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Korovkin Type Approximation Theorem via $A_2^{\mathcal{I}}$-Summability Methods

Sudipta Dutta[a], Sevda Akdağ[b], Pratulananda Das[c]

[a]*Department of Mathematics, Jhargram Raj College, Jhargram, Paschim Medinipur, Pin-721507, West Bengal, India*
[b]*Department of Mathematics, Faculty of Arts and Sciences, Sinop University 57000 Sinop, Turkey*
[c]*Department of Mathematics, Jadavpur University, Jadavpur, Kol-700032, West Bengal, India*

Abstract. In this paper we consider the notion of $A_2^{\mathcal{I}}$-summability for real double sequences which is an extension of the notion of $A^{\mathcal{I}}$-summability for real single sequences introduced by Savas, Das and Dutta. We primarily apply this new notion to prove a Korovkin type approximation theorem. In the last section, we study the rate of $A_2^{\mathcal{I}}$-summability.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. Approximation theory has important applications in the theory of polynomial approximation in various areas of functional analysis. For a sequence $\{T_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [23] first established the necessary and sufficient conditions for the uniform convergence of $\{T_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1$, $e_2 = x$, $e_3 = x^2$ [1]. The study of the Korovkin type approximation theory has a long history and is a well-established area of research. As is mentioned in [13] in particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of positive linear operators such as the interpolation operator of Hermit-Fejér [7]. In recent years, using the concept of uniform statistical convergence various statistical approximation results have been proved ([15, 16]). Erkuş and Duman [18] studied a Korovkin type approximation theorem via A-statistical convergence in the space $H_w(I^2)$ where $I^2 = [0,\infty) \times [0,\infty)$ which was extended for double sequences of positive linear operators of two variables in A-statistical sense by Demirci and Dirik in [13]. Further it was extended for double sequences of positive linear operators of two variables in $A_2^{\mathcal{I}}$-statistical sense by Dutta and Das [17]. Our primary interest, in this paper is to obtain a general Korovkin type approximation theorem for double sequences of positive linear operators of two variables from $C(\mathcal{K})$ to $C(\mathcal{K})$ where $\mathcal{K}$ is a compact subset of the real two dimensional space, in the sense of $A_2^{\mathcal{I}}$-summability method.

The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [20]. Further investigations started in this area after the pioneering works of Šalát [36] and Fridy [21]. The notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et. al. [26] as a generalization

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Email addresses: dutta.sudipta@ymail.com (Sudipta Dutta), skarakus@sinop.edu.tr (Sevda Akdağ), pratulananda@yahoo.co.in (Pratulananda Das)

of statistical convergence using the notion of ideals (see [4–6] for further references). Later the idea of $\mathcal{I}$-convergence was also studied in topological spaces in [27]. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk ([24, 25]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [2, 3, 8, 9, 11, 12, 19, 22, 24, 25, 28, 32, 33, 37]). In particular very recently in [38] and [39] the two above mentioned approaches were unified and the very general notion of $A^{\mathcal{I}}$-statistical convergence and $A^{\mathcal{I}}$-summability was introduced and studied. In this paper we consider an extension of this notion to double sequences, namely $A_2^{\mathcal{I}}$-summability convergence.

A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ for all $m, n > N(\varepsilon)$ and denoted by $\lim\limits_{m,n} x_{mn} = L$ ([34]). A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is statistically convergent to L if for every $\varepsilon > 0$, $\lim\limits_{j,k} \dfrac{\left|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon\}\right|}{jk} = 0$ ([29–31]).

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if $(i) A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; $(ii) A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e. $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N} \times \mathbb{N}$: there is $m(A) \in \mathbb{N}$ such that $i, j \geq m(A) \Longrightarrow (i, j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [10]. Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$ a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I}$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ ([38, 39]).

Let $A = (a_{jkmn})$ be a four dimensional summability matrix. For a given double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j, k) \in \mathbb{N}^2$. In 1926, Robison [35] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a convergent double sequence is not necessarily bounded.

Recall that a four dimensional matrix $A = (a_{jkmn})$ is said to be RH-regular if it maps every bounded convergent double sequence into a convergent double sequence with the same limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = (a_{jkmn})$ is RH-regular if and only if

(i) $\lim\limits_{j,k} a_{jkmn} = 0$ for each $(m, n) \in \mathbb{N}^2$,

(ii) $\lim\limits_{j,k} \sum\limits_{(m,n)\in\mathbb{N}^2} a_{jkmn} = 1$,

(iii) $\lim\limits_{j,k} \sum\limits_{m\in\mathbb{N}} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $\lim\limits_{j,k} \sum\limits_{n\in\mathbb{N}} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{jkmn}|$ is convergent for each $(j,k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers M_0 and N_0 such that $\sum_{m,n>N_0} |a_{jkmn}| < M_0$

holds for every $(j,k) \in \mathbb{N}^2$.

Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix and let $K \subset \mathbb{N}^2$. Then the A-density of K is given by

$$\delta_A^{(2)}\{K\} = \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}$$

provided the limit exists. A real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be A-statistically convergent to a number L if for every $\varepsilon > 0$

$$\delta_A^{(2)}\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\} = 0.$$

We denote $\mathcal{I}_{\delta_A^{(2)}} = \left\{C \subset \mathbb{N}^2 : \delta_A^{(2)}\{C\} = 0\right\}$ which is an admissible ideal in $\mathbb{N} \times \mathbb{N}$. Throughout this paper we use the symbol $\mathcal{I}$ as a non-trivial strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$.

2. A Korovkin Type Approximation Theorem

Recently the concept of $A^{\mathcal{I}}$-summability for real single sequences has been introduced by Savas, Das and Dutta [38, 39] which are strictly weaker than the notion of statistical summability. In this paper we consider the following natural extension of these summability for real double sequences.

The following definition is due to E. Savas (who has informed about it in a personal communication).

Definition 2.1. *A real double sequence $x = \{x_{m,n}\}_{m,n\in\mathbb{N}}$ is said to be $\mathcal{I}_2$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,*
$\left\{(j,k) \in \mathbb{N}^2 : \frac{1}{jk}|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$.

We now introduce the main definition of this paper.

Definition 2.2. *Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Then a real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-statistically convergent to a number L if for every $\varepsilon > 0$ and $\delta > 0$,*
$$\left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\right\} \in \mathcal{I}$$
where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\}$.

Definition 2.3. *Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Then a real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-summable to a number L if for every $\varepsilon > 0$,*
$\left\{(j,k) \in \mathbb{N}^2 : |(Ax)_{j,k} - L| \geq \varepsilon\right\} \in \mathcal{I}$.

Thus $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to a number L if and only if $(Ax)_{j,k}$ is $\mathcal{I}_2$-convergent to L. In this case, we write $\mathcal{I}_2\text{-}\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn} = L$.

It should be noted that, if we take $\mathcal{I} = \mathcal{I}_d$, the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero, then $A_2^{\mathcal{I}}$-summability reduces to the notion of statistical A-summability.

Now we prove the the following relation between $A_2^{\mathcal{I}}$-summability and $A_2^{\mathcal{I}}$-statistical convergence.

Theorem 2.4. *If a double sequence is bounded and $A_2^{\mathcal{I}}$-statistically convergent to L then it is $A_2^{\mathcal{I}}$-summable to L.*

Proof. Let $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ be bounded and $A_2^{\mathcal{I}}$-statistically convergent to L and for $\varepsilon > 0$, let $K(\frac{\varepsilon}{2}) := \{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \frac{\varepsilon}{2}\}$. Then

$$\begin{aligned}
|(Ax)_{j,k} - L| &\leq \left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}(x_{mn} - L)\right| + |L|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} - 1\right| \\
&\leq \left|\sum_{(m,n)\notin K(\frac{\varepsilon}{2})} a_{jkmn}(x_{mn} - L)\right| + \left|\sum_{(m,n)\in K(\frac{\varepsilon}{2})} a_{jkmn}(x_{mn} - L)\right| \\
&\quad + |L|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} - 1\right| \\
&\leq \frac{\varepsilon}{2}\sum_{(m,n)\notin K(\frac{\varepsilon}{2})} a_{jkmn} + \sup_{(m,n)\in\mathbb{N}^2} |x_{mn} - L| \sum_{(m,n)\in K(\frac{\varepsilon}{2})} a_{jkmn} \\
&\quad + |L|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} - 1\right|
\end{aligned}$$

Then since $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-statistically convergent to L and the matrix A is a RH-regular matrix so $\left\{(j,k) \in \mathbb{N}^2 : |(Ax)_{j,k} - L| \geq \varepsilon\right\} \in \mathcal{I}$ and this consequently implies that x is $A_2^{\mathcal{I}}$-summable to L. □

The next example disproves the converse of the Theorem 2.1.

Example 2.5. *Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$. Choose an infinite subset $C = \{(p_i, q_i) : i \in \mathbb{N}\}$ from $\mathcal{I}$ where $p_i \neq q_i$, $p_1 < p_2 < \ldots$, and $q_1 < q_2 < \ldots$. Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by*

$$u_{mn} = \begin{cases} 1 & m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ be given by

$$a_{jkmn} = \begin{cases} \frac{1}{4} & \text{if } (j,k) \neq (p_i, q_i) \text{ for any } i \in \mathbb{N},\ m = j^2, j^2+1, n = k^2, k^2+1 \\ 1 & \text{if } j = p_i,\ k = q_i,\ m = p_i^2,\ n = q_i^2 \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$y_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}u_{mn} = \begin{cases} \frac{1}{4} & \text{if } (j,k) \neq (p_i, q_i) \text{ for any } i \in \mathbb{N} \\ 0 & \text{if } j = p_i,\ k = q_i,\ m = p_i^2,\ n = q_i^2 \text{ are odd} \\ 1 & \text{if } j = p_i,\ k = q_i,\ m = p_i^2,\ n = q_i^2 \text{ are even.} \end{cases}$$

Let $\varepsilon > 0$ be given. Then $\{(j,k) \in \mathbb{N}^2 : |y_{j,k} - \frac{1}{4}| \geq \varepsilon\} = C \in \mathcal{I}$. Then the sequence $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to $\frac{1}{4}$.

Again for $0 < \varepsilon < \frac{1}{4}$, $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : |u_{mn} - \frac{1}{4}| \geq \varepsilon\} = \mathbb{N}\times\mathbb{N}$. Therefore

$$\sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} = \frac{1}{4} \text{ for all } (i,j) \in (\mathbb{N}\times\mathbb{N}) \setminus C.$$

Consequently for $0 < \delta < \frac{1}{4}$, $(\mathbb{N}\times\mathbb{N}) \setminus C \subset \left\{(i,j) \in \mathbb{N}^2 : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\right\}$. This shows that $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is not $A_2^{\mathcal{I}}$-statistically convergent.

We consider the set of all real valued continuous functions on any compact subset $\mathcal{K}$ of the real two dimensional space. This space is endowed with the supremum norm $\|f\| = \sup_{(x,y)\in\mathcal{K}} |f(x,y)|,\ f \in C(\mathcal{K})$. Throughout the paper we will use the following test functions $f_0(x,y) = 1,\ \ f_1(x,y) = x,\ \ f_2 = y,\ \ f_3(x,y) = x^2 + y^2$ and we denote the value of Tf at a point $(u,v) \in \mathcal{K}$ by $T(f;u,v)$.

Now we establish the Korovkin type approximation theorem by using $A_2^{\mathcal{I}}$-summability method.

Theorem 2.6. *Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix and $\{T_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators from $C(\mathcal{K})$ into $C(\mathcal{K})$. Then for any $f \in C(\mathcal{K})$,*

$$\mathcal{I}_2\text{-}\lim_{j,k} \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f) - f\| = 0 \tag{1}$$

is satisfied if the following hold

$$\mathcal{I}_2\text{-}\lim_{j,k} \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_i) - f_i\| = 0,\ i = 0,1,2,3. \tag{2}$$

Proof. Assume that (2) holds. Let $f \in C(\mathcal{K})$. Our objective is to show that for given $\varepsilon > 0$ there exist constants $C_0,\ C_1,\ C_2,\ C_4$ (depending on $\varepsilon > 0$) such that

$$\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f) - f\| \leq \varepsilon + \sum_{i=0}^{3} C_i \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_i) - f_i\|.$$

If this is done then our hypothesis implies that for any $\delta > 0$,

$$\left\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f) - f\| \geq \delta\right\} \in \mathcal{I}.$$

Let $f \in C(\mathcal{K})$. Since f is continuous on the compact set $\mathcal{K}$, so $|f(x,y)| \leq M$ where $M = \|f\|$. Also since f is continuous on $\mathcal{K}$ then for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|f(x,y) - f(u,v)| < \varepsilon$ for all $(x,y) \in \mathcal{K}$ satisfying $|x - u| < \delta$ and $|v - y| < \delta$. Hence we get $|f(x,y) - f(u,v)| < \varepsilon + \frac{2M}{\delta^2}\left\{(x-u)^2 + (y-v)^2\right\}$.

Since each T_{mn} is a positive linear operator then we have for each $(u,v) \in \mathcal{K}$, $\Big|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f;u,v) -$

$$\begin{aligned}
&f(u,v)\Big| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(|f(x,y) - f(u,v)|;u,v) \\
&+ |f(u,v)|\Big|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0;u,v) - f_0(u,v)\Big| \\
&\leq \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(x-u)^2 + (y-v)^2\right\};u,v\right) \\
&+ M\Big|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0;u,v) - f_0(u,v)\Big| \\
&\leq \varepsilon + \left(\varepsilon + M + \frac{2M}{\delta^2}(E^2 + F^2)\right)\Big|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0;u,v) - f_0(u,v)\Big| \\
&+ \frac{2M}{\delta^2}\Big|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_3;u,v) - f_3(u,v)\Big| \\
&+ \frac{4ME}{\delta^2}\Big|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_1;u,v) - f_1(u,v)\Big|
\end{aligned}$$

$+ \frac{4MF}{\delta^2}\Big| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_2; u, v) - f_2(u, v)\Big|$
where $E = \max_{\mathcal{K}}|x|$ and $F = \max_{\mathcal{K}}|y|$.
Taking supremum over $(u, v) \in \mathcal{K}$

$$\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f\| \leq \varepsilon + \sum_{i=0}^{3} C_i \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_i) - f_i\|$$

where $C_0 = \varepsilon + M + \frac{2M}{\delta^2}(E^2 + F^2),\ C_1 = \frac{4ME}{\delta^2},\ C_2 = \frac{4MF}{\delta^2},\ C_3 = \frac{2M}{\delta^2}$.
Hence

$$\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f\| \leq \varepsilon + C\left\{\sum_{i=0}^{3} \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_i) - f_i\|\right\}$$

where $C = max\{C_0, C_1, C_2, C_3\}$.

For a given $\gamma > 0$, choose $\varepsilon < \gamma$. Now let

$$U = \left\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f\| \geq \gamma\right\},$$

and

$$U_i = \left\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_i) - f_i\| \geq \frac{\gamma - \varepsilon}{4C}\right\},\ i = 0, 1, 2, 3.$$

It follows that $U \subset \bigcup_{i=0}^{3} U_i$. By hypothesis each $U_i \in \mathcal{I},\ i = 0, 1, 2, 3$ and consequently $U \in \mathcal{I}$ i.e.

$$\left\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f\| \geq \gamma\right\} \in \mathcal{I}.$$

This completes the proof of the theorem. □

Remark 2.7. *We now show that our theorem is stronger than the statistical A-summable version [14] (and so the classical version). Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$. Choose an infinite subset $C = \{(p_i, q_i) : i \in \mathbb{N}\}$ (where $p_i \neq q_i,\ p_1 < p_2 < \ldots$, and $q_1 < q_2 < \ldots$) from $\mathcal{I} \setminus \mathcal{I}_d$ where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero. Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by*

$$u_{mn} = \begin{cases} 1 & \text{if } m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ be given by

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i, m = 2j+1, n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$y_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} u_{mn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}. \end{cases}$$

Let $\varepsilon > 0$ be given. Then $\{(j,k) \in \mathbb{N}^2 : |y_{j,k} - 0| \geq \varepsilon\} = C \in \mathcal{I}$. Then the sequence $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to 0. Evidently this sequence is not statistically A-summable to 0.

Let $\mathcal{K} = [0,1] \times [0,1]$. Now we consider the double sequence $\{T_{mn}\}_{m,n\in\mathbb{N}}$ of positive linear operators defined by $T_{mn}(f;x,y) = (1 + u_{mn})B_{mn}(f;x,y)$ where $\{B_{mn}\}_{m,n\in\mathbb{N}}$ are the Berstein Polynomial of the two variables defined on $C(\mathcal{K})$ by $B_{mn}(f;x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n}\right)\binom{m}{j} x^j(1-x)^{m-j}\binom{n}{k} y^k(1-y)^{n-k}$. Then observe that

$$T_{mn}(f_0;x,y) = (1 + u_{mn})f_0(x,y),$$

$$T_{mn}(f_1;x,y) = (1 + u_{mn})f_1(x,y),$$

$$T_{mn}(f_2;x,y) = (1 + u_{mn})f_2(x,y),$$

$$T_{mn}(f_3;x,y) = (1 + u_{mn})\left[f_3(x,y) + \frac{x - x^2}{m} + \frac{y - y^2}{n}\right].$$

Now as A is a nonnegative RH-regular summability matrix and $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to 0 then for any $\varepsilon > 0$,

$$\left\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_i) - f_i\| \geq \varepsilon\right\} \in \mathcal{I},\ i = 0,1,2,3.$$

Therefore by previous theorem

$$\left\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f\| \geq \varepsilon\right\} \in \mathcal{I}.$$

But since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is not usual convergent and statistical A-summable so we can say that the classical version and statistical A-summable version of the previous theorem do not work for the operator defined above.

3. Rate of $A_2^{\mathcal{I}}$-summability

In this section we present a way to compute the rate of $A_2^{\mathcal{I}}$-summability methods of positive linear operators using the modulus of continuity in Theorem 2.2. Let $f \in C(\mathcal{K})$. Then the modulus of continuity for $\delta > 0$ is given by

$$w(f;\delta) = \sup\{|f(x,y) - f(u,v)| : (u,v), (x,y) \in \mathcal{K}, \sqrt{(x-u)^2 + (y-v)^2} \leq \delta\}.$$

Theorem 3.1. *Let $\{T_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators from $C(\mathcal{K})$ into $C(\mathcal{K})$. Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Assume that the following conditions hold*

(i) $\mathcal{I}_2\text{-}\lim_{j,k} \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0) - f_0\| = 0,$

(ii) $\mathcal{I}_2\text{-}\lim_{j,k} w(f;\delta) = 0$ *where* $\delta := \delta_{(j,k)} = \sqrt{\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(\psi)\|}$ *with* $\psi(x,y) = (x-u)^2 + (y-v)^2$. *Then*

$$\mathcal{I}_2\text{-}\lim_{j,k} \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f\| = 0$$

for any $f \in C(\mathcal{K})$.

Proof. Let $(u,v) \in \mathcal{K}$ and $f \in C(\mathcal{K})$ be fixed. Then for all $(m,n) \in \mathbb{N}^2$ and any $\delta > 0$ we have

$$
\begin{aligned}
\left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f;u,v) - f(u,v) \right| \leq\; & \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(|f(x,y) - f(u,v)|;u,v) \\
& + |f(u,v)| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0;u,v) - f_0(u,v) \right| \\
\leq\; & w(f;\delta) \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn} \left(1 + \frac{\sqrt{(x-u)^2 + (y-v)^2}}{\delta}; u,v \right) \\
& + \|f\| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0;u,v) - f_0(u,v) \right| \\
\leq\; & w(f;\delta) \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0;u,v) - w(f;\delta) f_0(u,v) \\
& + w(f;\delta) + \frac{w(f;\delta)}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(\psi;u,v) \\
& + \|f\| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0;u,v) - f_0(u,v) \right| \\
\leq\; & w(f;\delta) \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0;u,v) - f_0(u,v) \right| + w(f;\delta) \\
& + \frac{w(f;\delta)}{\delta^2} \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(\psi;u,v) \right| \\
& + \|f\| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0;u,v) - f_0(u,v) \right|.
\end{aligned}
$$

Taking supremum over $(u,v) \in \mathcal{K}$

$$
\begin{aligned}
\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f) - f \| \leq\; & w(f;\delta) \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0) - f_0 \| \\
& + w(f;\delta) + \frac{w(f;\delta)}{\delta^2} \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(\psi) \| \\
& + \|f\| \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} T_{mn}(f_0) - f_0 \|.
\end{aligned}
$$

If we take $\delta := \delta_{(j,k)} = \sqrt{\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(\psi)\|}$ then

$$\begin{aligned}
\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f) - f\| &\leq w(f;\delta_{(j,k)})\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0) - f_0\| + 2w(f;\delta_{(j,k)}) \\
&\quad +\|f\|\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0) - f_0\| \\
&\leq M\Big\{w(f;\delta_{(j,k)})\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}f_0 - f_0\| + w(f;\delta_{(j,k)}) \\
&\quad +\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0) - f_0\|\Big\}
\end{aligned}$$

where $M = max\{2, \|f\|\}$. Let $\mu > 0$ be given. Now consider the following sets

$$U = \Big\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f) - f\| \geq \mu\Big\},$$

$$U_1 = \Big\{(j,k) \in \mathbb{N}^2 : w(f;\delta_{(j,k)}) \geq \frac{\mu}{3M}\Big\},$$

$$U_2 = \Big\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0) - f_0\| \geq \frac{\mu}{3M}\Big\},$$

$$U_3 = \Big\{(j,k) \in \mathbb{N}^2 : w(f;\delta_{(j,k)})\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0) - f_0\| \geq \frac{\mu}{3M}\Big\}.$$

Therefore $U \subset U_1 \cup U_2 \cup U_3$. Also define $U_3{}' = \Big\{(j,k) \in \mathbb{N}^2 : w(f;\delta_{(j,k)}) \geq \sqrt{\frac{\mu}{3M}}\Big\}$ and $U_3{}'' = \Big\{(j,k) \in \mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f_0) - f_0\| \geq \sqrt{\frac{\mu}{3M}}\Big\}$. Hence $U \subset U_1 \cup U_2 \cup U_3{}' \cup U_3{}''$. Now U_1, U_2, $U_3{}'$, $U_3{}''$ belong to $\mathcal{I}$. So U is also belong to $\mathcal{I}$. Therefore $\mathcal{I}_2\text{-}\lim_{j,k} \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}T_{mn}(f) - f\| = 0$. This completes the proof. □

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