Some Geometric Characteristics and Perturbations of Semi-Fredholm Operators

Snežana Č. Živković-Zlatanović[a]

[a]University of Niš
Faculty of Sciences and Mathematics
P.O. Box 224, 18000 Niš, Serbia

Abstract. We consider some geometric characteristics of bounded operators on Banach spaces concerning the sets of upper and lower semi-Browder operators and left and right Browder operators. Using various operational quantities we give some perturbation results for upper and lower semi-Fredholm, Weyl and semi-Browder operators as well as for left and right Fredolm, Weyl and Browder operators.

1. Introduction

Let $\mathbb{C}$ be the set of all complex numbers and let X, Y, Z be infinite dimensional complex Banach spaces. Let $B(X, Y)$ be the set of all linear bounded operators from X to Y, and denote the set of all compact (finite rank) operators from X to Y by $K(X, Y)$ ($F(X, Y)$).

For $A \in B(X, Y)$ we use $N(A)$ and $R(A)$, respectively, to denote the null-space and the range of A. Let $\alpha(A) = \dim N(A)$ if $N(A)$ is finite dimensional, and let $\alpha(A) = \infty$ if $N(A)$ is infinite dimensional. Similarly, let $\beta(A) = \dim Y/R(A) = \operatorname{codim}R(A)$ if $Y/R(A)$ is finite dimensional, and let $\beta(A) = \infty$ if $Y/R(A)$ is infinite dimensional.

An operator $A \in B(X, Y)$ is called upper semi-Fredholm, or $A \in \Phi_+(X, Y)$, if $\alpha(A) < \infty$ and $R(A)$ is closed, while $A \in B(X, Y)$ is called lower semi-Fredholm, or $A \in \Phi_-(X, Y)$, if $\beta(A) < \infty$. The set of semi-Fredholm operators is defined by $\Phi_\pm(X, Y) = \Phi_+(X, Y) \cup \Phi_-(X, Y)$, while the set of Fredholm operators is defined by $\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y)$. If $A \in \Phi_\pm(X, Y)$, the index is defined by $i(A) = \alpha(A) - \beta(A)$.

The set of upper Weyl operators, denoted by $\mathcal{W}_+(X, Y)$, is the set of upper semi-Fredholm operators with non-positive index, and the set of lower Weyl operators, denoted by $\mathcal{W}_-(X, Y)$, is the set of lower semi-Fredholm operators with non-negative index. The set of Weyl operators is defined by $\mathcal{W}(X, Y) = \mathcal{W}_+(X, Y) \cap \mathcal{W}_-(X, Y) = \{A \in \Phi(X, Y) : i(A) = 0\}$.

An operator $A \in B(X, Y)$ is relatively regular (or g-invertible) if there exists $B \in B(Y, X)$ such that $ABA = A$. It is well-known that A is relatively regular if and only if $R(A)$ and $N(A)$ are closed and complemented subspaces of Y and X, respectively.

An operator $A \in B(X, Y)$ is called left Fredholm, or $A \in \Phi_l(X, Y)$, if A is relatively regular upper semi-Fredholm, i.e.

$$\Phi_l(X, Y) = \{A \in B(X, Y) : R(A) \text{ is a closed and complemented}$$
$$\text{subspace of } Y \text{ and } \alpha(A) < \infty\}.$$

Also, $A \in B(X, Y)$ is called right Fredholm, or $A \in \Phi_r(X, Y)$, if A is relatively regular lower semi-Fredholm, i.e.

$$\Phi_r(X, Y) = \{A \in B(X, Y) : N(A) \text{ is a complemented subspace of } X$$
$$\text{and } \beta(A) < \infty\}.$$

An operator $A \in B(X, Y)$ is left (right) Weyl if A is left (right) Fredholm operator with non-positive (non-negative) index. We use $\mathcal{W}_l(X, Y)$ ($\mathcal{W}_r(X, Y)$) to denote the set of all left (right) Weyl operators. Evidently, A is left (right) Weyl iff A is upper (lower) Weyl and relatively regular.

For $H = B$, F, K, Φ_+, Φ_-, $\Phi_\pm$, Φ, Φ_l, Φ_r, $\mathcal{W}_+$, $\mathcal{W}_-$, $\mathcal{W}$, $\mathcal{W}_l$, $\mathcal{W}_r$, instead of $H(X, X)$ we write $H(X)$.

The ascent of $A \in B(X)$, denoted by asc(A), is the smallest $n \in N$ such that $N(A^n) = N(A^{n+1})$. If such n does not exist, then asc$(A) = \infty$. The descent of A, denoted by dsc(A), is the smallest $n \in N$ such that $R(A^n) = R(A^{n+1})$. If such n does not exist, then dsc$(A) = \infty$. An operator $A \in B(X)$ is upper semi-Browder if it is upper semi-Fredholm of finite ascent, and A is lower semi-Browder if it is lower semi-Fredholm of finite descent. Let $\mathcal{B}_+(X)$ ($\mathcal{B}_-(X)$) denote the set of all upper (lower) semi-Browder operators. The set of Browder operators is defined as $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

The operator $A \in B(X)$ is left Browder if it is left Fredholm of finite ascent, and A is right Browder if it is right Fredholm of finite descent. Let $\mathcal{B}_l(X)$ ($\mathcal{B}_r(X)$) denote the set of all left (right) Browder operators. These classes of operators were introduced in [47]. Evidently, $A \in B(X)$ is left (right) Browder iff A is upper (lower) semi-Browder and relatively regular.

The Calkin algebra over X is the quotient algebra $C(X) = B(X)/K(X)$, and $\pi : B(X) \to C(X)$ denotes the natural homomorphism. Let $r_e(A)$ denote spectral radius of the element $\pi(A)$ in $C(X)$, $A \in B(X)$, i.e. $r_e(A) = \lim_{n \to \infty} (\|\pi(A^n)\|)^{\frac{1}{n}}$ and it is called essential spectral radius of A. An operator $A \in B(X)$ is Riesz, or $A \in R(X)$, if $\{\lambda \in \mathbb{C} : A - \lambda \in \Phi(X)\} = \mathbb{C}\backslash\{0\}$, i.e. $r_e(A) = 0$.

Let S be a subset of a Banach space A. The perturbation class of S, denoted by $P(S)$, is the set

$$P(S) = \{a \in A : a + s \in S \text{ for every } s \in S\}.$$

It is known that ([28] Theorem 3.7)

$$P(\mathcal{W}_+(X)) = P(\Phi_+(X)) \text{ and } P(\mathcal{W}_-(X)) = P(\Phi_-(X)), \tag{1}$$

and ([20] Theorem 2.7), ([6] (Chapter 5.2, Corollary 3)),

$$P(\Phi_l(X)) = P(\Phi(X)) = P(\Phi_r(X)). \tag{2}$$

Also, it is known that ([20], Theorems 2.7 and 2.8)

$$P(\mathcal{W}_l(X)) = P(\mathcal{W}_r(X)) = P(\mathcal{W}(X)) = P(\Phi(X)). \tag{3}$$

and, $P(\Phi(X))$, $P(\Phi_+(X))$ and $P(\Phi_-(X))$ are closed two-sided ideals ([20], Theorem 2.4).

For $a \in X$ we set $K(a, \epsilon) = \{b \in X : \|a - b\| < \epsilon\}$ and $K[a, \epsilon] = \{b \in X : \|a - b\| \leq \epsilon\}$.

In this paper an operational quantity a is a procedure explaining how to get a non-negative number $a(A)$ for every $A \in B(X, Y)$ and every infinite dimensional Banach spaces X and Y.

We list some operational quantities.

The minimum modulus $m(A)$ of $A \in B(X, Y)$ is defined by

$$m(A) = \{\|Ax\| : x \in X, \|x\| = 1\},$$

and the surjection modulus is defined by

$$q(A) = \{\epsilon \geq 0 : \epsilon K[0, 1]_Y \subset A(K[0, 1]_X)\},$$

where $K[0, 1]_X$ ($K[0, 1]_Y$) denotes the closed unit ball of the space X (Y).

An operator $A \in B(X, Y)$ is strictly singular if, for every infinite dimensional (closed) subspace M of X, the restriction $A|_M$ is not an isomorphism, i.e. $m(AJ_M) = 0$, where J_M is the natural inclusion of M into X. An operator $A \in B(X, Y)$ is strictly cosingular if, for every infinite codimensional closed subspace V of Y, the composition $Q_V A$ is not surjective, i.e. $q(Q_V A) = 0$, where Q_V is the quotient map from Y to Y/V. Let $SS(X, Y)$ ($CS(X, Y)$) denote the set of all strictly singular (cosingular) operators from X to Y, and $SS(X) = SS(X, X)$ ($CS(X) = CS(X, X)$).

Recall that

$$K(X, Y) \subset SS(X, Y) \subset P(\Phi_+(X, Y)) \tag{4}$$

and

$$K(X, Y) \subset CS(X, Y) \subset P(\Phi_-(X, Y)). \tag{5}$$

For the first inclusion in (4) see [12] Theorem III.1.3 and the second inclusion in (4) was proved by Kato [18]. The first inclusion in (5) follows from [26] Theorem 1.10.3, and the second inclusion was proved by Vladimirskii [39].

The paper is organized as follows. In Section 2 we consider some geometric characteristics of bounded operators on Banach spaces concerning specially sets of upper and lower semi-Browder operators, as well as sets of left and right Browder operators. In Section 3 we give general perturbation theorems related to upper (lower) semi-Fredholm and Weyl operators, and also to left (right) Fredholm and Weyl operators. We show that many pairs of known operational quantities satisfy the conditions of those theorems. Also we construct the new operational quantities which satisfy those conditions (Theorem 3.17). Finally we give some perturbation results related to upper and lower semi-Browder operators and also to left and right Browder operators.

2. Geometric Characteristics

For $H = \Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}, \mathcal{B}_+, \mathcal{B}_-, , \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}$, the corresponding spectrum of $A \in B(X)$ is defined by

$$\sigma_H(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin H(X)\}.$$

We define upper, lower, left and right Fredholm, Weyl and Browder radius, as well as Fredholm, Weyl and Browder radius of $A \in B(X)$:

$$s_H(A) = \sup\{\epsilon \geq 0 : |\lambda| < \epsilon \Rightarrow A - \lambda \in H(X)\},$$

where $H = \Phi_+, \Phi_-, \Phi_l, \Phi_r, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{B}_+, \mathcal{B}_-, \mathcal{B}_l, \mathcal{B}_r, \Phi, \mathcal{W}, \mathcal{B}$. Clearly

$$s_H(A) = \max\{\epsilon \geq 0 : |\lambda| < \epsilon \Rightarrow A - \lambda \in H(X)\}$$
$$= \min\{|\lambda| : \lambda \in \sigma_H(A)\} = \text{dist}(0, \sigma_H(A)).$$

Theorem 2.1. *If $A \in B(X)$, then for each $* = +, -, l, r$ there is an implication*

$$A \in \mathcal{B}_*(X) \Longrightarrow s_{\mathcal{B}_*}(A) = s_{\mathcal{W}_*}(A) = s_{\Phi_*}(A). \tag{6}$$

Proof. Since $\mathcal{B}_*(X) \subset \mathcal{W}_*(X) \subset \Phi_*(X)$ (the first inclusion follows from [17], Proposition 38.5 (a)), we have

$$s_{\mathcal{B}_*}(A) \leq s_{\mathcal{W}_*}(A) \leq s_{\Phi_*}(A). \tag{7}$$

Suppose that $A \in \mathcal{B}_+(X)$. For $\lambda \in \mathbb{C}$, $|\lambda| < s_{\Phi_+}(A)$, we have $A - \lambda I \in \Phi_+(X)$ and by [13], Theorem 3, the function $\lambda \to \overline{N^\infty(A - \lambda I)} \cap R^\infty(A - \lambda I)$ is locally constant on the ball $K(0, s_{\Phi_+}(A)) \subset \mathbb{C}$. Since $K(0, s_{\Phi_+}(A))$ is a path connected set, it follows that the function $\lambda \to \overline{N^\infty(A - \lambda I)} \cap R^\infty(A - \lambda I)$ is constant on $K(0, s_{\Phi_+}(A))$. Therefore we conclude that

$$\overline{N^\infty(A - \lambda I)} \cap R^\infty(A - \lambda I) = \overline{N^\infty(A)} \cap R^\infty(A), \ \lambda \in K(0, s_{\Phi_+}(A)).$$

Since $\mathrm{asc}(A) < \infty$, from [41], Proposition 1.6(i) it follows that $\overline{N^\infty(A)} \cap R^\infty(A) = N^\infty(A) \cap R^\infty(A) = \{0\}$. Therefore, for every $\lambda \in K(0, s_{\Phi_+}(A))$ we have $\overline{N^\infty(A - \lambda I)} \cap R^\infty(A - \lambda I) = \{0\}$, and hence $N^\infty(A - \lambda I) \cap R^\infty(A - \lambda I) = \{0\}$ which again by [41], Proposition 1.6(i) implies $\mathrm{asc}(A - \lambda I) < \infty$. Consequently,

$$s_{\Phi_+}(A) \leq s_{\mathcal{B}_+}(A). \tag{8}$$

Now, from (7) and (8) it follows $s_{\mathcal{B}_+}(A) = s_{\mathcal{W}_+}(A) = s_{\Phi_+}(A)$.

If $A \in \mathcal{B}_-(X)$, then the adjoint operator $A' \in \mathcal{B}_+(X')$. For $\lambda \in K(0, s_{\Phi_-}(A))$ we have $A' - \lambda I \in \Phi_+(X')$, and, as above, we conclude that $\mathrm{asc}(A' - \lambda I) < \infty$. Since $\mathrm{dsc}(A - \lambda I) = \mathrm{asc}(A' - \lambda I)$, we have $\mathrm{dsc}(A - \lambda I) < \infty$ for $\lambda \in K(0, s_{\Phi_-}(A))$, which implies

$$s_{\mathcal{B}_-}(A) \geq s_{\Phi_-}(A). \tag{9}$$

From (7) and (9) we get $s_{\mathcal{B}_-}(A) = s_{\mathcal{W}_-}(A) = s_{\Phi_-}(A)$.

Analogously we can prove (6) for $* = l, r$. $\quad\square$

Therefore, for $A \in B(X)$, the left (resp. right, upper, lower) Browder radius of A is either equal to 0 or coincides with the left (resp. right, upper, lower) Fredholm radius of A. The argument can be also based on the punctured neighbourhood theorem [25], Theorem 18.7.

To show that we shall say that an operator $A \in B(X)$ is almost bounded below if there exists $\delta > 0$ such that $A - \lambda I$ is bounded below for $0 < |\lambda| < \delta$. If $A \in \mathcal{B}_l(X)$, then by [47], Theorem 5 there exists $\delta > 0$ such that $A - \lambda I$ is injective for $0 < |\lambda| < \delta$. From the punctured neighbourhood theorem ([25], Theorem 18.7) it follows that $\lambda \to \alpha(A - \lambda I)$ is equal to 0 everywhere in the open ball $K(0, s_{\Phi_l}(A)) \subset \mathbb{C}$ except possibly in the set whose all points are isolated. Consequently, for all $\lambda \in K(0, s_{\Phi_l}(A))$, $A - \lambda I \in \Phi_l(X)$ and $A - \lambda I$ is almost bounded below and hence again by [47], Theorem 5 we conclude $A - \lambda I \in \mathcal{B}_l(X)$. Therefore, $s_{\mathcal{B}_l}(A) \geq s_{\Phi_l}(A)$, and from (7) it follows $s_{\mathcal{B}_l}(A) = s_{\mathcal{W}_l}(A) = s_{\Phi_l}(A)$.

Analogously, we can prove the rest assertions for right, upper and lower Browder radius (see [47] Theorem 6, [41] Propositions 2.6 and 2.7).

Corollary 2.2. *Let $A \in \mathcal{B}(X)$. Then*

$$\begin{aligned}
s_{\Phi_+}(A) = s_{\Phi_-}(A) = s_{\Phi_l}(A) = s_{\Phi_r}(A) = s_\Phi(A) = \\
s_{\mathcal{W}_+}(A) = s_{\mathcal{W}_-}(A) = s_{\mathcal{W}_l}(A) = s_{\mathcal{W}_r}(A) = s_{\mathcal{W}}(A) = \\
s_{\mathcal{B}_+}(A) = s_{\mathcal{B}_-}(A) = s_{\mathcal{B}_l}(A) = s_{\mathcal{B}_r}(A) = s_{\mathcal{B}}(A).
\end{aligned} \tag{10}$$

Proof. From $A \in \mathcal{B}(X)$, because the index is locally constant, it follows that

$$s_{\Phi_+}(A) = s_{\Phi_-}(A) = s_\Phi(A) = s_{\Phi_l}(A) = s_{\Phi_r}(A).$$

Now from Theorem 2.1 we get (10). $\quad\square$

For $A \in B(X)$, the injectivity radius of A is defined by

$$s_{inj}(A) = \max\{\epsilon \geq 0 \, : \, |\lambda| < \epsilon \Longrightarrow A - \lambda \text{ is bounded below}\}.$$

The surjectivity radius of the operator A is defined by

$$s_{sur}(A) = \max\{\epsilon \geq 0 \, : \, |\lambda| < \epsilon \Longrightarrow A - \lambda \text{ is onto}\}.$$

The set of all left (right) invertible operators on X is denoted by $\mathcal{G}_l(X)$ ($\mathcal{G}_r(X)$). For $A \in B(X)$ we define the $\mathcal{G}_l$-radius $s_{\mathcal{G}_l}(A)$ and $\mathcal{G}_r$-radius $s_{\mathcal{G}_r}(A)$:

$$s_{\mathcal{G}_l}(A) \quad = \max\{\epsilon \geq 0 \, : \, |\lambda| < \epsilon \Longrightarrow A - \lambda \in \mathcal{G}_l(X)\},$$
$$s_{\mathcal{G}_r}(A) \quad = \max\{\epsilon \geq 0 \, : \, |\lambda| < \epsilon \Longrightarrow A - \lambda \in \mathcal{G}_r(X)\}.$$

Zemánek ([43], Theorem 7.1) proved the following: if $A \in B(X)$ is surjective, then

$$s_{\Phi_-}(A) = \sup_{F \in F(X)} s_{sur}(A + F).$$

From the proof of this theorem it follows that

$$s_{\Phi_-}(A) = \sup_{F \in F(X)} s_{sur}(A + F) = \sup_{F \in F(X), AF=FA} s_{sur}(A + F). \tag{11}$$

Similarly, if $A \in B(X)$ is bounded below, then ([43], Theorem 7.2)

$$s_{\Phi_+}(A) = \sup_{F \in F(X)} s_{inj}(A + F) = \sup_{F \in F(X), AF=FA} s_{inj}(A + F). \tag{12}$$

By using Zemánek's method of removing jumping points, it can be proved that (11) holds for upper semi-Browder operators, while (12) holds for lower semi-Browder operators.

Theorem 2.3. *Let $A \in B(X)$.*

(2.3.1) If A is upper semi-Browder, then

$$\begin{aligned} s_{\mathcal{B}_+}(A) \quad &= \quad s_{\Phi_+}(A) = \sup_{F \in F(X)} s_{inj}(A + F) \\ &= \quad \sup_{F \in F(X), AF=FA} s_{inj}(A + F) \\ &= \quad \sup_{E \in R(X), AE-EA \in P(\Phi_+(X))} s_{inj}(A + E) \\ &= \quad \sup_{E \in R(X), AE=EA} s_{inj}(A + E). \end{aligned} \tag{13}$$

(2.3.2) If A is lower semi-Browder, then

$$\begin{aligned} s_{\mathcal{B}_-}(A) \quad &= \quad s_{\Phi_-}(A) = \sup_{F \in F(X)} s_{sur}(A + F) \\ &= \quad \sup_{F \in F(X), AF=FA} s_{sur}(A + F) \\ &= \quad \sup_{E \in R(X), AE-EA \in P(\Phi_-(X))} s_{sur}(A + E) \\ &= \quad \sup_{E \in R(X), AE=EA} s_{sur}(A + E). \end{aligned} \tag{14}$$

Proof. (2.3.1): Let $E \in R(X)$ and $AE - EA \in P(\Phi_+(X))$. Then $\sigma_{\Phi_+}(A) = \sigma_{\Phi_+}(A + E)$ (the comment after Theorem 8 in [47]) and hence we have

$$s_{\Phi_+}(A) = \text{dist}(0, \sigma_{\Phi_+}(A)) = \text{dist}(0, \sigma_{\Phi_+}(A + E)) = s_{\Phi_+}(A + E) \geq s_{inj}(A + E).$$

Therefore

$$
\begin{aligned}
s_{\Phi_+}(A) &\geq \sup_{E \in R(X), AE - EA \in P(\Phi_+(X))} s_{inj}(A + E) \\
&\geq \sup_{E \in R(X), AE = EA} s_{inj}(A + E) \\
&\geq \sup_{F \in F(X), AF = FA} s_{inj}(A + F).
\end{aligned}
\tag{15}
$$

Suppose that A is upper semi-Browder. Analogously to the comment after Theorem 2.1, from [41], Proposition 2.6 ([47], Theorem 3) and the punctured neighbourhood theorem ([25], Theorem 18.7) it follows $s_{\mathcal{B}_+}(A) = s_{\Phi_+}(A)$, and for $D = K(0, s_{\Phi_+}(A))$, $\lambda \to \alpha(A - \lambda I)$ is equal to 0 everywhere in D except possibly in the set whose all points are isolated. These points are called jumping points. Therefore, the set of jumping points is at most countable and can have accumulation points only in the boundary of D.

If A does not have any jumping point in D, then

$$s_{\Phi_+}(A) = s_{inj}(A) \leq \sup_{AF = FA, F \in F(X)} s_{inj}(A + F). \tag{16}$$

From (15) and (16) we get (13).

Suppose that A has the jumping points in D. Denote the jumping points such that

$$|\lambda_1| \leq |\lambda_2| \leq \ldots |\lambda_n| \leq \cdots < s_{\Phi_+}(A).$$

Therefore, $s_{inj}(A) = |\lambda_1|$.

Since $A - \lambda_1$ is upper semi-Browder, from [25], Theorem 20.10 it follows that X is a direct sum of closed subspaces X_1 and X_2 in X, which are invariant for $A - \lambda_1$, i.e. they are invariant for A, $\dim X_1 < \infty$, $A - \lambda_1$ is nilpotent on X_1, and for the reduction $A_2 = A_{|X_2} : X_2 \to X_2$ we have $A_2 - \lambda_1$ is injective.

Let $\mu \in \mathbb{C}$ such that $|\mu| > \|A\| + s_{\Phi_+}(A)$ and $F = \mu P$, where P is the projection from X onto X_1 along X_2. Clearly, $F \in F(X)$ and $AF = FA$. Let $\lambda \in D$. Then $\|A - \lambda\| \leq \|A\| + s_{\Phi_+}(A) < |\mu|$, so $A - \lambda + \mu$ is invertible. Hence the reduction $(A + \mu - \lambda)_{|X_1} = (A + F - \lambda)_{|X_1} : X_1 \to X_1$ is invertible on X_1 and for all $\lambda \in D \setminus \{\lambda_2, \ldots, \lambda_n, \ldots\}$ we have

$$
\begin{aligned}
N(A + F - \lambda) &= N((A + F - \lambda)_{|X_1}) \oplus N((A + F - \lambda)_{|X_2}) \\
&= \{0\} \oplus N((A_2 - \lambda)_{|X_2}) = \{0\}.
\end{aligned}
$$

For all $\lambda \in D$, $A - \lambda \in \Phi_+(X)$, and so $A + F - \lambda \in \Phi_+(X)$ ([25], Theorem 16.9). Therefore, $A + F - \lambda$ is bounded below for all $\lambda \in D \setminus \{\lambda_2, \ldots, \lambda_n, \ldots\}$.

Let $\epsilon > 0$. Then there exist only finitely many jumping points λ_i such that $|\lambda_i| < s_{\Phi_+}(A) - \epsilon$. Therefore, applying the previous method finitely many times, we obtain the operator $F_1 \in F(X)$ such that $AF_1 = F_1 A$ and $A + F_1 - \lambda$ is bounded below for $|\lambda| < s_{\Phi_+}(A) - \epsilon$, i.e.

$$s_{inj}(A + F_1) \geq s_{\Phi_+}(A) - \epsilon. \tag{17}$$

From (15) and (17) we get (13).

(2.3.2) can be proved similarly. $\quad\square$

Theorem 2.4. *Let $A \in B(X)$.*

(2.4.1) If A is left Browder, then

$$
\begin{aligned}
s_{\mathcal{B}_l}(A) &= s_{\Phi_l}(A) = \sup_{F \in F(X)} s_{\mathcal{G}_l}(A + F) \\
&= \sup_{F \in F(X),\, AF=FA} s_{\mathcal{G}_l}(A + F) \\
&= \sup_{E \in R(X),\, AE-EA \in P(\Phi(X))} s_{\mathcal{G}_l}(A + E) \\
&= \sup_{E \in R(X),\, AE=EA} s_{\mathcal{G}_l}(A + E).
\end{aligned}
\tag{18}
$$

(2.4.2) If A is right Browder, then

$$
\begin{aligned}
s_{\mathcal{B}_r}(A) &= s_{\Phi_r}(A) = \sup_{F \in F(X)} s_{\mathcal{G}_r}(A + F) \\
&= \sup_{F \in F(X),\, AF=FA} s_{\mathcal{G}_r}(A + F) \\
&= \sup_{E \in R(X),\, AE-EA \in P(\Phi(X))} s_{\mathcal{G}_r}(A + E) \\
&= \sup_{E \in R(X),\, AE=EA} s_{\mathcal{G}_r}(A + E).
\end{aligned}
\tag{19}
$$

Proof. The assertions (2.4.1) and (2.4.2) follow from Theorem 2.1, [47], Theorems 5, 6 and 8, and [6], Chap. 5.2, Theorem 7, analogously to the proof of Theorem 2.3. $\square$

It is well-known that the sets $\Phi_+(X), \Phi_-(X), \Phi_l(X), \Phi_r(X), \mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}_l(X), \mathcal{W}_r(X)$ are open ([7], Theorems 4.2.1, 4.2.2), ([6], Chapter 5.2, Theorem 6), as well $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ ([19], Satz 4) and consequently, $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$ are also open in $B(X)$. Therefore, for $H = \Phi_+, \Phi_-, \Phi_l, \Phi_r, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{B}_+, \mathcal{B}_-, \mathcal{B}_l, \mathcal{B}_r$, the quantities

$$
m_H(A) = \mathrm{dist}(A, B(X) \backslash H(X)),
$$

show how firmly a given operator $A \in B(X)$ belongs to each set $H(X)$. Clearly, $m_H(A)$ is the radius of the largest open ball centered at A and contained in the set $H(X)$ and

$$
m_H(A) > 0 \iff A \in H(X). \tag{20}
$$

We shall write

$$
\mathrm{comm}(A) = \{B \in B(X) : AB = BA\},
$$

for the commutant of $A \in B(X)$, and

$$
\begin{aligned}
K^{comm}(A, \epsilon) &= K(A, \epsilon) \cap \mathrm{comm}(A) \\
&= \{B \in B(X) : \|A - B\| < \epsilon,\ B \in \mathrm{comm}(A)\}
\end{aligned}
$$

for the "commutative open ball" centered at A with radius ϵ.

For $A \in B(X)$ set

$$
m_H^{comm}(A) = \sup\{\epsilon \geq 0 : K^{comm}(A, \epsilon) \subset H(X)\},
$$

where $H = \mathcal{B}_+, \mathcal{B}_-, \mathcal{B}_l, \mathcal{B}_r$.

The following result is proved in [31], Theorem 1.

Theorem 2.5. *Suppose that A, $B \in B(X)$ and $AB = BA$. Then*
(2.5.1) $A \in \mathcal{B}_+(X)$ and $r_e(B) < s_{\Phi_+}(A) \implies A + B \in \mathcal{B}_+(X)$,
(2.5.2) $A \in \mathcal{B}_-(X)$ and $r_e(B) < s_{\Phi_-}(A) \implies A + B \in \mathcal{B}_-(X)$.

In the following theorem we prove, for $A \in B(X)$, that the radius of the largest "commutative" open ball centered at A and contained in the set $\mathcal{B}_+(X)$ ($\mathcal{B}_-(X)$) is equal to the upper (lower) Browder radius of A.

Theorem 2.6. *Let $A \in B(X)$.*

(2.6.1) If $A \in \mathcal{B}_+(X)$, then

$$
\begin{aligned}
s_{\mathcal{B}_+}(A) &= s_{\mathcal{W}_+}(A) = s_{\Phi_+}(A) = m^{comm}_{\mathcal{B}_+}(A) \\
&\geq m_{\mathcal{W}_+}(A) = m_{\Phi_+}(A) \geq m_{\mathcal{B}_+}(A).
\end{aligned} \tag{21}
$$

(2.6.2) If $A \in \mathcal{B}_-(X)$, then

$$
\begin{aligned}
s_{\mathcal{B}_-}(A) &= s_{\mathcal{W}_-}(A) = s_{\Phi_-}(A) = m^{comm}_{\mathcal{B}_-}(A) \\
&\geq m_{\mathcal{W}_-}(A) = m_{\Phi_-}(A) \geq m_{\mathcal{B}_-}(A).
\end{aligned} \tag{22}
$$

Proof. (2.6.1): Clearly

$$
m^{comm}_{\mathcal{B}_+}(A) \leq s_{\mathcal{B}_+}(A). \tag{23}
$$

Let $A \in \mathcal{B}_+(X)$. If $\|B\| < s_{\Phi_+}(A)$ and $B \in \mathrm{comm}(A)$, then, since $r_e(B) \leq \|B\|$, from (2.5.1) it follows that $A + B \in \mathcal{B}_+(X)$. This implies

$$
s_{\Phi_+}(A) \leq m^{comm}_{\mathcal{B}_+}(A). \tag{24}
$$

Since $A \in \mathcal{W}_+(X)$, and because of the local constancy of the index, we get $m_{\mathcal{W}_+}(A) = m_{\Phi_+}(A)$. Since $s_{\Phi_+}(A) \geq m_{\Phi_+}(A) \geq m_{\mathcal{B}_+}(A)$, together with (23), (7) and (24) we get (21).

(2.6.2): Analogously to (2.6.1). □

Let us remark that from the previous proof we also get again the implication (6) for $* = +, -$.

Corollary 2.7. *Let $A \in B(X)$.*

(2.7.1) If $A \in \mathcal{B}_+(X)$, then for every $n \in \mathbb{N}$

$$
\begin{aligned}
s_{\mathcal{B}_+}(A) &= m^{comm}_{\mathcal{B}_+}(A) = m^{comm}_{\mathcal{B}_+}(A^n)^{\frac{1}{n}} \\
&= \lim_{n \to \infty} m_{\Phi_+}(A^n)^{\frac{1}{n}}.
\end{aligned} \tag{25}
$$

(2.7.2) If $A \in \mathcal{B}_-(X)$, then for every $n \in \mathbb{N}$

$$
\begin{aligned}
s_{\mathcal{B}_-}(A) &= m^{comm}_{\mathcal{B}_-}(A) = m^{comm}_{\mathcal{B}_-}(A^n)^{\frac{1}{n}} \\
&= \lim_{n \to \infty} m_{\Phi_-}(A^n)^{\frac{1}{n}}.
\end{aligned} \tag{26}
$$

Proof. To prove (2.7.1), recall that ([42], Theorem 1), ([8], Proposition 10), ([37], Theorem 8)

$$
s_{\Phi_+}(A) = \lim_{n \to \infty} m_{\Phi_+}(A^n)^{\frac{1}{n}}. \tag{27}
$$

Therefore

$$
s_{\Phi_+}(A^n) = (s_{\Phi_+}(A))^n, \quad n \in \mathbb{N}. \tag{28}
$$

From $A \in \mathcal{B}_+(X)$ we have $A^n \in \mathcal{B}_+(X)$ by [16], Theorem 7.9.2, and (25) follows from (2.6.1),(27) and (28).

(2.7.2) can be proved similarly. □

At the end of the third section (Corollary 3.23) we get the inequalities $s_{\mathcal{B}_l}(A) \geq m^{comm}_{\mathcal{B}_l}(A) \geq m_{\Phi_l}(A)$ and $s_{\mathcal{B}_l}(A) \geq \limsup_{n \to \infty} m^{comm}_{\mathcal{B}_l}(A^n)^{\frac{1}{n}}$ for $A \in \mathcal{B}_l(X)$, and analogously for the right version.

3. Perturbations

In this section we generalize and extend some results of Abdelmoumen and Baklouti [1], as well as some results from [2].

We shall write Poly = $\mathbf{C}[z]$ for the algebra of complex polynomials.

Theorem 3.1. *For $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \mathcal{W}_+, \mathcal{W}_l, \mathcal{W}_-, \mathcal{W}_r$, let a_1 and a_2 be operational quantities such that*

$$a_1(\lambda A) = |\lambda| a_1(A), \ \lambda \in \mathbb{C}, \tag{29}$$

$$a_1(P) < a_2(A) \Longrightarrow A + P \in H(X), \quad A, \ P \in B(X). \tag{30}$$

Then:

(3.1.1) *If $a_1(P) < a_2(A)$, then A, $A + P \in H(X)$ and $i(A + P) = i(A)$.*

(3.1.2) *If $a_1(P) < a_2(I)$, then $I - P \in \Phi(X)$ and $i(I - P) = 0$.*

(3.1.3) *If p, $q \in$ Poly such that q divides $p - 1$, then*

$$a_1(p(P)) < a_2(I) \Longrightarrow q(P) \in \Phi(X).$$

(3.1.4) *If $p \in$ Poly such that $p(0) = 1$, then*

$$a_1(p(P)) < a_2(I) \Longrightarrow P \in \Phi(X).$$

(3.1.5) *If $a_1(P^n) < a_2(I)$ for some $n \in \mathbb{N}$, then $I - P \in \Phi(X)$ and $i(I - P) = 0$.*

(3.1.6) *If $AP - PA \in P(H(X))$ and $a_1(P^n) < a_2(A^n)$ for some $n \in \mathbb{N}$, then $A + P \in H(X)$ and $i(A + P) = i(A)$.*

Proof. (3.1.1): Let $\lambda \in [0, 1]$ and $a_1(P) < a_2(A)$. By (29), $a_1(\lambda P) = \lambda a_1(P) \le a_1(P) < a_2(A)$, and by (30) it follows $A + \lambda P \in H(X)$. Hence A, $A + P \in H(X)$ and from the local constancy of the index we get $i(A + P) = i(A)$.

(3.1.2): From (3.1.1).

(3.1.3): Let p, $q \in$ Poly such that q divides $p - 1$. Then there exists $r \in$ Poly such that $p(z) - 1 = q(z)r(z) = r(z)q(z)$. Suppose that $a_1(p(P)) < a_2(I)$. From (3.1.2) it follows that $p(P) - I = q(P)r(P) = r(P)q(P) \in \Phi(X)$, which implies $q(P) \in \Phi(X)$.

(3.1.4): Let $a_1(p(P)) < a_2(I)$. If $p(0) = 1$, then the polynomial $q(z) = z$ divides $p - 1$ and from (3.1.3) we obtain $P \in \Phi(X)$.

(3.1.5): Suppose that $a_1(P^n) < a_2(I)$ for some $n \in \mathbb{N}$. Then for $\lambda \in [0, 1]$ we have $a_1((\lambda P)^n) < a_2(I)$ by (29). For $p(z) = z^n$, the polynomial $q(z) = z - 1$ divides $p(z) - 1$, and from (3.1.3) it follows that $I - \lambda P \in \Phi(X)$. Because of the local constancy of the index, we get $i(I - P) = 0$.

(3.1.6): Let $AP - PA \in P(H(X))$, $\lambda \in [0, 1]$ and $a_1(P^n) < a_2(A^n)$ for some $n \in \mathbb{N}$. Then, $a_1(-(-\lambda P)^n) < a_2(A^n)$ and by (30) we get $A^n - (-\lambda P)^n \in H(X)$. Since

$$
\begin{aligned}
A^n - (-\lambda P)^n &= (A + \lambda P)(A^{n-1} + A^{n-2}(-\lambda P) + \cdots + (-\lambda P)^{n-1}) + P_1 \\
&= (A^{n-1} + A^{n-2}(-\lambda P) + \cdots + (-\lambda P)^{n-1})(A + \lambda P) + P_2,
\end{aligned}
$$

where $P_1, P_2 \in P(H(X))$, it follows that

$$
\begin{aligned}
&(A + \lambda P)(A^{n-1} + A^{n-2}(-\lambda P) + \cdots + (-\lambda P)^{n-1}), \\
&(A^{n-1} + A^{n-2}(-\lambda P) + \cdots + (-\lambda P)^{n-1})(A + \lambda P) \in H(X).
\end{aligned} \tag{31}
$$

If $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r$, then from (31) we get $A + \lambda P \in H(X)$, and because of the local constancy of the index, $i(A + P) = i(A)$.

Let $H = \mathcal{W}_l$. From (31) we have $A + \lambda P \in \Phi_l(X)$ and consequently, $i(A + P) = i(A)$. As $a_2(A^n) > 0$, it holds $A^n \in \mathcal{W}_l(X)$, which implies $A \in \mathcal{W}_l(X)$. Therefore, $A + P \in \mathcal{W}_l(X)$.

The cases when $H = \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_r$ can be proved analogously. $\square$

The following theorem shows that the assertion (3.1.6) extends to analytic functions, but only for $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r$.

Theorem 3.2. *For $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r$, let a_1 and a_2 be operational quantities which satisfy the conditions (29) and (30).*

For A, $P \in B(X)$, if $AP - PA \in P(H(X))$ and if g is an analytic function in a neighborhood U of $\sigma(A) \cup \sigma(P)$ such that $a_1(g(P)) < a_2(g(A))$, then $A - P \in H(X)$.

Moreover, if $g(A) \in \Phi(X)$, then $A - P \in \Phi(X)$.

Proof. Suppose that g is an analytic function in a neighborhood U of $\sigma(A) \cup \sigma(P)$ such that $a_1(g(P)) < a_2(g(A))$ and $AP - PA \in P(H(X))$. By (29), $a_1(-g(P)) = a_1(g(P)) < a_2(g(A))$ and by (30), $g(A) - g(P) \in H(X)$. If Ω is an open set such that $\sigma(A) \cup \sigma(P) \subset \Omega \subset \overline{\Omega} \subset U$ and whose boundary $\partial\Omega$ consists of a finite numbers of simple closed curves which do not intersect, then the following holds

$$g(A) - g(P) = \frac{1}{2\pi i} \int_{\partial\Omega} ((\lambda - A)^{-1} - (\lambda - P)^{-1}) g(\lambda) d\lambda. \tag{32}$$

Since $P(H(X))$ is a two-sided ideal and $AP - PA \in P(H(X))$, then for every $\lambda \in \partial\Omega$ there are $S_1(\lambda), S_2(\lambda) \in P(H(X))$ such that

$$
\begin{aligned}
(\lambda - A)^{-1} - (\lambda - P)^{-1} &= (\lambda - A)^{-1}(A - P)(\lambda - P)^{-1} \\
&= (\lambda - A)^{-1}(\lambda - P)^{-1}(A - P) + S_1(\lambda) \\
&= (A - P)(\lambda - A)^{-1}(\lambda - P)^{-1} + S_2(\lambda).
\end{aligned}
\tag{33}
$$

As $P(H(X))$ is closed, from (32) and (33) we get

$$g(A) - g(P) = A_1(A - P) + P_1 = (A - P)A_1 + P_2, \tag{34}$$

where $A_1 = \frac{1}{2\pi i} \int_{\partial\Omega} ((\lambda - A)^{-1}(\lambda - P)^{-1}) g(\lambda) d\lambda \in B(X)$ and

$$
\begin{aligned}
P_1 &= \frac{1}{2\pi i} \int_{\partial\Omega} S_1(\lambda) g(\lambda) d\lambda \in P(H(X)), \\
P_2 &= \frac{1}{2\pi i} \int_{\partial\Omega} S_2(\lambda) g(\lambda) d\lambda \in P(H(X)).
\end{aligned}
$$

It follows that $A_1(A - P), (A - P)A_1 \in H(X)$ which implies $A - P \in H(X)$. Further, if $g(A) \in \Phi(X)$, then from (3.1.1) it follows that $g(A) - g(P) \in \Phi(X)$. Since $P(H(X)) \subset P(\Phi(X))$, from (34) we obtain $A_1(A - P), (A - P)A_1 \in \Phi(X)$, and so $A - P \in \Phi(X)$. $\square$

Analogously to the terminology introduced by Schechter and Whitley in [35], Definition 5 and Definition 16, we will say that an operational quantity $a(P, A)$, A, $P \in B(X)$ is a H perturbation function, where $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r$, if

$$a(\lambda P, A) = |\lambda| a(P, A), \ \lambda \in \mathbb{C},$$
$$a(P, A) < 1 \implies A + P \in H(X).$$

If a H perturbation function can be written in the form

$$a(P, A) = \frac{a_1(P)}{a_2(A)},$$

then we will say that a is factored.

Clearly, the quantities a_1 and a_2 satisfy the conditions (29) and (30) if and only if $\frac{a_1(P)}{a_2(A)}$ is a factored H perturbation function.

Example 1. For $A \in B(X, Y)$ set

$$u(A) = \sup\{m(AJ_M) : M \text{ infinite dimensional subspace of } X\},$$
$$v(A) = \sup\{q(Q_V A) : V \text{ closed infinite codimensional subspace of } Y\}.$$

The quantity u was introduced by Schechter [34], and it characterizes strictly singular operators. The quantity v was introduced by Zemánek [43], and it characterizes strictly cosingular operators. Precisely,

$$u(A) = 0 \Longleftrightarrow A \in SS(X, Y),$$
$$v(A) = 0 \Longleftrightarrow A \in CS(X, Y).$$

For $A \in B(X, Y)$ set

$$B(A) = \sup\{m(AJ_M) : M \text{ finite codimensional subspace of } X\},$$
$$M(A) = \sup\{q(Q_V A) : V \text{ finite dimensional subspace of } Y\}.$$

The quantity B was introduced in [34], and the quantity M in [43].

The quantities $a_1 = u$ and $a_2 = B$ satisfy the conditions of Theorem 3.1 and Theorem 3.2 for the case $H = \Phi_+$ ([34], Theorem 2.14), ([43], Theorem 5.1), i.e. $\frac{u(P)}{B(A)}$ is a Φ_+ perturbation function. The quantities $a_1 = v$ and $a_2 = M$ satisfy the conditions of Theorems 3.1 and 3.2 for the case $H = \Phi_-$ ([43], Theorem 5.1), i.e. $\frac{v(P)}{M(A)}$ is a Φ_- perturbation function.

Remark 3.3. *For $H = \Phi_+, \Phi_-, \Phi_l, \Phi_r, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r$, if a_1 and a_2 are operational quantities such that (29) holds and*

$$a_2(T) > 0 \Longrightarrow T \in H(X) \tag{35}$$
$$a_2(T + S) \leq a_1(T) + a_2(S), \ T, \ S \in B(X), \tag{36}$$

then they satisfy the condition (30).

In order to show this fact, we suppose that $a_1(P) < a_2(A)$. Then, by (36) and (29), $a_2(A) = a_2((-P) + A + P) \leq a_1(-P) + a_2(A + P) = a_1(P) + a_2(A + P) < a_2(A) + a_2(A + P)$. Therefore $a_2(A + P) > 0$ and by (35), $A + P \in H(X)$. Hence $\frac{a_1(P)}{a_2(A)}$ is a H perturbation function.

Similarly, it can be proved that if a_1, a_2 and a_3 are operational quantities such that (29) holds and

$$a_3(T) > 0 \Longrightarrow T \in H(X) \tag{37}$$
$$a_2(T + S) \leq a_1(T) + a_3(S), \ T, \ S \in B(X), \tag{38}$$

then the quantities a_1 and a_2 satisfy the condition (30), i.e. $\frac{a_1(P)}{a_2(A)}$ is a H perturbation function.

Example 2. We shall write $\mathcal{M}_X$ for the class of all nonempty bounded subsets of X. The Hausdorff measure of noncompactness of $\Omega \in \mathcal{M}_X$, denoted by $\chi(\Omega)$, is given by

$$\chi(\Omega) = \inf\{\epsilon > 0 : \quad \text{there exists a finite set } F \subset X \text{ such that}$$
$$\Omega \subset F + \epsilon K[0, 1]_X\}.$$

For $A \in B(X, Y)$ the Hausdorff measure of noncopactness of A is defined by

$$\|A\|_\chi = \inf\{k \geq 0 : \chi_Y(A\Omega) \leq k \chi_X(\Omega), \ \Omega \in \mathcal{M}_X\}.$$

Clearly,

$$\|A\|_\chi = \sup\{\chi_Y(A\Omega) : \Omega \in \mathcal{M}_X, \ \chi_X(\Omega) = 1\},$$

and set (see [8], p. 185, [42], p.70)

$$m_\chi(A) = \inf\{\chi_Y(A\Omega) : \Omega \in \mathcal{M}_X, \ \chi_X(\Omega) = 1\}.$$

Recall that ([42], Theorem 4.10)

$$m_\chi(A) > 0 \iff A \in \Phi_+(X, Y),$$

and

$$m_\chi(A + B) \le m_\chi(A) + \|B\|_\chi.$$

According to Remark 3.3, the quantities $a_1 = \| \cdot \|_\chi$ and $a_2 = m_\chi$ satisfy the conditions of Theorems 3.1 and 3.2 for $H = \Phi_+$ (see also [8], p. 186).

It is well known [8] that

$$\|A\|_\chi = \inf\{\|Q_V A\| \,:\, V \text{ finite dimensional subspace of } Y\}.$$

In [43], p. 223 and [8], p. 186, the following quantity was introduced

$$q_\chi(A) = \sup\{q(Q_V A) \,:\, V \text{ finite dimensional subspace of } Y\}.$$

The quantity $a_1 = \| \cdot \|_\chi$ and $a_2 = q_\chi$ satisfy the conditions of Theorems 3.1 and 3.2 for $H = \Phi_-$ ([8], p. 186).

Example 3. The Kuratowski measure of non-compactness of $\Omega \in \mathcal{M}_X$, denoted by $\alpha(\Omega)$, is given by

$$\alpha(\Omega) = \inf\{d > 0 : \Omega \subset \cup_{k=1}^n D_k, \text{ where each } D_k \text{ has diametar } \le d\}.$$

For $A \in B(X, Y)$ the Kuratowski measure of non-compactness of A is defined by

$$\|A\|_\alpha = \inf\{k \ge 0 : \alpha_Y(A\Omega) \le k\, \alpha_X(\Omega), \ \Omega \in \mathcal{M}_X\}.$$

Obviously,

$$\|A\|_\alpha = \sup\{\alpha_Y(A\Omega) : \Omega \in \mathcal{M}_X, \ \alpha_X(\Omega) = 1\},$$

and set

$$m_\alpha(A) = \inf\{\alpha_Y(A\Omega) : \Omega \in \mathcal{M}_X, \ \alpha_X(\Omega) = 1\}.$$

It is known that ([1] Proposition 2.1, [10])

$$m_\alpha(A) > 0 \iff A \in \Phi_+(X, Y),$$

and

$$m_\alpha(A + B) \le \|A\|_\alpha + m_\alpha(B), \ A, \ B \in B(X, Y).$$

Now according to Remark 3.3 the quantities $a_1 = \| \cdot \|_\alpha$ and $a_2 = m_\alpha$ satisfy the conditions of Theorems 3.1 and 3.2 for $H = \Phi_+$. So we get an improvement of Theorem 2.2 in [1].

Since $m_\alpha(I) = 1$, from Theorem 3.1 ((3.1.3) and (3.1.4)) we get the following corollaries.

Corollary 3.4. *Let $P \in B(X)$. If p, $q \in$ Poly such that q divides $p - 1$, then $\|p(P)\|_\alpha < 1$ implies $q(P) \in \Phi(X)$.*

Corollary 3.5. *Let $P \in B(X)$. If $p \in$ Poly such that $p(0) = 1$, then $\|p(P)\|_\alpha < 1$ implies $P \in \Phi(X)$.*

We remark that Corollaries 3.4 and 3.5, respectively, improve Theorem 3.1 and Corollary 3.3 in [2]. From (3.1.5) we obtain Corollary 3.4 in [2].

Example 4. Let $A \in B(X, Y)$. The following quantity was introduced by Sedaev [36] and Lebow and Schechter [20]:

$$\|A\|_\mu = \inf\{\|A J_M\| : M \text{ subspace of } X, \text{codim} M < \infty\}.$$

Schechter [34] introduced the following quantity:

$$m_\mu(A) = \sup\{m(A J_M) : M \text{ subspace of } X, \text{codim} M < \infty\}.$$

Since ([34] Lemma 2.13)

$$m_\mu(A) > 0 \Longleftrightarrow A \in \Phi_+(X, Y)$$

and ([22] Proposicion 25.8(5))

$$m_\mu(A + B) \leq m_\mu(A) + \|B\|_\mu, \ A, \ B \in B(X, Y),$$

the quantities $a_1 = \| \cdot \|_\mu$ and $a_2 = m_\mu$ satisfy the conditions of Theorems 3.1 and 3.2 for $H = \Phi_+$, according to Remark 3.3.

Example 5. Let $l_\infty(X)$ denote the Banach space obtained from the space of all bounded sequences $x = (x_n)$ in X by imposing term-by-term linear combination and the supremum norm $\|x\| = \sup_n \|x_n\|$. Let

$$m(X) = \{(x_n) \in l_\infty(X) : \{x_n : n \in \mathbb{N}\} \text{ relatively compact in } X\}.$$

$m(X)$ is a closed subspace of $l_\infty(X)$. If X^+ denotes the quotient $l_\infty(X)/m(X)$, then $A \in B(X, Y)$ induces an operator $A^+ : X^+ \to Y^+$, $(x_n) + m(X) \mapsto (Ax_n) + m(Y)$, $(x_n) \in l_\infty(X)$ [4], [32], [5]. Recall that ([5] Theorem 2, [9] Theorem 3.4)

$$A \in \Phi_+(X, Y) \Longleftrightarrow m(A^+) > 0,$$
$$A \in \Phi_-(X, Y) \Longleftrightarrow q(A^+) > 0,$$

and

$$m(A^+ + B^+) \leq m(A^+) + \|B^+\| \text{ and } q(A^+ + B^+) \leq q(A^+) + \|B^+\|.$$

Set $\|A\|_s = \|A^+\|$, $m_s(A) = m(A^+)$ and $q_s(A) = q(A^+)$. According to Remark 3.3, the quantities $a_1 = \| \cdot \|_s$ and $a_2 = m_s$ satisfy the conditions of Theorems 3.1 and 3.2 for $H = \Phi_+$, as well as the quantities $a_1 = \| \cdot \|_s$ and $a_2 = q_s$ for the case $H = \Phi_-$.

Example 6. For $A \in B(X, Y)$ set

$$
\begin{aligned}
n_F(A) &= \inf\{\|A + F\| : F \in F(X, Y)\}, \\
n_K(A) &= \inf\{\|A + K\| : K \in K(X, Y)\}, \\
n_{SS}(A) &= \inf\{\|A + P\| : P \in SS(X, Y)\}, \\
n_{CS}(A) &= \inf\{\|A + P\| : P \in CS(X, Y)\}, \\
n_{P\Phi_+}(A) &= \inf\{\|A + P\| : P \in P(\Phi_+(X, Y))\}, \\
n_{P\Phi_-}(A) &= \inf\{\|A + P\| : P \in P(\Phi_-(X, Y))\},
\end{aligned}
$$

and

$$
\begin{aligned}
m_F(A) &= \sup\{m(A + F) : F \in F(X, Y)\}, \\
q_F(A) &= \sup\{q(A + F) : F \in F(X, Y)\}, \\
m_K(A) &= \sup\{m(A + K) : K \in K(X, Y)\}, \\
q_K(A) &= \sup\{q(A + K) : K \in K(X, Y)\}, \\
m_{SS}(A) &= \sup\{m(A + P) : P \in SS(X, Y)\}, \\
q_{CS}(A) &= \sup\{q(A + P) : P \in CS(X, Y)\}, \\
m_{P\Phi_+}(A) &= \sup\{m(A + P) : P \in P(\Phi_+(X, Y))\}, \\
q_{P\Phi_-}(A) &= \sup\{q(A + P) : P \in P(\Phi_-(X, Y))\}.
\end{aligned}
$$

Recall that ([43] p. 231-232), ([27] Lemma 1), ([44] p. 84-86)

$$m_F(A) > 0 \Leftrightarrow m_K(A) > 0 \Leftrightarrow m_{SS}(A) > 0 \Leftrightarrow m_{P\Phi_+}(A) > 0 \Leftrightarrow A \in \mathcal{W}_+(X, Y),$$
$$q_F(A) > 0 \Leftrightarrow q_K(A) > 0 \Leftrightarrow q_{CS}(A) > 0 \Leftrightarrow q_{P\Phi_-}(A) > 0 \Leftrightarrow A \in \mathcal{W}_-(X, Y).$$

It is easy to see that

$$m_F(A + P) \leq m_F(A) + n_F(P),$$
$$q_F(A + P) \leq q_F(A) + n_F(P),$$
$$m_K(A + P) \leq m_K(A) + n_K(P),$$
$$q_K(A + P) \leq q_K(A) + n_K(P),$$
$$m_{SS}(A + P) \leq m_{SS}(A) + n_{SS}(P),$$
$$q_{CS}(A + P) \leq q_{CS}(A) + n_{CS}(P),$$
$$m_{P\Phi_+}(A + P) \leq m_{P\Phi_+}(A) + n_{P\Phi_+}(P),$$
$$q_{P\Phi_-}(A + P) \leq q_{P\Phi_-}(A) + n_{P\Phi_-}(P), \quad P \in B(X, Y).$$

Therefore, according to Remark 3.3, $\frac{n_F(P)}{m_F(A)}$, $\frac{n_K(P)}{m_K(A)}$, $\frac{n_{SS}(P)}{m_{SS}(A)}$ and $\frac{n_{P\Phi_+}(P)}{m_{P\Phi_+}(A)}$ are $\mathcal{W}_+$ perturbation functions, and $\frac{n_F(P)}{q_F(A)}$, $\frac{n_K(P)}{q_K(A)}$, $\frac{n_{CS}(P)}{q_{CS}(A)}$ and $\frac{n_{P\Phi_-}(P)}{q_{P\Phi_-}(A)}$ are $\mathcal{W}_-$ perturbation functions, $A, P \in B(X)$.

Example 7. We use the following result ([48], Theorem 3.7):

Theorem 3.6. *Let $\mathcal{U}$ be an open subset of $\Phi_{\pm}(X)$ such that $\mu\mathcal{U} \subset \mathcal{U}$ for every $\mu \neq 0$. For $A \in B(X)$, set*

$$m_{\mathcal{U}}(A) = \text{dist}(A, B(X)\backslash\mathcal{U}),$$
$$\|A\|_{P(\mathcal{U})} = \inf\{\|A + P\| : P \in P(\mathcal{U})\},$$

where $P(\mathcal{U})$ is the perturbation class of $\mathcal{U}$.
Then, for $A, B \in B(X)$, the following hold:
(3.6.1) $m_{\mathcal{U}}(A) > 0 \Longleftrightarrow A \in \mathcal{U}$;
(3.6.2) $m_{\mathcal{U}}(A + B) = m_{\mathcal{U}}(A)$ for every $A \in B(X) \Longleftrightarrow B \in P(\mathcal{U})$;
(3.6.3) $m_{\mathcal{U}}(A + B) \leq m_{\mathcal{U}}(A) + \|B\|$;
(3.6.4) $m_{\mathcal{U}}(A + B) \leq m_{\mathcal{U}}(A) + \|B\|_{P(\mathcal{U})}$.

We remark that Theorem 3.6 holds more generally, for $B(X, Y)$ instead of $B(X)$.

If we apply Theorem 3.6 on $\mathcal{U} = \Phi_+(X), \Phi_-(X), \mathcal{W}_+(X), \mathcal{W}_-(X)$, using also (1), we get the following inequalities:

$$
\begin{aligned}
m_{\Phi_+}(A + P) &\leq m_{\Phi_+}(A) + n_{P\Phi_+}(P), \\
m_{\Phi_-}(A + P) &\leq m_{\Phi_-}(A) + n_{P\Phi_-}(P), \\
m_{\mathcal{W}_+}(A + P) &\leq m_{\mathcal{W}_+}(A) + n_{P\Phi_+}(P), \\
m_{\mathcal{W}_-}(A + P) &\leq m_{\mathcal{W}_-}(A) + n_{P\Phi_-}(P), \quad A, P \in B(X).
\end{aligned}
$$

Therefore from (20) and Remark 3.3 it follows that $\frac{n_{P\Phi_+}(P)}{m_{\Phi_+}(A)}$ is a Φ_+ perturbation function and $\frac{n_{P\Phi_+}(P)}{m_{\mathcal{W}_+}(A)}$ is a $\mathcal{W}_+$ perturbation function. Also, $\frac{n_{P\Phi_-}(P)}{m_{\Phi_-}(A)}$ is a Φ_- perturbation function and $\frac{n_{P\Phi_-}(P)}{m_{\mathcal{W}_-}(A)}$ is a $\mathcal{W}_-$ perturbation function.

For $A \in B(X)$ let

$$n_{P\Phi}(A) = \inf\{\|A + P\| : P \in P(\Phi(X))\}.$$

According to (2), (3) and Theorem 3.6 applied to $\mathcal{U} = \Phi_l(X), \Phi_r(X), \mathcal{W}_l(X), \mathcal{W}_r(X)$ (see also [48], Theorem 3.4(3.4.4), Theorem 3.5(3.5.4)) we obtain

$$
\begin{aligned}
m_{\Phi_l}(A + P) &\leq m_{\Phi_l}(A) + n_{P\Phi}(P), \\
m_{\Phi_r}(A + P) &\leq m_{\Phi_r}(A) + n_{P\Phi}(P), \\
m_{\mathcal{W}_l}(A + P) &\leq m_{\mathcal{W}_l}(A) + n_{P\Phi}(P), \\
m_{\mathcal{W}_r}(A + P) &\leq m_{\mathcal{W}_r}(A) + n_{P\Phi}(P), \quad A, P \in B(X).
\end{aligned}
$$

Consequently, we conclude that $\frac{n_{P\Phi}(P)}{m_{\Phi_l}(A)}$ is a Φ_l perturbation function and $\frac{n_{P\Phi}(P)}{m_{\Phi_r}(A)}$ is a Φ_r perturbation function. Also, $\frac{n_{P\Phi}(P)}{m_{\mathcal{W}_l}(A)}$ is a $\mathcal{W}_l$ perturbation function and $\frac{n_{P\Phi}(P)}{m_{\mathcal{W}_r}(A)}$ is a $\mathcal{W}_r$ perturbation function.

Let us notice that for each $* = +, -, l, r$, if $m_{\mathcal{W}_*}(A) > 0$, i.e. if $A \in \mathcal{W}_*(X)$, then because of the local constancy of the index it follows that $m_{\mathcal{W}_*}(A) = m_{\Phi_*}(A)$.

Example 8. Let $S(X)$ denotes the family of all infinite dimensional subspace of X and let $Q(Y)$ denotes the family of all closed infinite codimensional subspace of Y. Gramsch introduced the following quantity (see [34])

$$G(A) = \inf\{\|AJ_M\| : M \in S(X)\}, \ A \in B(X, Y)$$

which characterizes the set of upper semi-Fredholm operators ([34], Theorem 2.11):

$$G(A) > 0 \iff A \in \Phi_+(X, Y). \tag{39}$$

The following quantity was introduced by Schechter [34]:

$$\Delta(A) = \sup\{G(AJ_M) : M \in S(X)\}.$$

Δ is a semi-norm which characterizes the strictly singular operators ([34], Corollary 3.2):

$$\Delta(A) = 0 \iff A \in SS(X, Y).$$

Weis [40] introduced the following quantities:

$$K(A) = \inf\{\|Q_V A\| : V \in Q(Y)\},$$
$$\nabla(A) = \sup\{K(Q_V A) : V \in Q(Y)\}.$$

The quantity K characterizes the lower semi-Fredholm operators ([40]):

$$K(A) > 0 \iff A \in \Phi_-(X, Y), \tag{40}$$

and the quantity ∇ characterizes the strictly cosingular operators ([40]):

$$\nabla(A) = 0 \iff A \in CS(X, Y).$$

The quantities $a_1 = \Delta$ and $a_2 = G$ satisfy also the condition (36) ([34], Theorem 2.1.) and therefore the conditions of Theorems 3.1 and 3.2 with $H = \Phi_+$ according to Remark 3.3 and (39).

The quantities $a_1 = \nabla$ and $a_2 = K$ satisfy the conditions (29) and (30) with $H = \Phi_-$ ([40]).

Let us mention that it holds more generally:

For an operational quantity a and $A \in B(X, Y)$ set

$$G_a(A) = \inf\{a(AJ_M) : M \in S(X)\},$$
$$\Delta_a(A) = \sup\{G_a(AJ_M) : M \in S(X)\},$$

and

$$K_{a,V}(A) = \inf_{W \supset V} a(Q_W A), \quad K_a(A) = K_{a,\{0\}}(A) = \inf_V a(Q_V A),$$
$$\nabla_a(A) = \sup_V K_{a,V}(A) = \sup_V \inf_{W \supset V} a(Q_W A), \quad V, W \in Q(Y).$$

If the operational quantity a has the property that $a(TA) = a(A)$ where $A \in B(X, Y)$ and $T \in B(Y, Z)$ is an isometric isomorphism, then $\nabla_a(A) = \inf_V K_a(Q_V A)$, where V is as above (the norm, the injection modulus and the surjection modulus have this property). In order to prove that we need the following auxiliary assertions.

Lemma 3.7. *Let X_1 and Y_1 be vector spaces and let $A : X_1 \to Y_1$ be a linear operator. If V is a subspace of Y_1, then*

$$\operatorname{codim} A^{-1}(V) \le \operatorname{codim} V. \tag{41}$$

Moreover, if A is surjective, then $\operatorname{codim} A^{-1}(V) = \operatorname{codim} V$.

Proof. Let $T : X_1/A^{-1}(V) \to Y_1/V$ be the map defined by

$$T(x + A^{-1}(V)) = Ax + V, \ x \in X_1.$$

Since T is linear and injective, it follows that $\dim X_1/A^{-1}(V) \le \dim Y_1/V$, i.e., (41) holds. Moreover, if A is surjective, then T is surjective and so, $\dim X_1/A^{-1}(V) = \dim Y_1/V$, i.e. $\operatorname{codim} A^{-1}(V) = \operatorname{codim} V$. $\square$

Lemma 3.8. *Let $V \in Q(X)$. Then $U \in Q(X/V)$ if and only if there exists $W \in Q(X)$ such that $W \supset V$ and $U = W/V$.*

Proof. Suppose that U a closed infinite codimensional subspace of X/V. Then $W = Q_V^{-1}(U)$ is a closed subspace of X and $W \supset Q_V^{-1}(\{0\}) = V$. Since Q_V is surjective, it follows that $U = Q_V(Q_V^{-1}(U)) = Q_V(W) = W/V$ and by Lemma 3.7, $\operatorname{codim} W = \operatorname{codim} Q_V^{-1}(U) = \operatorname{codim} U < \infty$.

Conversely, let $U = W/V$ where W is a closed infinite codimensional subspace W of X such that $W \supset V$. Then W/V is complete and hence, it is a closed subspace of Y/V. From $W \supset V$ it follows that $Q_V^{-1}(Q_V(W)) = W + N(Q_V) = W + V = W$ and therefore, by Lemma 3.7, $\operatorname{codim} U = \operatorname{codim} Q_V(W) = \operatorname{codim} Q_V^{-1}(Q_V(W)) = \operatorname{codim} W < \infty$. $\square$

Proposition 3.9. *Let a be an operational quantity such that*

$$a(TA) = a(A), \tag{42}$$

where $A \in B(X, Y)$ and $T \in B(Y, Z)$ is an isometric isomorphism. Then for every $V \in Q(Y)$,

$$K_{a,V}(A) = K_a(Q_V A) \tag{43}$$

and

$$\nabla_a(A) = \sup\{K_a(Q_V A) : V \in Q(Y)\}. \tag{44}$$

Proof. Let V and W be closed infinite codimensional subspaces of Y and $W \supset V$. Then the operator $T : (Y/V)/(W/V) \to Y/W$ defined by

$$T((y + V) + W/V) = y + W, \ y \in Y,$$

is an isometric isomorphism and $TQ_{W/V}Q_V = Q_W$. Because of the property (42), we get $a(Q_{W/V}Q_V A) = a(TQ_{W/V}Q_V A) = a(Q_W A)$ and hence, by Lemma 3.8 it follows that

$$\begin{aligned}
K_a(Q_V A) &= \inf\{a(Q_U Q_V A) : U \in Q(Y/V)\} \\
&= \inf\{a(Q_{W/V}Q_V A) : W \in Q(Y), \ W \supset V\} \\
&= \inf\{a(Q_W A) : W \in Q(Y), \ W \supset V\} = K_{a,V}(A).
\end{aligned}$$

(44) follows from (43). $\square$

If an operational quantity a satisfies the property (42), then every operational quantity derived from a, in the sense of Definitions 2.1 and 3.1 in [24], satisfies that property. We show this in the case of the operational quantity K_a and for the other ones the proof is analogous.

Proposition 3.10. *Let a be an operational quantity with the property (42). Then*

$$K_a(TA) = K_a(A),$$

where $A \in B(X, Y)$ and $T \in B(Y, Z)$ is an isometric isomorphism.

Proof. Let $V \in Q(Z)$. From Lemma 3.7 it follows that $\mathrm{codim} T^{-1}(V) = \mathrm{codim}(V)$. Define the operator $\hat{T}_V : X/T^{-1}(V) \mapsto Y/V$ by

$$\hat{T}_V(x + T^{-1}(V)) = Tx + V, \quad x \in X.$$

$\hat{T}_V$ is well-defined, linear operator, and

$$\hat{T}_V Q_{T^{-1}(V)} = Q_V T.$$

As T is surjective, then $\hat{T}_V$ is surjective. Since

$$
\begin{aligned}
\|\hat{T}_V(x + T^{-1}(V))\| &= \|Tx + V\| = \inf\{\|Tx + v\| : v \in V\} \\
&= \inf\{\|T(x + T^{-1}v)\| : v \in V\} \\
&= \inf\{\|x + T^{-1}v\| : v \in V\} \\
&= \|x + T^{-1}(V)\|,
\end{aligned}
$$

it follows that $\hat{T}_V$ is an isometric isomorphism and consequently, $a(Q_V TA) = a(\hat{T}_V Q_{T^{-1}V} A) = a(Q_{T^{-1}V} A)$. From Lemma 3.7 it follows that $V \in Q(Z)$ if and only if $T^{-1}V \in Q(Y)$, and hence,

$$
\begin{aligned}
K_a(TA) &= \inf\{a(Q_V TA) : V \in Q(Z)\} = \inf\{a(Q_{T^{-1}V} A) : V \in Q(Z)\} \\
&= \inf\{a(Q_U A) : U \in Q(Y)\} = K_a(A).
\end{aligned}
$$

$\square$

The proof of the following result can be found in [11], Theorem 6 and [46], Theorem 2.17. For the convenience of the reader, we give the complete proof and remark that, for a change of the formulations in [11] and [46], it is enough that an operational quantity satisfies only two conditions, (29) and the following condition (45).

Proposition 3.11. *Let a be an operational quantity such that the condition (29) holds and*

$$a(A + K) = a(A), \ A \in K(X, Y). \tag{45}$$

Then if $\Delta_a(P) < G_a(A)$, it follows $A + P \in \Phi_+(X, Y)$. Also, if $\nabla_a(P) < K_a(A)$, then $A + P \in \Phi_-(X, Y)$.

Proof. Let $\Delta_a(P) < G_a(A)$ and suppose that $A + P \notin \Phi_+(X, Y)$. By [33], Theorem 23 ([7], Theorem 4.4.7), there exists an infinite dimensional subspace M of X and a compact operator $K \in K(X, Y)$ such that

$$(A + P)J_M = KJ_M. \tag{46}$$

From (46), (45) and (29) it follows

$$
\begin{aligned}
G_a(A) &\leq G_a(AJ_M) \\
&= \inf\{a(AJ_N) : N \in S(M)\} \\
&= \inf\{a(KJ_N - PJ_N) : N \in S(M)\} \\
&= \inf\{a(PJ_N) : N \in S(M)\} \\
&= G_a(PJ_M) \leq \Delta_a(P),
\end{aligned}
$$

which is a contradiction.

In order to prove the second assertion, suppose that $\nabla_a(P) < K_a(A)$ and $A + P \notin \Phi_-(X, Y)$. From [7], Theorem 4.4.10 it follows that there exists $K \in K(X, Y)$ such that $\mathrm{codim}\overline{R(A + P - K)} = \infty$. Let $V = \overline{R(A + P - K)}$ and let W be an infinite codimensional subspace of Y such that $W \supset V$. Then $Q_W(A + P - K) = 0$. Hence, from (46), (45) and (29) it follows

$$
\begin{aligned}
K_a(A) &\leq K_{a,V}(A) = \inf\{a(Q_W A) : W \in Q(Y), \ W \supset V\} \\
&= \inf\{a(Q_W K - Q_W P) : W \in Q(Y), \ W \supset V\} \\
&= \inf\{a(Q_W P) : W \in Q(Y), \ W \supset V\} \\
&= K_{a,V}(P) \leq \nabla_a(P),
\end{aligned}
$$

which is a contradiction. □

Obviously, if a satisfies the condition (29), then Δ_a i ∇_a also satisfy that condition. Therefore, from Proposition 3.11 it follows that if a satisfies the conditions (29) and (45), then the quantities Δ_a and G_a satisfy the conditions of Theorems 3.1 and 3.2 for $H = \Phi_+$, as well as ∇_a and K_a for $H = \Phi_-$.

The quantities u and v satisfy the conditions (29) and (45) according to [34], Theorem 3.7, (4), [22] Proposicion 26.13(6) and (5). The quantities G_u and K_v were introduced by Martinón [21], [22] . Since $\Delta_u = u$ and $\nabla_v = v$ (see [22], [23], [24], [46] Remark 2.6) we conclude that $u(P)/G_u(A)$ is a Φ_+ perturbation function, while $v(P)/K_v(A)$ is a Φ_- perturbation function. These facts follow also from the following inequalities ([22], Proposicion 25.8(4) and Proposicion 26.8(4)):

$$
\begin{aligned}
G_u(T + S) &\leq u(T) + G(S), \\
K_v(T + S) &\leq v(T) + K(S), \ T, \ S \in B(X, Y),
\end{aligned}
$$

and the second part of Remark 3.3, (39) and (40).

Let us mention that also from these facts, since $B \leq G_u$ and $M \leq K_v$ ([15], Theorem 2.7 and Theorem 4.3), we get, in one more way, that the quantities from Example 1, $a_1 = u$ and $a_2 = B$, satisfy the conditions 3.1 and 3.2 for $H = \Phi_+$, while $a_1 = v$ and $a_2 = M$ satisfy that conditions for $H = \Phi_-$. In other words, the Φ_+ perturbation function $\frac{u(P)}{G_u(A)}$ is smaller, and therefore better than $\frac{u(P)}{B(A)}$, and also $\frac{v(P)}{K_v(A)}$ is a better Φ_- perturbation function than $\frac{v(P)}{M(A)}$.

For each operational quantity a we consider the *kernel* of a:

$$\ker(a) = \{A : a(A) = 0\}.$$

We say that an operational quantity a is *sub-additive* if

$$a(S + T) \leq a(S) + a(T) \text{ for every } T, S \in B(X, Y).$$

An operational quantity a is a *semi-norm* if it is sub-additive and satisfies for all $\lambda \in \mathbb{C}$ and $A \in B(X, Y)$: $a(\lambda A) = |\lambda| a(A)$.

Proposition 3.12. *Suppose that a is a semi-norm such that* $\ker(a)$ *contains compact operators. Then $\Delta_a(P)/G_a(A)$ is a Φ_+ perturbation function and $\nabla_a(P)/K_a(A)$ is a Φ_- perturbation function.*

Proof. Let A, $K \in K(X, Y)$. Since a is sub-additive and K, $-K \in \ker(a)$, then

$$
\begin{aligned}
a(A + K) &\leq a(A) + a(K) = a(A), \\
a(A) = a(A + K + (-K)) &\leq a(A + K) + a(-K) = a(A + K),
\end{aligned}
$$

and so, $a(A + K) = a(A)$. From Proposition 3.11 it follows that $\Delta_a(P)/G_a(A)$ is a Φ_+ perturbation function and $\nabla_a(P)/K_a(A)$ is a Φ_- perturbation function. □

We shall say that an operational quantity a is *s-increasing on $B(X, Y)$* if for every $A \in B(X, Y)$ and every ifinite dimensional subspace N and M of X the following implication holds

$$N \subset M \Longrightarrow a(AJ_N) \leq a(AJ_M).$$

An operational quantity a is *s-increasing* if for every X, Y, a is s-increasing on $B(X, Y)$. Also we shall say that an operational quantity a is *q-increasing on $B(X, Y)$* if for every $A \in B(X, Y)$ and every infinite codimensional subspace V and U of Y the following implication holds

$$V \supset U \Longrightarrow a(Q_V A) \leq a(Q_U A),$$

and a is *q-increasing* if for every X, Y, a is q-increasing on $B(X, Y)$.

The proof of the following lemma is analogous to the proof of Theorem 2.1 in [34]. For the sake of completeness we give the proof.

Lemma 3.13. *Let a be a sub-additive operational quantity, A, $P \in B(X, Y)$.*

(3.13.1) If a is s-increasing, then

$$G_a(A + P) \leq \Delta_a(A) + G_a(P). \tag{47}$$

(3.13.2) If a is q-increasing, then

$$K_a(A + P) \leq \nabla_a(A) + K_a(P). \tag{48}$$

Proof. (3.13.1) follows from [24] Proposition 4.2(1).

(3.13.2): Suppose that a is q-increasing and sub-additive. Let V be an infinite codimensional closed subspace of Y and let $\epsilon > 0$. There exists an infinite codimensional closed subspace W of Y such that $W \supset V$ and

$$a(Q_W A) < K_{a,V}(A) + \epsilon.$$

Since a is sub-additive and q-increasing we get

$$a(Q_W(A + P)) \leq a(Q_W A) + a(Q_W P) < K_{a,V}(A) + \epsilon + a(Q_V P),$$

and consequently,

$$K_{a,V}(A + P) < K_{a,V}(A) + \epsilon + a(Q_V P).$$

Since $\epsilon > 0$ is arbitrary, we get

$$K_{a,V}(A + P) \leq K_{a,V}(A) + a(Q_V P) \leq \nabla_a(A) + a(Q_V P),$$

and therefore,

$$\inf_V K_{a,V}(A + P) \leq \nabla_a(A) + \inf_V a(Q_V P) = \nabla_a(A) + K_a(P).$$

Since $\inf_V K_{a,V}(A + P) = K_a(A + P)$, we get (48). $\quad\square$

For $T \in B(X, Y)$, let $n(T) = \|T\|$.

Theorem 3.14. *Let a be a semi-norm such that $a \leq n$ and let a be s-increasing and q-increasing. Then the quantities Δ_a i G_a satisfy the conditions (29), (35) and (36) for $H = \Phi_+$, as well as the quantities ∇_a i K_a for $H = \Phi_-$, and therefore $\frac{\Delta_a(P)}{G_a(A)}$ is a Φ_+ perturbation function and $\frac{\nabla_a(P)}{K_a(A)}$ is a Φ_- perturbation function.*

Proof. Since a is a semi-norm, then Δ_a and ∇_a satisfy the condition (29).

From $a \leq n$ it follows $G_a \leq G$ and $K_a \leq K$. Therefore, from $G_a(T) > 0$ it follows $G(T) > 0$ and by (39) $T \in \Phi_+(X, Y)$. Similarly, $K_a(T) > 0$ implies $T \in \Phi_-(X, Y)$.

Now from Lemma 3.13 and Remark 3.3 it follows that $\frac{\Delta_a(P)}{G_a(A)}$ is a Φ_+ perturbation function and $\frac{\nabla_a(P)}{K_a(A)}$ is a Φ_- perturbation function. $\quad\square$

We say that an operational quantity a is sub-multiplicative if

$$a(ST) \leq a(S)a(T) \text{ for every } T \in B(X, Y) \text{ and every } S \in B(Y, Z).$$

It is not difficult to see that if a is a sub-multiplicative operational quantity such that $a \leq n$, then a is s-increasing and q-increasing. Therefore, from Theorem 3.14, we obtain the following corollary.

Corollary 3.15. *Let a be a sub-multiplicative semi-norm such that $a \leq n$. Then the quantities Δ_a i G_a satisfy the conditions (29), (35) and (36) for $H = \Phi_+$, while the quantities ∇_a i K_a satisfy that conditions for $H = \Phi_-$, and hence $\frac{\Delta_a(P)}{G_a(A)}$ is a Φ_+ perturbation function and $\frac{\nabla_a(P)}{K_a(A)}$ is a Φ_- perturbation function.*

If $a = \|\cdot\|_\chi, \|\cdot\|_\alpha, \|\cdot\|_\mu, \|\cdot\|_s$, it is well known that a is a sub-multiplicative semi-norm such that $a \leq n$ and by Corollary 3.15 it follows that $\Delta_a(P)/G_a(A)$ is a Φ_+ perturbation function and $\nabla_a(P)/K_a(A)$ is a Φ_- perturbation function. This also follows from Proposition 3.12 since $\ker(a)$ is equal to the set of all compact operators. We remark that a has the property (42). Indeed, if $A \in B(X, Y)$ and $T \in B(Y, Z)$, then since a is sub-multiplicative and $a \leq n$,

$$a(TA) \leq a(T)a(A) \leq n(T)a(A) = a(A), \quad a(A) = a(T^{-1}TA) \leq a(TA),$$

and so, $a(TA) = a(A)$.

If $a = n_F, n_K, n_{SS}, n_{CS}$, then a is a semi-norm, s-increasing and q-increasing ([24] Example 5.2), ([46] Examples 2.2 and 2.12) and $a \leq n$, and, according to Theorem 3.14, it follows that $\frac{\Delta_a(P)}{G_a(A)}$ is a Φ_+ perturbation function and $\frac{\nabla_a(P)}{K_a(A)}$ is a Φ_- perturbation function.

Since $B(Y, Z)H(X, Y) \subset H(X, Z)$ for $H = F, K, SS, CS$, it follows that $n_H(BA) \leq \|B\|n_H(A)$, $A \in B(X, Y)$, $B \in B(Y, Z)$. It follows that the quantity n_H has the property (42).

Let us mention that if $a = \|\cdot\|_\mu, n_K, n_{SS}$, then $G_a = G$ and $\Delta_a = \Delta$, and if $a = \|\cdot\|_\chi, n_K, n_{CS}$, then $K_a = K$ and $\nabla_a = \nabla$ ([38], Summary and discussion, Remark 2), ([46], p. 81 and p. 87), ([24], p. 482). Since $n_K \leq n_F \leq n$, it follows that $G = G_{n_K} \leq G_{n_F} \leq G$, $K = K_{n_K} \leq K_{n_F} \leq K$ and so, $G_{n_F} = G$, $\Delta_{n_F} = \Delta$, $K_{n_F} = K$ and $\nabla_{n_F} = \nabla$.

In [45] the operational quantities $\Delta_{n_{P\Phi_+}}$ and $\nabla_{n_{P\Phi_-}}$ were considered and the following equivalences were proved for $A \in B(X, Y)$:

$$\Delta_{n_{P\Phi_+}}(A) = 0 \iff A \in SS(X, Y), \tag{49}$$
$$\nabla_{n_{P\Phi_-}}(A) = 0 \iff A \in CS(X, Y). \tag{50}$$

Proposition 3.16. *(3.16.1) The operational quantity $n_{P\Phi_+}$ is s-increasing on $B(X, Y)$ if and only if $SS(X, Y) = P(\Phi_+(X, Y))$.*

(3.16.2) The operational quantity $n_{P\Phi_-}$ is q-increasing on $B(X, Y)$ if and only if $CS(X, Y) = P(\Phi_-(X, Y))$.

Proof. (3.16.1): Suppose that $n_{P\Phi_+}$ is s-increasing on $B(X, Y)$. Then

$$\Delta_{n_{P\Phi_+}} \leq n_{P\Phi_+}. \tag{51}$$

Let $A \in P(\Phi_+(X, Y))$. Then $n_{P\Phi_+}(A) = 0$ and from (51) it follows that $\Delta_{n_{P\Phi_+}}(A) = 0$. By (49) we obtain $A \in SS(X, Y)$. Thus $P(\Phi_+(X, Y)) \subset SS(X, Y)$. According to (4) we get $SS(X, Y) = P(\Phi_+(X, Y))$.

To prove opposite suppose that $SS(X, Y) = P(\Phi_+(X, Y))$. Then $n_{P\Phi_+} = n_{SS}$ and since n_{SS} is s-increasing it follows that $n_{P\Phi_+}$ is s-increasing.

(3.16.2) can be proved similarly. $\square$

Theorem 3.17. *$\Delta_{n_{CS}}(P)/G_{n_{CS}}(A)$, $\Delta_{n_{P\Phi_+}}(P)/G_{n_{P\Phi_+}}(A)$ and $\Delta_{n_{P\Phi_-}}(P)/G_{n_{P\Phi_-}}(A)$ are Φ_+ perturbation functions.*
$\nabla_{n_{SS}}(P)/K_{n_{SS}}(A)$, $\nabla_{n_{P\Phi_+}}(P)/K_{n_{P\Phi_+}}(A)$ and $\nabla_{n_{P\Phi_-}}(P)/K_{n_{P\Phi_-}}(A)$ are Φ_- perturbation functions.

Proof. The assertions follow from Proposition 3.12, since $n_{SS}, n_{CS}, n_{P\Phi_+}$ and $n_{P\Phi_-}$ are semi-norms and their kernels contain compact operators according to (4) and (5). $\square$

We remark that if $A \in P(\Phi_+(X, Y))$ and $T \in B(Y, Z)$ is an isomorphism, then $TA \in P(\Phi_+(X, Z))$. Indeed, if $B \in \Phi_+(X, Y)$, then $TA + B = T(A + T^{-1}B) \in \Phi_+(X)$, and hence $TA \in P(\Phi_+(X, Z))$.

Therefore,

$$
\begin{aligned}
n_{P\Phi_+}(TA) &= \inf\{\|TA - P\| : P \in P(\Phi_+(X, Z))\} \\
&\leq \inf\{\|TA - TS\| : S \in P(\Phi_+(X, Y))\} \\
&\leq \|T\| n_{P\Phi_+}(A),
\end{aligned}
$$

and now, it is easy to see that $n_{P\Phi_+}$ has the property (42) and so, for $A \in B(X, Y)$,

$$\nabla_{n_{P\Phi_+}}(A) = \sup_{V \in Q(Y)} \inf_{W \supset V, W \in Q(Y)} n_{P\Phi_+}(Q_W A) = \sup_{V \in Q(Y)} K_{n_{P\Phi_+}}(Q_V A).$$

Analogously, $n_{P\Phi_-}$ also has the property (42).

Theorem 3.18. *For $H = \Phi_+, \Phi_l, \mathcal{W}_+, \mathcal{W}_l$, let a_1 and a_2 be operational quantities which satisfy the conditions (29) and (30). Then if A, $P \in B(X)$ and*

$$AP = PA \text{ and } a_1(P^n) < a_2(A^n) \text{ for some } n \in \mathbb{N},$$

there is implication

$$\operatorname{asc}(A) < \infty \Longrightarrow \operatorname{asc}(A + P) < \infty.$$

Proof. Let $AP = PA$, $\operatorname{asc}(A) < \infty$ and $a_1(P^n) < a_2(A^n)$ for some $n \in \mathbb{N}$ and $\lambda \in [0,1]$. Then $a_1((\lambda P)^n) < a_2(A^n)$ and from Theorem 3.1 it follows that $A + \lambda P \in H(X)$ and hence $A + \lambda P \in \Phi_+(X)$. Since A and P commute, from [13], Theorem 3 it follows that the function $\lambda \to \overline{N^\infty(A + \lambda P)} \cap R^\infty(A + \lambda P)$ is a locally constant function on the set $[0,1]$ and therefore this function is constant on $[0,1]$. As $\operatorname{asc}(A) < \infty$, from [41], Proposition 1.6(i) it follows $\overline{N^\infty(A)} \cap R^\infty(A) = N^\infty(A) \cap R^\infty(A) = \{0\}$ and hence $\overline{N^\infty(A + P)} \cap R^\infty(A + P) = \{0\}$. It implies $N^\infty(A + P) \cap R^\infty(A + P) = \{0\}$, and again by [41], Proposition 1.6(i), we get $\operatorname{asc}(A + P) < \infty$. $\square$

The next theorem is a dual part of Theorem 3.4.

Theorem 3.19. *For $H = \Phi_-, \Phi_r, \mathcal{W}_-, \mathcal{W}_r$, let a_1 and a_2 be operational quantities which satisfy the conditions (29) and (30). Then if A, $P \in B(X)$ and*

$$AP = PA \text{ and } a_1(P^n) < a_2(A^n) \text{ for some } n \in \mathbb{N},$$

there is implication

$$\operatorname{dsc}(A) < \infty \Longrightarrow \operatorname{dsc}(A + P) < \infty.$$

Proof. From $AP = PA$, and $a_1(P^n) < a_2(A^n)$ for some $n \in \mathbb{N}$, it follows $A + \lambda P \in H(X)$, $\lambda \in [0,1]$, by Theorem 3.1. Therefore, $A + \lambda P \in \Phi_-(X)$ and so $A' + \lambda P' \in \Phi_+(X')$. Since $\operatorname{asc}(A') = \operatorname{dsc}(A) < \infty$, from the proof of Theorem 3.18 it follows $\operatorname{asc}(A' + P') < \infty$, i.e. $\operatorname{dsc}(A + P) < \infty$. $\square$

Theorem 3.20. *For $H = \Phi_+, \Phi_l, \mathcal{W}_+, \mathcal{W}_l, \Phi_-, \Phi_r, \mathcal{W}_-, \mathcal{W}_r$, let a_1 and a_2 be operational quantities which satisfy the conditions (29) and (30). If A, $P \in B(X)$ and*

$$AP = PA \text{ and } a_1(P^n) < a_2(A^n) \text{ for some } n \in \mathbb{N},$$

there is implication

$$A \in \mathcal{B}(X) \Longrightarrow A + P \in \mathcal{B}(X).$$

Proof. Suppose that a_1 and a_2 satisfy the conditions (29) and (30) with $H = \Phi_l$ (the rest of the cases for H can be proved similarly). Let $a_1(P^n) < a_2(A^n)$ for some $n \in \mathbb{N}$, $AP = PA$ and $A \in \mathcal{B}(X)$. From Theorem 3.18 it follows $\operatorname{asc}(A + P) < \infty$ and from Theorem 3.1 we get $A + P \in \Phi_l(X)$ and $i(A + P) = i(A)$. Since $i(A) = 0$, we have $\beta(A + P) = \alpha(A + P) < \infty$. It follows $\operatorname{dsc}(A + P) < \infty$ ([17], Proposition 38.6 (b)), ([3], Theorem 3.4. (iv)) and therefore, $A + P \in \mathcal{B}(X)$. $\square$

Corollary 3.21. *Let A, $P \in B(X)$, $AP = PA$ and $n_{P\Phi}(P^n) < m_{\Phi_l}(A^n)$ for some $n \in \mathbb{N}$. Then:*

(3.21.1) If $A \in \mathcal{B}_l(X)$, then $A + P \in \mathcal{B}_l(X)$.
(3.21.2) If $A \in \mathcal{B}(X)$, then $A + P \in \mathcal{B}(X)$.

Proof. Since $n_{P\Phi}(P)/m_{\Phi_l}(A)$ is a Φ_l perturbation function, (3.21.1) follows from Theorem 3.18 and (3.21.2) follows from Theorem 3.20. $\square$

Corollary 3.22. *Let A, $P \in B(X)$, $AP = PA$ and $n_{P\Phi}(P^n) < m_{\Phi_r}(A^n)$ for some $n \in \mathbb{N}$. Then:*

(3.22.1) If $A \in \mathcal{B}_r(X)$, then $A + P \in \mathcal{B}_r(X)$.
(3.22.2) If $A \in \mathcal{B}(X)$, then $A + P \in \mathcal{B}(X)$.

Proof. Since $n_{P\Phi}(P)/m_{\Phi_r}(A)$ is a Φ_r perturbation function, (3.22.1) follows from Theorem 3.19 and (3.22.2) follows from Theorem 3.20. $\square$

Corollary 3.23. *Let $A \in B(X)$.*
(3.23.1) If $A \in \mathcal{B}_l(X)$, then

$$s_{\mathcal{B}_l}(A) \geq m_{\mathcal{B}_l}^{comm}(A) \geq m_{\mathcal{W}_l}(A) = m_{\Phi_l}(A) \geq m_{\mathcal{B}_l}(A) \tag{52}$$

and

$$s_{\mathcal{B}_l}(A) \geq \limsup_{n \to \infty} m_{\mathcal{B}_l}^{comm}(A^n)^{\frac{1}{n}} \geq \limsup_{n \to \infty} m_{\Phi_l}(A^n)^{\frac{1}{n}}. \tag{53}$$

(3.23.2) If $A \in \mathcal{B}_r(X)$, then

$$s_{\mathcal{B}_r}(A) \geq m_{\mathcal{B}_r}^{comm}(A) \geq m_{\mathcal{W}_r}(A) = m_{\Phi_r}(A) \geq m_{\mathcal{B}_r}(A) \tag{54}$$

and

$$s_{\mathcal{B}_r}(A) \geq \limsup_{n \to \infty} m_{\mathcal{B}_r}^{comm}(A^n)^{\frac{1}{n}} \geq \limsup_{n \to \infty} m_{\Phi_r}(A^n)^{\frac{1}{n}}. \tag{55}$$

Proof. (3.23.1): Clearly $s_{\mathcal{B}_l}(A) \geq m_{\mathcal{B}_l}^{comm}(A)$. Let $A \in \mathcal{B}_l(X)$, let $\|P\| < m_{\Phi_l}(A)$ and $AP = PA$. By Theorem 3.6, the quantities $a_1 = \| \cdot \|$ and $a_2 = m_{\Phi_l}$ satisfy the conditions (29), (35) and (36) with $H = \Phi_l$, and hence, from Remark 3.3 and Theorem 3.18 we get $A + P \in \mathcal{B}_l(X)$. It implies that

$$m_{\mathcal{B}_l}^{comm}(A) \geq m_{\Phi_l}(A). \tag{56}$$

Obviously, $m_{\Phi_l}(A) \geq m_{\mathcal{B}_l}(A)$ and since $A \in \mathcal{W}_l(X)$, because of the local constancy of the index, it holds $m_{\mathcal{W}_l}(A) = m_{\Phi_l}(A)$. These, together with (56), imply (52).

To prove (53), recall that $f(\sigma_{\mathcal{B}_l}(A)) = \sigma_{\mathcal{B}_l}(f(A))$ for every analytic function f defined in a neighborhood of $\sigma(A)$ ([48], Theorem 3.6). Putting $f(z) = z^n$, $n \in \mathbb{N}$, we conclude

$$s_{\mathcal{B}_l}(A^n) = (s_{\mathcal{B}_l}(A))^n. \tag{57}$$

From $A \in \mathcal{B}_l(X)$ it follows $A^n \in \mathcal{B}_l(X)$, $n \in \mathbb{N}$ ([48], Lemma 2.5), and so from (52) and (57) we get (53).
(3.23.2) can be proved similarly. $\square$

Finally, we set several questions:

If $A \in \mathcal{B}_l(X)$, does the following hold:

$$s_{\mathcal{B}_l}(A) = m_{\mathcal{B}_l}^{comm}(A)?$$

If $A \in \mathcal{B}_r(X)$, does the following hold:

$$s_{\mathcal{B}_r}(A) = m_{\mathcal{B}_r}^{comm}(A)?$$

If $A \in \mathcal{B}_l(X)$, does the following hold:

$$s_{\mathcal{B}_l}(A) = \lim_{n \to \infty} m_{\mathcal{B}_l}^{comm}(A^n)^{\frac{1}{n}} = \lim_{n \to \infty} m_{\Phi_l}(A^n)^{\frac{1}{n}}?$$

If $A \in \mathcal{B}_r(X)$, does the following hold:

$$s_{\mathcal{B}_r}(A) = \lim_{n \to \infty} m_{\mathcal{B}_r}^{comm}(A^n)^{\frac{1}{n}} = \lim_{n \to \infty} m_{\Phi_r}(A^n)^{\frac{1}{n}}?$$

References

[1] B. Abdelmoumen and H. Baklouti, *Perturbation results on semi-Fredholm operators and applications*, J. Inequal. Appl. (2009), doi:10.1155/2009/284526, Article ID 284526, 13 pages.

[2] B. Abdelmoumen, A. Dehici, A. Jeribi and M. Mnif, *Some new properties in Fredholm theory, Schechter essential spetrum, and application to transport theory*, J. Inequal. Appl. (2008), doi:10.1155/2008/852676, Article ID 852676, 14 pages.

[3] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer (2004)

[4] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, *Measures of noncompactness and condensing operators* (in Russian), Nauka, Novosibirsk, 1986.

[5] J. J. Buoni, R. Harte and T. Wickstead, *Upper and Lower Fredholm spectra*, Proc. Amer. Math. Soc.,**66** (1977), 309-314.

[6] S. R. Caradus, *Generalized inverses and operator theory*, Queen's Papers in Pure and Applied Mathematics, No. 38, Queen's University, Kingston, Ontario, 1978.

[7] S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, 1974.

[8] A. S. Fainshtein, *Measures of noncompactness of linear operators and analogues of the minimum modulus for semi-Fredholm operators* (in Russian), Spectral Theory of Operators 6, 182-195, Elm, Baku, 1985 .

[9] K. -H. Förster and E. -O. Liebetrau, *Semi-Fredholm operators and sequence conditions*, Manuscripta Math. **44** (1983) 35-44.

[10] M. Furi, M. Martelli and A. Vignoli, *Contributions to the spectral theory for nonlinear operators in Banach spaces*, Annali di Matematica Pura ed Applicata, vol. 118, no. 1, pp. 229-294, 1978.

[11] F. Galaz-Fontes, *Measures of noncompatness and upper semi-Fredholm pertirbation theorems*, Proc. Amer. Math. Soc. 118 (1993), 891-897.

[12] S. Goldberg, *Unbounded linear operators with applications*, McGraw-Hill, New York, 1966.

[13] M. A. Goldman and S. N. Kračkovskii, *Behaviour of the space of zero elements with finite-dimensional salient on the Riesz kernel under perturbations of the operator*, Dokl. Akad. Nauk SSSR **221**(1975), 532-534 (in Russian); English transl.: Soviet Math. Dokl. **16**(1975), 370-373.

[14] M. González and A. Martinón, *Fredholm theory and space ideals*, Boll. Unione Mat. Ital. Sez. B (7) 7 (1993) 473-488.

[15] M. González and A. Martinón, *Operational quantities characterizing semi-Fredholm operators* , Studia Math. 114 (1995), 13-27.

[16] R. E. Harte, *Invertibility and singularity*, Marcel Dekker, Inc., New York/Basel, 1988.

[17] H. Heuser, *Functional Analysis*, Wiley Interscience, Chichester, 1982.

[18] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math., **6**(1958), 261-322.

[19] H. Kroh and P. Volkman, *Störungssätze für Semifredholmoperatoren*, Math. Z. 148 (1976), 295-297.

[20] A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Funct. Anal. **7**(1971), 1-26.

[21] A. Martinón, *Generating operational quantities in Fredholm theory*, Actas XIV Jornadas Hispano-Lusas Mat. (1989), Univ. La Laguna, vol. I, 1990, 449-453.

[22] A. Martinón, *Cantidades operacionales en teoria de Fredholm*, Doctoral thesis, University of La Laguna, 1989.

[23] A. Martinón, *Generating real maps on a biordered set*, Comment. Math. Univ. Carolinae 32,2 (1991), 265-272.

[24] A. Martinón, *Operational quantities*, Comment. Math. Univ. Carolinae 38,3 (1997), 471-484.

[25] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Birkhäuser 2007.

[26] A. Pietsch, *Operator ideals*, VEB Deutscher Verlag der Wissenschaften, Berlin 1978.

[27] V. Rakočević, *On one subset of M. Schechter's essential spectrum*, Mat.Vesnik, **33**(1981), 389-391.

[28] V. Rakočević, *On the essential approximate point spectrum* II, Mat.Vesnik, **36**(1984), 89-97.

[29] V. Rakočević, *Essential spectrum in Banach algebras*, Doctoral dissertation (in Srbian), Niš, 1983.

[30] V. Rakočević, *Approximate point spectrum and commuting comact perturbations*, Glasgow Math. J., **28**(1986), 193-198.

[31] V. Rakočević, *Semi-Browder operators and perturbations*, Studia Math. **122**(1996), 131-137.

[32] B. N. Sadovskii, *Limit-compact and condensing operators*, Uspekhi Mat. Nauk **27**(1972), 81-146 (Russian), English transl. Russian Math. Surveys 27 (1972), 85-155.

[33] M. Schechter, *Riesz operators and Fredholm perturbations*, Bull. Amer. Math. Soc. **74**(1968), 1139-1144.

[34] M. Schechter, *Quantities related to strictly singular operators*, Indiana Univ. Math. J., vol. **21**, No 11 (1972), 1061-1071.

[35] M. Schechter and R. Whitley, *Best Fredholm perturbation theorems*, Studia Math., T. XC. (1988).

[36] A. A. Sedaev, *The structure of certain linear operators* (in Russian), Mat. Issled. **5** (1970), 166-175.

[37] H.-O. Tylli, *On the asymptotic behaviour of some quantities related to semifredholm operators*, J. London Math. Soc. (2) **31** (1985), 340-348.

[38] H.-O. Tylli, *On semifredholm operators, Calkin algebras and some related quantities* , Dissertation, Helsinki, 1986.

[39] J. I. Vladimirskii, *Strictly cosingular operators*, Sov. Math. Dokl. **8**(1967), 739-740.

[40] L. Weis, *Über strikt singuläre und strikt cosinguläre Operatoren in Banachräumen*, Dissertation, Bonn, 1974.

[41] T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. Roy. Irish Acad. Sect. A **87**(1987), 137-146.

[42] J. Zemánek, *The Semi-Fredholm Radius of a linear Operator*, Bull. Pol. Acad. Sci., Math. **32**(1984), 67-76.

[43] J. Zemánek, *Geometric characteristics of semi-Fredholm operators and their asymptotic behaviour*, Studia Math., **80**(1984), 219-234.

[44] S. Živković, *Semi-Fredholm operators and perturbations*, Publ. Inst. Math., Beograd, **61**(1997), 73-89.

[45] S. Živković, *On measures of non-strict-singularity and non-strict-cosingularity of operators*, Mat. Vesnik **54**(2002), 1-7.

[46] S. Živković-Zlatanović, *Semi-Fredholm operators and perturbation functions*, Filomat, **11**(1997), 77-88.

[47] S.Č. Živković-Zlatanović, D.S. Djordjević and R.E. Harte, *On left and right Browder operators*, Jour. Korean Math. Soc. **48**(2011), 1053-1063.

[48] S. Č. Živković-Zlatanović, D. S. Djordjević and R.E. Harte, *Left-right Browder and left-right Fredholm operators*, Integr. Equ. Oper. T. **69**(2011), 347-363.