measure, no matter how big the order α of potentials is. As seen from the following theorem, the using of the additional parameter β (order of the semigroup $W^{(\beta)}_t f$, $t > 0$) in the characterization of the generalized Riesz potential spaces gives rise to minimize the number of vanishing moments, more precisely, only one vanishing moment of measure μ is sufficient.

Theorem 4.1. *Let* $0 < \alpha < n + 2\nu$, $1 < p < \frac{n+2\nu}{\alpha}$ *and* $\beta = 2k > \alpha$, $(k \in \mathbb{N})$. *Suppose that* μ *is a finite Borel measure on* $[0,\infty)$ *satisfying the following conditions:*

$$\textbf{(a)} \int_0^\infty d\mu(\eta) = 0;\ \ \textbf{(b)} \int_1^\infty \eta d\left|\mu\right|(\eta) < \infty;\ \ \textbf{(c)}\ c_{\frac{\alpha}{\beta},\mu} \neq 0, \tag{44}$$

where $c_{\frac{\alpha}{\beta},\mu}$ *is defined by (33):* $c_{\frac{\alpha}{\beta},\mu} = \Gamma(-\frac{\alpha}{\beta}) \int_0^\infty \eta^{\frac{\alpha}{\beta}} d\mu(\eta)$.

Denote

$$(D^\alpha_\varepsilon \varphi)(x) \equiv \left(D^\alpha_{\varepsilon,\beta}\varphi\right)(x) = \int_\varepsilon^\infty t^{-\frac{\alpha}{\beta}-1}(A\varphi)(x,t)dt,\ (\varepsilon > 0), \tag{45}$$

where the wavelet-type transform $A\varphi$ *is defined as in (27). Then,*

$$\varphi \in I^\alpha_\nu\left(L_{p,\nu}\right) \Leftrightarrow \varphi \in L_{q,\nu},\ q = \frac{p(n+2\nu)}{n+2\nu-\alpha p} \text{ and } \sup_{\varepsilon>0}\left\|D^\alpha_\varepsilon\varphi\right\|_{p,\nu} < \infty.$$

Proof. Let $\varphi \in I_\nu^\alpha\left(L_{p,\nu}\right)$. Then $\varphi = I_\nu^\alpha f$, for some $f \in L_{p,\nu}$. The suitable analog of the Hardy-Littlewood-Sobolev's theorem [3] claimed that $\varphi \in L_{q,\nu}$, where $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2\nu}$, i.e. $q = \frac{p(n+2\nu)}{n+2\nu-\alpha p}$. Moreover, since the $\lim\limits_{\varepsilon\to 0} D_\varepsilon^\alpha \varphi$ exists in the $L_{p,\nu}$-sense (see, Theorem 3.6, formula (39)), then

$$\sup_{\varepsilon>0} \left\| D_\varepsilon^\alpha \varphi \right\|_{p,\nu} < \infty.$$

Let us prove the "sufficient part". We will use some ideas from [21], p.222 and [26] (see also [27]). Denote by $\phi_+ \equiv \phi_+(\mathbb{R}_+^n)$ the Semyanisty-Lizorkin type space of rapidly decreasing $C^\infty-$functions which are even with respect to x_n and such that

$$\omega \in \phi_+ \Leftrightarrow \int_{\mathbb{R}_+^n} \omega(x) x_1^{k_1} x_2^{k_2} ... x_n^{2k_n} x_n^{2\nu} dx = 0,\ \forall k_1, k_2, ..., k_n \in \mathbb{Z}^+.$$

The class ϕ_+ is dense in $L_{p,\nu}(\mathbb{R}_+^n)$ and the operator I_ν^α is an automorphism of ϕ_+([7]). (The density of classical Lizorkin spaces ϕ in $L_p(\mathbb{R}^n)$, and much more information about its generalizations can be found in the paper by S.G. Samko [25]; see also [23], p. 487). The action of a distribution f as a functional on the test function $\omega \in \phi_+$ will be denoted by (f, ω). For a locally integrable on $\mathbb{R}_+^n$ function f we set

$$(f, \omega) = \int_{\mathbb{R}_+^n} f(x)\omega(x) x_n^{2\nu} dx,$$

provided that the integral is finite for all $\omega \in \phi_+$. It is not difficult to show that, being a convolution-type operator, I_ν^α has the following property:

$$(I_\nu^\alpha f, \omega) = (f, I_\nu^\alpha \omega),\ \forall \omega \in \phi_+,\ \alpha > 0,\ f \in L_{p,\nu}. \tag{46}$$

It is known that if $(f, \omega) = (g, \omega)$, $\forall \omega \in \phi_+$, then $f = g + P$, where $P = P(x)$, $x \in \mathbb{R}_+^n$ is a polynomial which is even with respect to the last variable x_n (see [7]). Now, denote $\mathbf{D}_\varepsilon^\alpha \varphi = \frac{1}{c_{\frac{\alpha}{\beta},\mu}} D_\varepsilon^\alpha \varphi$, where $D_\varepsilon^\alpha \varphi$ is defined by (45). Since $\sup\limits_{\varepsilon>0} \left\| \mathbf{D}_\varepsilon^\alpha \varphi \right\|_{p,\nu} < \infty$, by Banach-Alaoglu theorem, there exists a sequence (ε_k) and a function $f \in L_{p,\nu}$ such that

$$\lim_{\varepsilon_k\to 0} \left(\mathbf{D}_{\varepsilon_k}^\alpha \varphi, \omega \right) = (f, \omega),\ \forall \omega \in \phi_+. \tag{47}$$

From (45), (27) and (19) it follows that the integral operator $\mathbf{D}_{\varepsilon_k}^\alpha \varphi$ can be represented as generalized convolution with some radial kernel. Therefore, we have

$$\left(\mathbf{D}_{\varepsilon_k}^\alpha \varphi, v \right) = \left(\varphi, \mathbf{D}_{\varepsilon_k}^\alpha v \right),\ \forall v \in \phi_+. \tag{48}$$

Firstly, we are going to show that

$$(I_\nu^\alpha f, \omega) = (\varphi, \omega),\ \forall \omega \in \phi_+.$$

For this, we have for all $\omega \in \phi_+$:

$$(I_\nu^\alpha f, \omega) \overset{(46)}{=} (f, I_\nu^\alpha \omega) \overset{(47)}{=} \lim_{\varepsilon_k\to 0} \left(\mathbf{D}_{\varepsilon_k}^\alpha \varphi, I_\nu^\alpha \omega \right) \overset{(48)}{=} \lim_{\varepsilon_k\to 0} \left(\varphi, \mathbf{D}_{\varepsilon_k}^\alpha I_\nu^\alpha \omega \right)$$

$$\overset{(40)}{=} \lim_{\varepsilon_k\to 0} \left(\varphi, \frac{1}{c_{\frac{\alpha}{\beta},\mu}} \int_0^\infty \left(W_{\varepsilon_k \tau}^{(\beta)} \omega \right)(x) K_{\frac{\alpha}{\beta}}(\tau) d\tau \right). \tag{49}$$

We must show that the last limit is equal to (φ, ω). Using the Hölder's inequality and then Minkowski one, we have

$$\left|\left(\varphi, \frac{1}{c_{\frac{\alpha}{\beta},\mu}} \int_0^\infty \left(W^{(\beta)}_{\varepsilon_k\tau}\omega\right)(x) K_{\frac{\alpha}{\beta}}(\tau) d\tau\right) - (\varphi, \omega)\right| \le \frac{1}{\left|c_{\frac{\alpha}{\beta},\mu}\right|} \left\|\varphi\right\|_{p,\nu} \left\|\int_0^\infty \left(W^{(\beta)}_{\varepsilon_k\tau}\omega\right)(x) K_{\frac{\alpha}{\beta}}(\tau) d\tau - c_{\frac{\alpha}{\beta},\mu}\omega(x)\right\|_{p',\nu}$$

(we use the relation $c_{\frac{\alpha}{\beta},\mu} = \int_0^\infty K_{\frac{\alpha}{\beta}}(\tau) d\tau$)

$$\le \frac{1}{\left|c_{\frac{\alpha}{\beta},\mu}\right|} \left\|\varphi\right\|_{p,\nu} \int_0^\infty \left|K_{\frac{\alpha}{\beta}}(\tau)\right| \left\|W^{(\beta)}_{\varepsilon_k\tau}\omega - \omega\right\|_{p',\nu} d\tau, \ (\frac{1}{p} + \frac{1}{p'} = 1). \tag{50}$$

It follows from the Lebesgue convergence theorem, the last expression tends to zero as $\varepsilon_k \to 0$. Hence, $(I^\alpha_\nu f, \omega) = (\varphi, \omega), \forall \omega \in \phi_+$. This implies that, $I^\alpha_\nu f = \varphi + P$, where $P = P(x)$ is a polynomial (which is even with respect to the variable x_n). But, $\varphi \in L_{q,\nu}$ and $I^\alpha_\nu f \in L_{q,\nu}$ (with $q = \frac{p(n+2\nu)}{n+2\nu-\alpha p}$), then $P = 0$ and therefore, $I^\alpha_\nu f = \varphi$. Finally, $\varphi \in I^\alpha_\nu \left(L_{p,\nu}\right)$ and the proof is complete. □

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