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F-Evolution Algebra

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Abstract. We consider the evolution algebra of a free population generated by an F-quadratic stochastic operator. We prove that this algebra is commutative, not associative and necessarily power–associative. We show that this algebra is not conservative, not stationary, not genetic and not train algebra, but it is a Banach algebra. The set of all derivations of the F-evolution algebra is described. We give necessary conditions for a state of the population to be a fixed point or a zero point of the F-quadratic stochastic operator which corresponds to the F-evolution algebra. We also establish upper estimate of the ω-limit set of the trajectory of the operator. For an F-evolution algebra of Volterra type we describe the full set of idempotent elements and the full set of absolute nilpotent elements.

1. Introduction

The action of genes is manifested statistically in sufficiently large communities of matching individuals (belonging to the same species). These communities are called populations. The evolution (or dynamics) of a population comprises a determined change of state in the next generations as a result of reproduction and selection. This evolution of a population can be studied by a dynamical system of a quadratic stochastic operator [18].

The concept of quadratic stochastic operator (QSO) and its application in a biological context were first established by Bernstein in [1]. Since then, the theory has been further deepened as it frequently occurs in mathematical models of genetics, where QSOs serve as a tool for the study of dynamical properties and modeling, see [7–12, 14, 15, 18, 20, 21, 23–26]. QSOs were introduced as "evolutionary operators" to describe the dynamics of gene frequencies for given laws of heredity in mathematical population genetics.

In the description of the genetic evolution of large populations QSOs arise as follows: Consider a population with $m \in \mathbb{N}$ different genetic types, where every individual in this population belongs to precisely one of the species $E := \{1, 2, \ldots, m\}$. Let $\mathbf{x}^0 = (x_1^0, \ldots, x_m^0)$ be a probability distribution on E describing the relative frequencies of the genetic types within the whole population in the initial generation. Denote by $p_{ij,k}$ the conditional probability that two individuals of types i and j produce an offspring of

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type k given they interbreed and assume that the population is large enough for frequency fluctuations to be negligible. Presuming a free population, i.e. absence of sexual differentiation and the statistical independence of genotypes for breeding, the distribution $\mathbf{x}' = (x_1', \dots, x_m')$ of the (expected) gene frequencies in the next generation is given by

$$x_k' = \sum_{i,j\in E} p_{ij,k} x_i^0 x_j^0, \quad k \in E. \tag{1.1}$$

The association $\mathbf{x}^0 \mapsto \mathbf{x}'$ defines a map $V\colon S^{m-1} \to S^{m-1}$ called *evolutionary operator*. The population evolves by starting from an arbitrary frequency distribution $\mathbf{x}^0$, then passing to the state $\mathbf{x}' = V(\mathbf{x}^0)$ in the next "generation", then to the state $\mathbf{x}'' = V(V(\mathbf{x}^0))$, and so on. Thus the evolution of gene frequencies in this population can be considered as a dynamical system

$$\mathbf{x}^0, \ \mathbf{x}' = V(\mathbf{x}^0), \ \mathbf{x}'' = V^2(\mathbf{x}^0), \ \mathbf{x}''' = V^3(\mathbf{x}^0), \ \dots$$

Note that V as defined by (1.1) is a non-linear (quadratic) operator. Higher dimensional dynamical systems, as the one resulting from the observations above for $m \geq 3$, are important, but only relatively few dynamical phenomena are thoroughly comprehended (see [2, 3]).

One of the main motivations to study dynamical systems and QSOs is the asymptotic behaviour of their trajectories, depending on the initial value. However, this has only been determined for certain particular subclasses of QSOs so far. One such subclass that arises naturally in the biological context is given by the additional restriction

$$p_{ij,k} = 0, \ \text{ if } \ k \notin \{i, j\}, \ \ i, j, k \in E. \tag{1.2}$$

These QSOs, called *Volterra operators*, describe a reproductive behaviour where the offspring is a genetic copy of one of its parents. The asymptotic behaviour of trajectories of this kind of QSOs was analysed in [9–11] using the theory of Lyapunov functions and tournaments. However, in the non-Volterra case (i.e., where condition (1.2) is violated), many questions remain open and there seems to be no general theory available. See [12] for a recent review of QSOs.

There exists an algebraic approach in the study of laws of genetics. Several classes of non-associative algebras have provided a number of significant contributions to theoretical population genetics and have been defined different times by several authors, and all algebras belonging to these classes are generally called genetic. Etherington introduced the formal language of abstract algebra to the study of genetics in a series of seminal papers [4–6]. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. Recently in the book of Tian [22] a new type of evolution algebra was introduced. This algebra also describes some evolution laws of genetics. The study of evolution algebras constitutes a new subject both in algebra and the theory of dynamical systems. In the book [22] the foundations of evolution algebra theory and applications in non-Mendelian genetics are developed.

In the book [18] evolution algebras associated with a free population are studied. But there are few results devoted to evolution algebras corresponding to bisexual populations.

In [19] evolution algebras generated by Volterra quadratic stochastic operators in the case of small dimensions are considered.

Recently, in [17], the authors considered a bisexual population and defined an evolution algebra using inheritance coefficients of the population. This algebra is a natural generalization of the algebra of a free population. Moreover, in [16], an evolution algebra of a chicken population is considered. This algebra corresponds to a bisexual population with a set of females partitioned into finitely many different types and the males having only one type. The basic properties of this algebra are studied.

In the present paper we consider an evolution algebra generated by an F-quadratic stochastic operator, i.e., we define an evolution algebra using inheritance coefficients of the population. The paper is organized as follows. In Section 2 we recall the definition of an F-QSO as well as definitions and known results related to an evolution algebra of a free population. In Section 3 we define an F-evolution algebra, study its basic properties and therein we show an F-evolution algebra is different from associative, power–associative, Bernstein, genetic, train, conservative, Jordan, Jacobi and alternative algebras. We also prove that an F-evolution algebra is a Banach algebra and we describe the set of all derivations of an F-evolution algebra. In Section 4 we find necessary conditions for a state of the population to be a fixed point or a zero point of the evolution operator. We also establish upper estimate of the limit points set for trajectories of the evolution operator. Finally, in Section 5, we describe the full set of idempotents and absolute nilpotents for a special case.

2. Preliminaries and Known Results

F-quadratic stochastic operator. A quadratic stochastic operator (QSO) on a set $E = \{1, \dots, m\}$ is a mapping V of the simplex

$$S^{m-1} = \Big\{\mathbf{x} = \big(x_1, \dots, x_m\big) \in \mathbb{R}^m : x_i \geq 0,\ i \in E,\ \sum_{i=1}^{m} x_i = 1\Big\} \tag{2.1}$$

into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$x'_k = \sum_{i,j\in E} p_{ij,k} x_i x_j, \quad k \in E, \tag{2.2}$$

and the $p_{ij,k}$ satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \qquad \sum_{k\in E} p_{ij,k} = 1, \quad i, j, k \in E. \tag{2.3}$$

The *trajectory (orbit)* $\{\mathbf{x}^{(n)}\}_{n\in\mathbb{N}_0}$ of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{n+1}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots \tag{2.4}$$

A point $\mathbf{x} \in \mathbb{R}^{m+1}$ is called a *fixed point* of V if $V(\mathbf{x}) = \mathbf{x}$ and is called a *zero point* of V if $V(\mathbf{x}) = 0$.

We recall the definition of an *F-quadratic stochastic operator* following [20]. Let us extend the set E by adding the element "0", i.e., we shall consider the set $E_0 = \{0, 1, \dots, m\}$. Let us fix a set $F \subset E$ and call it the set of "women", while the set $M = E \setminus F$ is called the set of "men". The element 0 plays the role of an "empty body".

The coefficients $p_{ij,k}$ of the matrix $\mathbf{P}$ are defined as follows:

$$p_{ij,k} = \begin{cases} 1, & \text{if } k = 0, i, j \in F \cup \{0\} \text{ or } i, j \in M \cup \{0\}; \\ 0, & \text{if } k \neq 0, i, j \in F \cup \{0\} \text{ or } i, j \in M \cup \{0\}; \\ \geq 0, & \text{if } i \in F, j \in M, k \in E. \end{cases} \tag{2.5}$$

The biological interpretation of the coefficients (2.5) is obvious: the "child" k can be born only if its parents are taken from different classes F and M. Generally, $p_{ij,0}$ can be strictly positive for $i \in F$ and $j \in M$, which corresponds, for example, to the case in which "woman" i with "man" j cannot have a "child", because one of them is ill or both are.

Definition 2.1. *For any fixed $F \subset E$, a QSO satisfying conditions* (2.2), (2.3) *and* (2.5) *is called an F-*quadratic stochastic operator *(F-QSO).*

Consider

$$E_0 = \{0, 1, \dots, m\}, \quad F = \{1, 2, 3, \dots, m_1\}, \quad M = \{m_1 + 1, \dots, m\}.$$

It is evident that the corresponding F-QSO is of the form

$$V:\begin{cases} x_0' = x_0^2 + 2x_0\sum_{i\in E} x_i + \sum_{i,j\in F} x_ix_j + 2\sum_{i\in F}\sum_{j\in M} p_{ij,0}x_ix_j + \sum_{i,j\in M} x_ix_j; \\ x_k' = 2\sum_{i\in F}\sum_{j\in M} p_{ij,k}x_ix_j, \qquad k = 1,2,\dots,m, \end{cases} \tag{2.6}$$

where

$$p_{ij,k} = p_{ji,k} \geq 0,\ \ k\in E_0;\ \sum_{k\in E_0} p_{ij,k} = 1,\ \ i\in F, j\in M. \tag{2.7}$$

It is shown in [20] that the fixed point is unique and that all trajectories approach this fixed point exponentially rapid.

Evolution algebra of a free population. Let us recall the definition of an evolution algebra of a free population following [18]. The quadratic stochastic operator is closely related to an algebra structure on $\mathbb{R}^m$ containing the unit simplex (2.1). Let $\{\mathbf{e}_k\}_{k=1}^m$ be the canonical basis of $\mathbb{R}^m$ and we introduce a multiplication as follows

$$\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i = \sum_{k=1}^m p_{ij,k}\mathbf{e}_k. \tag{2.8}$$

Thus we identify the coefficients of inheritance as the structure of an algebra, i.e. a bilinear mapping of $\mathbb{R}^m\times\mathbb{R}^m$ to $\mathbb{R}^m$.

Suppose that $\mathbf{x} = (x_1,\dots,x_m)\in\mathbb{R}^m$ and $\mathbf{y} = (y_1,\dots,y_m)\in\mathbb{R}^m$. Then the general formula for the multiplication is the extension of (2.8) on $\mathbb{R}^m$ generated by QSO (2.2) and it has the form

$$\mathbf{x}\circ\mathbf{y} = \sum_{i,j,k=1}^m (p_{ij,k}x_iy_j)\mathbf{e}_k = \frac{1}{4}(V(\mathbf{x}+\mathbf{y}) - V(\mathbf{x}-\mathbf{y})). \tag{2.9}$$

Using (2.3) it is easy to see that $\mathbf{x}\circ\mathbf{y} = \mathbf{y}\circ\mathbf{x}$, i.e. the multiplication (2.9) has the commutative property. It is also easy to check that

$$\mathbf{x}\mathbf{x} = \mathbf{x}^2 = \sum_{i,j,k=1}^m (p_{ij,k}x_ix_j)\mathbf{e}_k = V(\mathbf{x}) \ \text{ for any }\ \mathbf{x}\in S^{m-1}.$$

This algebraic interpretation is useful, e.g. a state $\mathbf{x}$ is an equilibrium precisely when $\mathbf{x}$ is an idempotent element of the unit simplex S^{m-1}.

If we write $\mathbf{x}^{[n]}$ for the power $(\cdots(\mathbf{x}^2)^2\cdots)$ (n times) with $\mathbf{x}^{[0]}\equiv\mathbf{x}$, then the trajectory with initial state $\mathbf{x}$ is $V^n(\mathbf{x}) = \mathbf{x}^{[n]}$.

The algebra $\mathcal{A}_V$ generated by the evolution operator (2.2) is called the *evolutionary algebra*.

A *character* for an algebra $\mathcal{A}$ is a nonzero multiplicative linear form on $\mathcal{A}$, that is, a nonzero algebra homomorphism from $\mathcal{A}$ to $\mathbb{R}$. A pair $(\mathcal{A},\sigma)$ consisting of an algebra $\mathcal{A}$ and a character σ on $\mathcal{A}$ is called a *baric algebra*.

Also in [18], for the evolution algebra of a free population, it is proven that there is a character $\sigma(\mathbf{x}) = \sum_{k=1}^m x_k$.

Denote $H_0 = \{\mathbf{x} : \sigma(\mathbf{x}) = 0\}$, $H_1 = \{\mathbf{x} : \sigma(\mathbf{x}) = 1\}$ the hyperplanes in $\mathbb{R}^m$ and $H_\infty = \{\mathbf{x} : \sigma(\mathbf{x}) = +\infty\}$. We also denote the sets $I_F = \{\mathbf{x} : x_i = 0,\ \text{for all } i\in F\}$, $I_M = \{\mathbf{x} : x_i = 0,\ \text{for all } i\in M\}$ and $I_E = I_F\cap I_M$.

A subset I_L (I_R) is called *left ideal* (resp. *right ideal*) of an algebra $\mathcal{A}$ if

(i) I_L (I_R) is a subalgebra of the algebra $\mathcal{A}$;

(ii) $\mathbf{a}\mathbf{x}\in I_L$, for all $\mathbf{x}\in I_L$, for all $\mathbf{a}\in\mathcal{A}$ (resp. $\mathbf{x}\mathbf{a}\in I_R$, for all $\mathbf{x}\in I_R$, for all $\mathbf{a}\in\mathcal{A}$).

A subset I is called *ideal* (two–sided ideal) if I is a left ideal and a right ideal, simultaneously.
The *invariant linear form* is a linear form f on a baric algebra $\mathcal{A}$ which satisfies

$$f(\mathbf{xy}) = \frac{\sigma(\mathbf{y})f(\mathbf{x}) + \sigma(\mathbf{x})f(\mathbf{y})}{2} \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathcal{A}.$$

The set J of invariant forms is a subspace of the dual space $\mathcal{A}^*$. Since the character $\sigma \neq 0$ itself is clearly invariant, we have $\dim J \geq 1$. Denote

$$J^{\perp} = \{\mathbf{x} : f(\mathbf{x}) = 0, \ \text{for all } f \in J\}, \ \ \operatorname{ann}\mathcal{A} = \{\mathbf{y} : \mathbf{yx} = 0, \ \text{for all } \mathbf{x} \in \mathcal{A}\}.$$

Definition 2.2 ([18]).

(i) An algebra is called flexible algebra *if it satisfies* $\mathbf{x}(\mathbf{yx}) = (\mathbf{xy})\mathbf{x}$ *for any* $\mathbf{x}, \mathbf{y} \in \mathcal{A}$*;*

(ii) An algebra is called conservative algebra *if* $J = \operatorname{ann}\mathcal{A}$*;*

(iii) An baric algebra is called genetic algebra *if*

$$\mathbf{e}_i\mathbf{e}_j = \sum_{k=0}^{m} \lambda_{ij,k}\mathbf{e}_k,$$

where coefficients satisfy

$$\begin{aligned} \lambda_{00,0} &= 1, \quad \lambda_{0i,k} = 0, \ \ 0 \leq k < i \leq m, \\ \lambda_{ij,k} &= 0, \quad 0 \leq k \leq \max(i,j), \ \ i,j = 1,\ldots,m. \end{aligned} \tag{2.10}$$

(iv) An algebra $\mathcal{A}$ *is called* Bernstein (or stationary) algebra *if for any element* $\mathbf{x} \in \mathcal{A}$ *it satisfies*

$$(\mathbf{x}^2)^2 = (\sigma(\mathbf{x}))^2\mathbf{x}^2. \tag{2.11}$$

(v) For each element we have a linear operator $M_{\mathbf{x}} \colon \mathcal{A} \to \mathcal{A}$ *defined by* $M_{\mathbf{x}}(\mathbf{y}) = \mathbf{xy}$*. A baric algebra* $\mathcal{A}$ *is called a* train algebra *if for each* $\mathbf{x} \in \mathcal{A}$ *the characteristic polynomial of* $M_{\mathbf{x}}$ *on* $\mathcal{A}$ *depends only of the character* $\sigma(\mathbf{x})$*.*

The conservative algebras are characterized by the following theorem.

Theorem 2.3 ([18]). *The baric algebra* $(\mathcal{A}, \sigma)$ *is conservative if and only if the following identity holds*

$$\mathbf{x}^2\mathbf{y} = \sigma(\mathbf{x})\mathbf{xy}, \ \textit{for all } \ \mathbf{x}, \mathbf{y} \in \mathcal{A}. \tag{2.12}$$

3. *F*-Evolution Algebra

Suppose that $\mathbf{x} = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ and $\mathbf{y} = (y_0, y_1, \ldots, y_m) \in \mathbb{R}^{m+1}$. Then we consider the multiplication on $\mathbb{R}^{m+1}$ generated with QSO (2.6) as in (2.9)

$$\mathbf{x} \circ \mathbf{y} = \frac{1}{4}(V(\mathbf{x}+\mathbf{y}) - V(\mathbf{x}-\mathbf{y})).$$

Let $\{\mathbf{e}_k\}_{k=0}^m$ be the canonical basis on $\mathbb{R}^{m+1}$. Then (2.8) has the form

$$\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i = \begin{cases} \sum\limits_{k \in E_0} p_{ij,k}\mathbf{e}_k, & \text{if } i \in F, \ j \in M, \\ \mathbf{e}_0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Definition 3.1. *A linear space generated by* $\mathbf{e}_k, k = 0, 1, \ldots, m$*, over the real number field* $\mathbb{R}$*, with the multiplication defined as* (3.1) *by using the coefficients of inheritance* (2.7) *is called an* F-evolution algebra *and denoted by* $\mathcal{F} = \mathcal{F}_V$*.*

Remark 3.2.

(i) It is evident that an F-evolution algebra is different from the evolution algebra introduced in [22].

(ii) It is also easy to see that an F-evolution algebra is different from the evolution algebra of the bisexual and "chicken" populations (see [16] and [17]).

Example 3.3. *Consider the F-evolution algebra $\mathcal{F}_V = \langle \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\rangle$, where $E_0 = \{0,1,2\}$, $F = \{1\}$ and $M = \{2\}$. Then, in this case,* (2.9) *has the form*

$$\mathbf{x} \circ \mathbf{y} = \Big(x_0y_0 + x_1y_1 + x_2y_2 + x_0y_1 + x_1y_0 + x_0y_2 + x_2y_0 + a(x_1y_2 + x_2y_1), b(x_1y_2 + x_2y_1), c(x_1y_2 + x_2y_1)\Big),$$

where $a, b, c \geq 0$, $a + b + c = 1$.

Then (3.1) *has the form*

$$\mathbf{e}_j\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_j = \begin{cases} a\mathbf{e}_0 + b\mathbf{e}_1 + c\mathbf{e}_2, & \text{if } i = 1,\ j = 2, \\ \mathbf{e}_0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Basic properties. The following theorem gives basic properties of an F-evolution algebra.

Theorem 3.4.

(i) F-evolution algebra is not associative, in general.

(ii) F-evolution algebra is flexible.

(iii) F-evolution algebra is not power-associative, in general.

Proof. (i) To see that an F-evolution algebra is not associative, we consider the F-evolution algebra of Example 3.3. We suppose that $b > 0$ or $c > 0$. Then a simple analysis shows that

$$\mathbf{e}_0 = \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_2) \neq (\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2 = (a + ab + c)\mathbf{e}_0 + b^2\mathbf{e}_1 + bc\mathbf{e}_2.$$

(ii) It is evident that a commutative algebra is flexible. As mentioned above that an F-evolution algebra is a commutative algebra, so it is flexible.

(iii) To show that an F-evolution algebra is not power-associative, in general, we shall construct an example of $\mathbf{x}$ such that $(\mathbf{xx})(\mathbf{xx}) \neq \big((\mathbf{xx})\mathbf{x}\big)\mathbf{x}$. Let the F-evolution algebra defined in Example 3.3. Taking $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2$, then we have

$$\mathbf{x}^2 = 2(a+1)\mathbf{e}_0 + 2b\mathbf{e}_1 + 2c\mathbf{e}_2, \tag{3.3}$$

$$\mathbf{x}^2\mathbf{x}^2 = 4\Big((a+1)^2 + b^2 + c^2 + 2(a+1)(b+c) + 2abc\Big)\mathbf{e}_0 + 8b^2c\mathbf{e}_1 + 8bc^2\mathbf{e}_2, \tag{3.4}$$

$$\mathbf{x}^2\mathbf{x} = 2(a+1)(b+c+2)\mathbf{e}_0 + 2b(b+c)\mathbf{e}_1 + 2c(b+c)\mathbf{e}_2,$$

$$(\mathbf{x}^2\mathbf{x})\mathbf{x} = 2(a+1)\Big((b+c+1)^2 + 3\Big)\mathbf{e}_0 + 2b(b+c)^2\mathbf{e}_1 + 2c(b+c)^2\mathbf{e}_2. \tag{3.5}$$

Then from (3.4) and (3.5) follows $(\mathbf{xx})(\mathbf{xx}) \neq \big((\mathbf{xx})\mathbf{x}\big)\mathbf{x}$, i.e. the algebra generated by an F-QSO is not power-associative. □

Since $\mathcal{F}$ is a baric algebra there is the character of the algebra $\sigma(\mathbf{x}) = \sum\limits_{k\in E_0} x_k$.

Proposition 3.5.

(i) The sets H_0, I_F, I_M, I_E are ideals of the F-evolution algebra;

(ii) H_1 is a closed set respect to the multiplication.

Proof. The proof immediately follows from that $\mathcal{F}$ is a baric algebra and from the definition of ideal. □

Proposition 3.6.

(i) An F-evolution algebra is not conservative, in general;

(ii) An F-evolution algebra is not Bernstein, in general;

(iii) An F-evolution algebra is not genetic.

(iv) An F-evolution algebra is not train, in general;

(v) An F-evolution algebra is not Jordan, in general;

(vi) An F-evolution algebra is not alternative, in general;

(vii) An F-evolution algebra is not Jacobi, in general;

Proof. (i) Let us consider the F-evolution algebra defined in Example 3.3. Taking $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$, it is easy to check that

$$\mathbf{x}^2\mathbf{y} = \mathbf{e}_1^2\mathbf{e}_2 = \mathbf{e}_0\mathbf{e}_2 = \mathbf{e}_0,$$

and using (3.2) we have

$$\sigma(\mathbf{e}_1)\mathbf{e}_1\mathbf{e}_2 = a\sigma(\mathbf{e}_1)\mathbf{e}_0 + b\sigma(\mathbf{e}_1)\mathbf{e}_1 + c\sigma(\mathbf{e}_1)\mathbf{e}_2.$$

Consequently in this case we obtain that Equation (2.12) is not satisfied.

(ii) Again, in the considered F-evolution algebra in Example 3.3, taking the element $\mathbf{z} = \mathbf{e}_1 + \mathbf{e}_2$ from (3.3) and (3.4) we get that Equation (2.11) is not satisfied, that is, $(\mathbf{z}^2)^2 \neq (\sigma(\mathbf{z}))^2\mathbf{z}^2$.

(iii) From the definition of F-evolution algebras we can see that (2.7) and (2.10) cannot be satisfied simultaneously.

(iv) The definition of train algebra is equivalent to the following: there are real constants $\theta_0, \dots, \theta_m$ such that on H_1 we have

$$\det(\lambda I_n - M_{\mathbf{x}}) = \lambda^n - \theta_1\lambda^{n-1} + \cdots + (-1)^n\theta_n, \tag{3.6}$$

and beyond the H_1 we have

$$\det(\lambda I_n - M_{\mathbf{x}}) = \lambda^n - \theta_1\sigma\lambda^{n-1} + \cdots + (-1)^n\theta_n\sigma^n, \tag{3.7}$$

where $\sigma = \sigma(\mathbf{x})$ (see [18]).

Let us consider the F-evolution algebra defined in Example 3.3. Then we have

$$\det(\lambda I_3 - M_{\mathbf{x}}) = \lambda(\lambda - \sigma)(\lambda - (bx_2 + cx_1)).$$

The last equation does not satisfy either (3.6) or (3.7). Thus the F-evolution algebra is not train algebra.

(v) An algebra is called *Jordan* if for any elements $\mathbf{x}, \mathbf{y}$ from the algebra it is hold $(\mathbf{x}\mathbf{y})\mathbf{x}^2 = \mathbf{x}(\mathbf{y}\mathbf{x}^2)$. Again we consider the F-evolution algebra defined in Example 3.3 and suppose that $b \neq c$. Taking $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{y} = \mathbf{e}_1 - \mathbf{e}_2$ one has that

$$0 = (\mathbf{x}\mathbf{y})\mathbf{x}^2 \neq \mathbf{x}(\mathbf{y}\mathbf{x}^2) = 2(b^2 - c^2)(1 - a)\mathbf{e}_0 - 2(b^2 - c^2)b\mathbf{e}_1 - 2(b^2 - c^2)c\mathbf{e}_2.$$

(vi) An algebra is called *alternative* if for any elements $\mathbf{x}, \mathbf{y}$ from the algebra, $(\mathbf{x}\mathbf{x})\mathbf{y} = \mathbf{x}(\mathbf{x}\mathbf{y})$ and $(\mathbf{y}\mathbf{x})\mathbf{x} = \mathbf{y}(\mathbf{x}\mathbf{x})$ hold simultaneously. Again we consider the F-evolution algebra defined in Example 3.3. Taking $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$ one has that

$$\mathbf{e}_0 = (\mathbf{x}\mathbf{x})\mathbf{y} \neq \mathbf{x}(\mathbf{x}\mathbf{y}) = (a + b + ac)\mathbf{e}_0 + bc\mathbf{e}_1 + c^2\mathbf{e}_2.$$

Similarly it is easy to see that the second equation does not hold.

(vii) An algebra is called *Jacobi* if for any elements $\mathbf{x}, \mathbf{y}, \mathbf{z}$ from the algebra, $(\mathbf{xy})\mathbf{z} + (\mathbf{yz})\mathbf{x} + (\mathbf{zx})\mathbf{y} = 0$ holds. Again, we consider the F-evolution algebra defined in Example 3.3. Taking $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{y} = \mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{z} = \mathbf{e}_1$ one has that

$$(\mathbf{xy})\mathbf{z} + (\mathbf{yz})\mathbf{x} + (\mathbf{zx})\mathbf{y} = 2b(1-a)\mathbf{e}_0 - 2b^2\mathbf{e}_1 - 2bc\mathbf{e}_2 \neq 0.$$

Thus the F-evolution algebra is not Jacobi algebra. □

Banach algebra. Define a norm $\|\cdot\|$ in the F-evolution algebra $\mathcal{F}$ as follows

$$\|\mathbf{x}\| = \sum_{k\in E_0} \|x_k\mathbf{e}_k\| := \sum_{k\in E_0} |x_k|.$$

For a fixed $\mathbf{y} \in \mathcal{F}$ consider the operator $L_{\mathbf{y}} : \mathcal{F} \to \mathcal{F}$, left multiplication (resp. right multiplication $R_{\mathbf{y}}$), defined as

$$L_{\mathbf{y}}(\mathbf{x}) = \mathbf{yx} \ \ (\text{resp. } R_{\mathbf{y}}(\mathbf{x}) = \mathbf{xy}).$$

Theorem 3.7. *For any $\mathbf{y} \in \mathcal{F}$ the operator $L_{\mathbf{y}}$ is a bounded linear operator.*

Proof. For arbitrary $\mathbf{y} \in \mathcal{F}$ we have

$$L_{\mathbf{y}}(\mathbf{x}) = \sum_{k\in E_0} \sum_{i,j\in E_0} p_{ij,k} x_i y_j \mathbf{e}_k,$$

$$\begin{aligned}\|L_{\mathbf{y}}(\mathbf{x})\| &= \sum_{k\in E_0} \left| \sum_{i,j\in E_0} p_{ij,k} x_i y_j \right| \le \sum_{k\in E_0} \sum_{i,j\in E_0} p_{ij,k} |x_i y_j| \\ &\le \sum_{i,j\in E_0} |x_i y_j| \le \sum_{i\in E_0} |x_i| \sum_{j\in E_0} |y_j| \le \|\mathbf{x}\|\|\mathbf{y}\|.\end{aligned}$$

So $L_{\mathbf{y}}(\mathbf{x})$ is bounded for any fixed $\mathbf{y} \in \mathcal{F}$. □

Theorem 3.8. *An F-evolution algebra $\mathcal{F}$ is a Banach space.*

Proof. It easy to see that if $\mathbf{x}^n$ converges then its all coordinates also converge. So limit of $\mathbf{x}^k$ also will be an element of $\mathcal{F}$, and so the theorem is proved. □

Corollary 3.9. *An F-evolution algebra is a non associative Banach algebra.*

The derivations of an F-evolution algebra. In the study of genetic algebras the notion of derivations of algebras is useful. There are many papers dedicated to derivations of genetic algebras. In [13] an explanation of the genetic meaning of derivations of a genetic algebra is given. Below, we describe the set of all derivations of an F-evolution algebra.

Definition 3.10. *A linear map $D : \mathcal{F} \to \mathcal{F}$ is called a* derivation *if*

$$D(\mathbf{xy}) = D(\mathbf{x})\mathbf{y} + \mathbf{x}D(\mathbf{y}), \ \ \textit{for any} \ \ \mathbf{x}, \mathbf{y} \in \mathcal{F}.$$

Let $D \in \mathrm{Der}(\mathcal{F})$ be a derivation and suppose

$$D(\mathbf{e}_i) = \sum_{j\in E_0} d_{ij}\mathbf{e}_j.$$

$$D(\mathbf{e}_i\mathbf{e}_j) = D(\sum_{k\in E_0} p_{ij,k}\mathbf{e}_k) = \sum_{k\in E_0} p_{ij,k}D(\mathbf{e}_k) = \sum_{k,t\in E_0} d_{kt}p_{ij,k}\mathbf{e}_t,$$

$$D(\mathbf{e}_i)\mathbf{e}_j = \Big(\sum_{k\in E_0} d_{ik}\mathbf{e}_k\Big)\mathbf{e}_j = \sum_{k,t\in E_0} d_{ik}p_{kj,t}\mathbf{e}_t, \quad \mathbf{e}_iD(\mathbf{e}_j) = \mathbf{e}_i\Big(\sum_{k\in E_0} d_{jk}\mathbf{e}_k\Big) = \sum_{k,t\in E_0} d_{jk}p_{ki,t}\mathbf{e}_t.$$

By definition, we have

$$\sum_{k,t\in E_0} d_{kt}p_{ij,k}\mathbf{e}_t = \sum_{k,t\in E_0} d_{ik}p_{kj,t}\mathbf{e}_t + \sum_{k,t\in E_0} d_{jk}p_{ki,t}\mathbf{e}_t,$$

$$\sum_{k\in E_0}(d_{kt}p_{ij,k} - d_{ik}p_{kj,t} - d_{jk}p_{ki,t}) = 0, \quad i,j,t \in E_0.$$

Since for any $\mathbf{x} \in \mathcal{F}$ we have $D(\mathbf{x}) = \sum\limits_{k\in E_0} x_kD(\mathbf{e}_k)$, D is uniquely defined by the matrix $\mathcal{D} = \mathcal{D}(D) = (d_{ij})_{i,j\in E_0}$. Therefore

$$\mathrm{Der}(\mathcal{F}) = \{D : \sum_{k\in E_0}(d_{kt}p_{ij,k} - d_{ik}p_{kj,t} - d_{jk}p_{ki,t}) = 0, \quad i,j,t \in E_0\}.$$

Example 3.11. *Consider the F-evolution algebra defined in Example 3.3. Suppose $b > 0$, $c > 0$ and*

$$\begin{aligned} D(\mathbf{e}_0) &= d_{00}\mathbf{e}_0 + d_{01}\mathbf{e}_1 + d_{02}\mathbf{e}_2,\\ D(\mathbf{e}_1) &= d_{10}\mathbf{e}_0 + d_{11}\mathbf{e}_1 + d_{12}\mathbf{e}_2,\\ D(\mathbf{e}_2) &= d_{20}\mathbf{e}_0 + d_{21}\mathbf{e}_1 + d_{22}\mathbf{e}_2.\end{aligned}$$

From equations $D(\mathbf{e}_0) = D(\mathbf{e}_0\mathbf{e}_1) = D(\mathbf{e}_0)\mathbf{e}_1 + \mathbf{e}_0D(\mathbf{e}_1)$, $D(\mathbf{e}_0) = D(\mathbf{e}_0\mathbf{e}_2) = D(\mathbf{e}_0)\mathbf{e}_2 + \mathbf{e}_0D(\mathbf{e}_2)$ and $D(\mathbf{e}_1\mathbf{e}_2) = D(\mathbf{e}_1)\mathbf{e}_2 + \mathbf{e}_1D(\mathbf{e}_2)$, and after some computations, we obtain

$$\begin{cases} d_{01} + ad_{02} + d_{10} + d_{11} + d_{12} = 0\\ bd_{02} = d_{01}\\ cd_{02} = d_{02}\\ d_{02} + ad_{01} + d_{20} + d_{21} + d_{22} = 0\\ bd_{01} = d_{01}\\ cd_{01} = d_{02}\\ d_{10} + ad_{11} + d_{12} + d_{20} + d_{21} + ad_{22} = ad_{00} + bd_{10} + cd_{20}\\ bd_{22} = ad_{01} + cd_{21}\\ cd_{11} = ad_{02} + bd_{12}\end{cases}$$

and

$$\begin{cases} d_{00} = 0 & d_{01} = 0 & d_{02} = 0\\ d_{10} = -\dfrac{b+c}{b}\alpha & d_{11} = \alpha & d_{12} = \dfrac{c}{b}\alpha\\ d_{20} = -\dfrac{b+c}{c}\beta & d_{21} = \dfrac{b}{c}\beta & d_{22} = \beta\end{cases}$$

where $\alpha, \beta \in \mathbb{R}$.

Consequently

$$\begin{aligned}
&D(\mathbf{e}_0) = 0\\
&D(\mathbf{e}_1) = -\frac{b+c}{b}\alpha\mathbf{e}_0 + \alpha\mathbf{e}_1 + \frac{c}{b}\alpha\mathbf{e}_2\\
&D(\mathbf{e}_2) = -\frac{b+c}{c}\beta\mathbf{e}_0 + \frac{b}{c}\beta\mathbf{e}_1 + \beta\mathbf{e}_2
\end{aligned}$$

and for any element $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ *we obtain*

$$D(\mathbf{x}) = -(b+c)(\frac{\alpha x_1}{b} + \frac{\beta x_2}{c})\mathbf{e}_0 + (\alpha x_1 + \frac{b x_2}{c})\mathbf{e}_1 + (\frac{c x_1}{b} + \beta x_2)\mathbf{e}_2.$$

Thus

$$\mathrm{Der}(\mathcal{F}) = \Big\{D : D(\mathbf{x}) = -(b+c)\Big(\frac{\alpha x_1}{b} + \frac{\beta x_2}{c}\Big)\mathbf{e}_0 + \Big(\alpha x_1 + \frac{b x_2}{c}\Big)\mathbf{e}_1 + \Big(\frac{c x_1}{b} + \beta x_2\Big)\mathbf{e}_2, \mathbf{x} \in \mathbb{R}^3, \alpha, \beta \in \mathbb{R}\Big\}.$$

4. Dynamics of an F-QSO

Let us consider the quadratic operator $V : \mathcal{F}_V \to \mathcal{F}_V$ defined by formula (2.6).

Proposition 4.1.

(i) If $\mathbf{x}$ *is a fixed point then* $\mathbf{x} \in H_0 \cup H_1$;

(ii) If $\mathbf{x}$ *is a zero point then* $\mathbf{x} \in H_0$.

Proof. (i) Let $\mathbf{x}$ be a fixed point of V. Then

$$\sigma(V(\mathbf{x})) = \sum_{k\in E_0} x'_k = \sum_{k,i,j\in E_0} p_{ij,k} x_i x_j = \sum_{i\in E_0} x_i \sum_{j\in E_0} x_j = (\sigma(\mathbf{x}))^2. \tag{4.1}$$

Therefore $\sigma(\mathbf{x}) = 0$ or $\sigma(\mathbf{x}) = 1$ and $\mathbf{x} \in H_0 \cup H_1$.

(ii) Let $\mathbf{x}$ be a zero point. Then from $0 = \sigma(V(\mathbf{x})) = (\sigma(\mathbf{x}))^2$ we obtain $\mathbf{x} \in H_0$. □

We denote by $\omega(\mathbf{x})$ the set of limit points of the trajectory (2.4).

Theorem 4.2. *For any initial* $\mathbf{x} \in \mathcal{F}_V$ *we have*

$$\omega(\mathbf{x}) \subset \begin{cases} H_0 & \textit{if} \quad |\sigma(\mathbf{x})| < 1,\\ H_1 & \textit{if} \quad |\sigma(\mathbf{x})| = 1,\\ H_\infty & \textit{if} \quad |\sigma(\mathbf{x})| > 1.\end{cases}$$

Proof. From (4.1) one easily has that

$$\sigma(\mathbf{x}^{n+1}) = \sigma(V(\mathbf{x}^n)) = (\sigma(\mathbf{x}^n))^2 = (\sigma(\mathbf{x}))^{2^n}.$$

Therefore

$$\lim_{n\to\infty} \sigma(\mathbf{x}^n) = \begin{cases} 0 & \text{if} \quad |\sigma(\mathbf{x})| < 1,\\ 1 & \text{if} \quad |\sigma(\mathbf{x})| = 1,\\ +\infty & \text{if} \quad |\sigma(\mathbf{x})| > 1.\end{cases}$$

□

5. *F*-Evolution Algebra Volterra Type

Let

$$E_0 = \{0, 1, \dots, m\}, \ \ F = \{1, 2, 3, \dots, m_1\}, \ \ M = \{m_1 + 1, \dots, m\}.$$

In this section we shall consider a special case of an F-evolution algebra giving the following additional condition on heredity coefficients

$$p_{ij,k} = 0 \ \text{ if } \ k \notin \{0, i, j\}, \ \text{ for all } \ i, j, k \in E_0. \tag{5.1}$$

The biological interpretation of condition (5.1) is clear: any pair of parents might have offspring which repeats one of them or might have not offspring.

An F-evolution algebra that satisfies condition (5.1) is called an *F-evolution algebra Volterra type* and denoted by $\mathcal{F}_V(m_1, m)$.

It is easy to see that the corresponding F-QSO Volterra type is of the form

$$V_{(m_1,m)} : \begin{cases} x_0' = x_0^2 + 2x_0 \sum\limits_{i\in E} x_i + \sum\limits_{i,j\in F} x_i x_j + 2\sum\limits_{i\in F}\sum\limits_{j\in M} p_{ij,0} x_i x_j + \sum\limits_{i,j\in M} x_i x_j; \\ x_i' = 2x_i(\mathbb{1}_{(i\in F)} \sum\limits_{j\in M} p_{ij,i} x_j + \mathbb{1}_{(i\in M)} \sum\limits_{j\in F} p_{ij,i} x_j), \qquad i = 1, 2, \dots, m, \end{cases} \tag{5.2}$$

where

$$p_{ij,0} = p_{ji,0} \geq 0, \ \ p_{ij,i} = p_{ji,i} \geq 0, \ \ p_{ij,0} + p_{ij,i} + p_{ij,j} = 1, \ \text{ for all } \ i, j \in E_0.$$

Let we shall describe the set of idempotent elements of an F-evolution algebra Volterra type. Denote by $\mathcal{I}d(\mathcal{F}_V(m_1, m))$ the set of all idempotent elements of an F-evolution algebra Volterra type. First we shall consider small cases to describe the full set of the idempotent elements.

Case 1: Let $E_0 = \{0, 1, 2\}$, $F = \{1\}$ and $M = \{2\}$. Then an idempotent element of the corresponding evolution algebra is a solution of the nonlinear system

$$\begin{cases} x_0 = x_0^2 + x_1^2 + x_2^2 + 2x_0x_1 + 2x_0x_2 + 2p_{12,0}x_1x_2, \\ x_1 = 2p_{12,1}x_1x_2, \\ x_2 = 2p_{12,2}x_1x_2, \end{cases}$$

where

$$p_{12,0}, p_{12,1}, p_{12,2} \in [0, 1]; \ \ p_{12,0} + p_{12,1} + p_{12,2} = 1.$$

It is easy to check that if $p_{12,1}, p_{12,2} > 0$ then

$$\mathcal{I}d(\mathcal{F}_V(1, 2)) = \{(0, 0, 0); (1, 0, 0); (\xi_0, \xi_1, \xi_2); (\eta_0, \eta_1, \eta_2)\}$$

and if $p_{12,1} = 0$ or $p_{12,2} = 0$ then

$$\mathcal{I}d(\mathcal{F}_V(1, 2)) = \{(0, 0, 0); (1, 0, 0)\},$$

where

$$\xi_0 = -\frac{1 - p_{12,0}}{2p_{12,1}p_{12,2}}, \ \ \eta_0 = 1 - \frac{1 - p_{12,0}}{2p_{12,1}p_{12,2}}, \ \ \xi_1 = \eta_1 = \frac{1}{2p_{12,2}}, \ \ \xi_2 = \eta_2 = \frac{1}{2p_{12,2}}.$$

Case 2: Let $E_0 = \{0,1,2,3\}$, $F = \{1,2\}$ and $M = \{3\}$. Then an idempotent element of the corresponding evolution algebra is a solution of the nonlinear system

$$\begin{cases} x_0 = x_0^2 + x_1^2 + x_2^2 + 2x_0x_1 + 2x_0x_2 + 2x_0x_3 + 2x_1x_2 + 2p_{13,0}x_1x_3 + 2p_{23,0}x_2x_3, \\ x_1 = 2p_{13,1}x_1x_3, \\ x_2 = 2p_{23,2}x_2x_3, \\ x_3 = 2x_3(p_{13,3}x_1 + p_{23,3}x_2), \end{cases}$$

where

$$p_{13,0}, p_{13,1}, p_{13,3}, p_{23,0}, p_{23,2}, p_{23,3} \in [0,1]; \ \ p_{13,0} + p_{13,1} + p_{13,3} = p_{23,0} + p_{23,2} + p_{23,3} = 1.$$

Subcase $x_1 = 0$: It is evident that if $p_{23,2}, p_{23,3} > 0$, then

$$\mathcal{I}_1 = \{(0,0,0,0); (1,0,0,0); (\xi_0, 0, \xi_2, \xi_3); (\eta_0, 0, \eta_2, \eta_3)\}$$

and if $p_{23,2} = 0$ or $p_{23,3} = 0$, then

$$\mathcal{I}_1 = \{(0,0,0,0); (1,0,0,0)\},$$

where

$$\xi_0 = -\frac{p_{23,2} + p_{23,3}}{2p_{23,2}p_{23,3}}, \ \ \eta_0 = 1 - \frac{p_{23,2} + p_{23,3}}{2p_{23,2}p_{23,3}}, \ \ \xi_2 = \eta_2 = \frac{1}{2p_{23,3}}, \ \ \xi_3 = \eta_3 = \frac{1}{2p_{23,2}}.$$

Subcase $x_2 = 0$: It is evident that if $p_{13,1}, p_{13,3} > 0$, then

$$\mathcal{I}_2 = \{(0,0,0,0); (1,0,0,0); (\xi_0, \xi_1, 0, \xi_3); (\eta_0, \eta_1, 0, \eta_3)\}$$

and if $p_{13,1} = 0$ or $p_{13,3} = 0$, then

$$\mathcal{I}_2 = \{(0,0,0,0); (1,0,0,0)\},$$

where

$$\xi_0 = -\frac{p_{13,1} + p_{13,3}}{2p_{13,1}p_{13,3}}, \ \ \eta_0 = 1 - \frac{p_{13,1} + p_{13,3}}{2p_{13,1}p_{13,3}}, \ \ \xi_1 = \eta_1 = \frac{1}{2p_{13,3}}, \ \ \xi_3 = \eta_3 = \frac{1}{2p_{13,1}}.$$

Subcase $x_1 \neq 0, x_2 \neq 0$: It is evident that if $p_{13,1} = p_{23,2} > 0$, then

$$\mathcal{I}_3 = \{\mathbf{x} : x_0 = -\sum_{k=1}^{3} x_k; \ p_{13,3}x_1 + p_{23,3}x_2 = \frac{1}{2}; \ x_3 = (2p_{13,1})^{-1}\}$$
$$\cup \{\mathbf{x} : x_0 = 1 - \sum_{k=1}^{3} x_k; \ p_{13,3}x_1 + p_{23,3}x_2 = \frac{1}{2}; \ x_3 = (2p_{13,1})^{-1}\}.$$

and if $p_{13,1} = 0$ or $p_{23,2} = 0$, then

$$\mathcal{I}_3 = \{(0,0,0,0); (1,0,0,0)\}$$

Consequently

$$\mathcal{I}d(\mathcal{F}_V(2,3)) = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$$

Case 3: Let us given the F-evolution algebras Volterra type $\mathcal{F}_V(m_1, m)$, we shall describe the set of all idempotent elements. Denote $\mathcal{F}_V(m_1, m)^{\neq} = \{\mathbf{x} : \prod_{i=0}^{m} x_i \neq 0\}$.

(a) We shall find the idempotents belonging to $I = \mathcal{I}d(\mathcal{F}_V(m_1, m)) \cap \mathcal{F}_V(m_1, m)^{\neq}$. Then, in this case, from (5.2) we obtain the following system of linear equations

$$\mathbb{1}_{(i\in F)} \sum_{j\in M} p_{ij,i}x_j + \mathbb{1}_{(i\in M)} \sum_{j\in F} p_{ij,i}x_j = \frac{1}{2}, \ \ i = 1, 2, \ldots, m,$$

or equivalently

$$\begin{cases} \displaystyle\sum_{j=m_1+1}^{m} p_{ij,i}x_j = \frac{1}{2}, & i = 1, 2, \ldots, m_1, \\ \displaystyle\sum_{j=1}^{m_1} p_{ij,i}x_j = \frac{1}{2}, & i = m_1+1, m_1+2, \ldots, m. \end{cases} \tag{5.3}$$

Denote

$$\mathbf{P} = \begin{pmatrix} 0 & \ldots & 0 & p_{1m_1+1,1} & \ldots & p_{1m,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & p_{m_1m_1+1,m_1} & \ldots & p_{m_1m,m_1} \\ p_{1m_1+1,m_1+1} & \ldots & p_{mm_1+1,m_1+1} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{1m,m} & \ldots & p_{mm,m} & 0 & \ldots & 0 \end{pmatrix} \tag{5.4}$$

and $\mathbf{1/2} = \frac{1}{2} \cdot \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)$. It is known by Kronecker-Capelli theorem that the system of linear equations (5.3) has a solution if and only if the rank of the matrix $\mathbf{P}$ is equal to the rank of its augmented matrix $(\mathbf{P}|\mathbf{1/2})$. Consequently:

– if $\operatorname{rank} \mathbf{P} = \operatorname{rank}(\mathbf{P}|\mathbf{1/2})$ and $\det(\mathbf{P}) \neq 0$ then $|I| = 1$;

– if $\operatorname{rank} \mathbf{P} = \operatorname{rank}(\mathbf{P}|\mathbf{1/2})$ and $\det(\mathbf{P}) = 0$ then $|I| = \infty$;

– if $\operatorname{rank} \mathbf{P} \neq \operatorname{rank}(\mathbf{P}|\mathbf{1/2})$ then $I = \emptyset$.

(b) Suppose $x_m = 0$. In this case using (5.2) one has that the evolution operator $V_{(m_1,m-1)}$ of the algebra $\mathcal{F}_V(m_1, m-1)$ is the restriction of the evolution operator $V_{(m_1,m)}$ of the algebra $\mathcal{F}_V(m_1, m)$. Using the above method we will find $\mathcal{I}d(\mathcal{F}_V(m_1, m-1)) \cap \mathcal{F}_V(m_1, m-1)^{\neq}$ and the elements of the form

$$\begin{aligned} &\mathbf{x} = (x_0, x_1, \ldots, x_{m-1}, 0) \in \mathcal{F}_V(m_1, m), \quad \text{where} \\ &\quad (x_0, x_1, \ldots, x_{m-1}) \in \mathcal{I}d(\mathcal{F}_V(m_1, m-1)) \cap \mathcal{F}_V(m_1, m-1)^{\neq} \end{aligned} \tag{5.5}$$

are idempotent elements of the $\mathcal{F}_V(m_1, m)$. Similarly one can consider the case $x_m = x_{m-1} = 0$ and find the idempotent elements for $x_m = 0$ and so on. We denote by I_{x_m} the set of all idempotents for $x_m = 0$. When we consider the case $x_{m-1} = 0$, we assume that $x_m > 0$ and repeat the above algorithm.

(c) Suppose $x_{m_1} = 0$. Similarly as in (a) using (5.2) one has that the evolution operator $V_{(m_1-1,m)}$ of the algebra $\mathcal{F}_V(m_1-1, m)$ is the restriction of the evolution operator $V_{(m_1,m)}$ of the algebra $\mathcal{F}_V(m_1, m)$. Using the above method we will find $\mathcal{I}d(\mathcal{F}_V(m_1-1, m)) \cap \mathcal{F}_V(m_1, m-1)^{\neq}$ and the elements of the form

$$\begin{aligned} &\mathbf{x} = (x_0, x_1, \ldots, x_{m_1-1}, 0, x_{m_1+1}, \ldots, x_m) \in \mathcal{F}_V(m_1, m), \quad \text{where} \\ &\quad (x_0, x_1, \ldots, x_{m_1-1}, x_{m_1+1}, \ldots, x_m) \in \mathcal{I}d(\mathcal{F}_V(m_1-1, m)) \cap \mathcal{F}_V(m_1, m-1)^{\neq} \end{aligned} \tag{5.6}$$

are idempotent elements of the $\mathcal{F}_V(m_1, m)$. We denote by $I_{x_{m_1}}$ the set of all idempotents of the form (5.6).

Assume $|M| = m - m_1 > |F| = m_1$. Then, in order to describe the set of all idempotent elements, first we make use of (a). Next step is to apply (b) $m - 2m_1$ times. Finally we apply items (b), (c) and so on and so forth. Since m is finite consequently we obtain the following sets of idempotent elements $I, I_{x_m}, \ldots, I_{x_1}$.

Thus

$$\mathcal{I}d(\mathcal{F}_V(m_1, m)) = \{\mathbf{0}\} \cup (I \cap H_0) \cup (I \cap H_1) \cup \bigcup_{i=1}^{m}(I_{x_i} \cap H_0) \cup \bigcup_{i=1}^{m}(I_{x_i} \cap H_1).$$

We have proved the following theorem.

Theorem 5.1. *The full set of the idempotents elements of an F-evolution algebra Volterra type has the form*

$$\mathcal{I}d(\mathcal{F}_V(m_1, m)) = \{\mathbf{0}\} \cup \Big((I \cup \bigcup_{i=1}^{m} I_{x_i}) \cap (H_0 \cup H_1)\Big).$$

An element $\mathbf{x}$ is called an *absolute nilpotent* if $\mathbf{x}^2 = 0$, i.e. $\mathbf{x}$ is zero point of the corresponding evolution operator. For an F-evolution algebra Volterra type $\mathcal{F}_V(m_1, m)$ the equation $\mathbf{x}^2 = 0$ is equivalent to the following system

$$\begin{cases} 0 = x_0^2 + 2x_0 \sum\limits_{i\in E} x_i + \sum\limits_{i,j\in F} x_i x_j + 2\sum\limits_{i\in F}\sum\limits_{j\in M} p_{ij,0} x_i x_j + \sum\limits_{i,j\in M} x_i x_j; \\ 0 = 2x_i(\mathbb{1}_{(i\in F)} \sum\limits_{j\in M} p_{ij,i} x_j + \mathbb{1}_{(i\in M)} \sum\limits_{j\in F} p_{ij,i} x_j), \qquad i = 1, 2, \ldots, m. \end{cases} \tag{5.7}$$

Denote by $\mathcal{N}(\mathcal{F}_V(m_1, m))$ the set of all absolute nilpotent elements of the algebra $\mathcal{F}_V(m_1, m)$.

(i) We shall find the absolute nilpotent elements belonging to $\mathcal{N}_0 = \mathcal{N}(\mathcal{F}_V(m_1, m)) \cap \mathcal{F}_V(m_1, m)^{\neq}$. In this case, from (5.7) we obtain the following system of linear equations

$$\mathbb{1}_{(i\in F)} \sum_{j\in M} p_{ij,i} x_j + \mathbb{1}_{(i\in M)} \sum_{j\in F} p_{ij,i} x_j = 0, \quad i = 1, 2, \ldots, m,$$

or equivalently

$$\mathbf{P}\mathbf{x} = 0, \tag{5.8}$$

where $\mathbf{P}$ is as in (5.4) and $\mathbf{x} = (x_1, \ldots, x_m)$.

It is very known that the system of linear equations (5.8) has a unique solution if $\det \mathbf{P} \neq 0$ and has infinitely many solutions if $\det \mathbf{P} = 0$. If $\det \mathbf{P} \neq 0$ then the system (5.8) has solution $(x_1, \ldots, x_m) = (0, \ldots, 0)$. By Proposition 4.1, a zero point belongs to H_0 and therefore $x_0 = -\sum\limits_{i=1}^{m} x_i = 0$ so $\mathbf{x} = 0$, and it contradicts the assumption $\mathbf{x} \in \mathcal{N}_0$. If $\det \mathbf{P} = 0$, then we obtain infinitely many solutions $(x_1^*, \ldots, x_m^*)$ of (5.8) and substituting this solution in $x_0 = -\sum\limits_{i=1}^{m} x_i = 0$ we get x_0^*, and consequently we obtain infinitely many elements of $\mathcal{N}_0$.

(ii) Suppose $x_m = 0$. Similarly as in (b) the evolution operator $V_{(m_1, m-1)}$ of the algebra $\mathcal{F}_V(m_1, m-1)$ is the restriction of the evolution operator $V_{(m_1, m)}$ of the algebra $\mathcal{F}_V(m_1, m)$. Using the above method (i), we will find $\mathcal{N}(\mathcal{F}_V(m_1, m-1)) \cap \mathcal{F}_V(m_1, m-1)^{\neq}$ and the elements of the form

$$\begin{aligned} &\mathbf{x} = (x_0, x_1, \ldots, x_{m-1}, 0) \in \mathcal{F}_V(m_1, m), \quad \text{where} \\ &\quad (x_0, x_1, \ldots, x_{m-1}) \in \mathcal{N}(\mathcal{F}_V(m_1, m-1)) \cap \mathcal{F}_V(m_1, m-1)^{\neq} \end{aligned} \tag{5.9}$$

are absolute nilpotent elements of $\mathcal{F}_V(m_1, m)$. Similarly one can consider the case $x_m = x_{m-1} = 0$ and find the absolute nilpotent elements for $x_m = 0$ and so on. We denote by $\mathcal{N}_{x_m}$ the set of all absolute nilpotent elements for $x_m = 0$. When we consider the case $x_{m-1} = 0$, we assume that $x_m > 0$ and repeat the above algorithm.

(iii) Suppose $x_{m_1} = 0$. Similarly as in (c) the evolution operator $V_{(m_1-1,m)}$ of the algebra $\mathcal{F}_V(m_1-1,m)$ is the restriction of the evolution operator $V_{(m_1,m)}$ of the algebra $\mathcal{F}_V(m_1,m)$. Using the above method of (i), we will find $\mathcal{N}(\mathcal{F}_V(m_1-1,m)) \cap \mathcal{F}_V(m_1,m-1)^{\neq}$ and the elements of the form

$$\begin{aligned} \mathbf{x} = (x_0, x_1, \ldots, x_{m_1-1}, 0, x_{m_1+1}, \ldots, x_m) \in \mathcal{F}_V(m_1,m), \quad \text{where} \\ (x_0, x_1, \ldots, x_{m_1-1}, x_{m_1+1}, \ldots, x_m) \in \mathcal{N}(\mathcal{F}_V(m_1-1,m)) \cap \mathcal{F}_V(m_1,m-1)^{\neq} \end{aligned} \tag{5.10}$$

are absolute nilpotent elements of $\mathcal{F}_V(m_1,m)$. We denote by $\mathcal{N}_{x_{m_1}}$ the set of all absolute nilpotent elements of the form (5.10).

Assume $|M| = m - m_1 > |F| = m_1$. Then, in order to describe the set of all absolute nilpotent elements, first we make use of (i). Next step is to apply (ii) $m - 2m_1$ times. Finally we apply items (ii), (iii) and so on and so forth. Since m is finite we get the following sets of absolute nilpotent elements $\mathcal{N}_0, \mathcal{N}_{x_m}, \ldots, \mathcal{N}_{x_1}$. Therefore

$$\mathcal{N}(\mathcal{F}_V(m_1,m)) = \{\mathbf{0}\} \cup \mathcal{N}_0 \cup \bigcup_{i=1}^{m} \mathcal{N}_{x_i}.$$

The following theorem describes the full set of absolute nilpotent elements.

Theorem 5.2. *The full set of absolute nilpotent elements of an F-evolution algebra Volterra type has the form*

$$\mathcal{N}(\mathcal{F}_V(m_1,m)) = \{\mathbf{0}\} \cup \mathcal{N}_0 \cup \bigcup_{i=1}^{m} \mathcal{N}_{x_i}.$$

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