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A "q-deformed" Generalization of the Hosszú-Gluskin Theorem

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Abstract. In this paper a new form of the Hosszú-Gluskin theorem is presented in terms of polyadic powers and using the language of diagrams. It is shown that the Hosszú-Gluskin chain formula is not unique and can be generalized ("deformed") using a parameter q which takes special integer values. A version of the "q-deformed" analog of the Hosszú-Gluskin theorem in the form of an invariance is formulated, and some examples are considered. The "q-deformed" homomorphism theorem is also given.

Contents

1. Introduction

Since the early days of "polyadic history" [1–3], the interconnection between polyadic systems and binary ones has been one of the main areas of interest [4, 5]. Early constructions were confined to building some special polyadic (mostly ternary [6, 7]) operations on elements of binary groups [8–10]. A very special form of n-ary multiplication in terms of binary multiplication and a special mapping as a chain formula was found in [11] and [12, 13]. The theorem that any n-ary multiplication can be presented in this form is called the Hosszú-Gluskin theorem (for review see [14, 15]). A concise and clear proof of the Hosszú-Gluskin chain formula was presented in [16].

In this paper we give a new form of the Hosszú-Gluskin theorem in terms of polyadic powers. Then we show that the Hosszú-Gluskin chain formula is not unique and can be generalized ("deformed") using a parameter q which takes special integer values. We present the "q-deformed" analog of the Hosszú-Gluskin

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theorem in the form of an invariance and consider some examples. The "q-deformed" homomorphism theorem is also given.

2. Preliminaries

We will use the concise notations from our previous review paper [17], while here we repeat some necessary definitions using the language of diagrams. For a non-empty set G, we denote its elements by lower-case Latin letters $g_i \in G$ and the *n-tuple* (or *polyad*) $g_1, \ldots, g_n$ will be written by $(g_1, \ldots, g_n)$ or using one bold letter with index $\mathbf{g}^{(n)}$, and an n-tuple with equal elements by g^n. In case the number of elements in the n-tuple is clear from the context or is not important, we denote it in one bold letter $\mathbf{g}$ without indices. We omit $g \in G$, if it is obvious from the context.

The Cartesian product $\overbrace{G \times \ldots \times G}^{n} = G^{\times n}$ consists of all n-tuples $(g_1, \ldots, g_n)$, such that $g_i \in G$, $i = 1, \ldots, n$. The *i-projection* of the Cartesian product G^n on its i-th "axis" is the map $\mathsf{Pr}_i^{(n)} : G^{\times n} \to G$ such that $(g_1, \ldots g_i, \ldots, g_n) \longmapsto g_i$. The *i-diagonal* $\mathsf{Diag}_n : G \to G^{\times n}$ sends one element to the equal element n-tuple $g \longmapsto (g^n)$. The one-point set $\{\bullet\}$ is treated as a unit for the Cartesian product, since there are bijections between G and $G \times \{\bullet\}^{\times n}$, where G can be on any place. In diagrams, if the place is unimportant, we denote such bijections by ϵ. On the Cartesian product $G^{\times n}$ one can define a polyadic (n-ary or n-adic, if it is necessary to specify n, its arity or rank) operation $\mu_n : G^{\times n} \to G$. For operations we use small Greek letters and place arguments in square brackets $\mu_n[\mathbf{g}]$. The operations with $n = 1, 2, 3$ are called *unary, binary and ternary*. The case $n = 0$ is special and corresponds to fixing a distinguished element of G, a "constant" $c \in G$, and it is called a *0-ary operation* $\mu_0^{(c)}$, which maps the one-point set $\{\bullet\}$ to G, such that $\mu_0^{(c)} : \{\bullet\} \to G$, and (formally) has the value $\mu_0^{(c)}[\{\bullet\}] = c \in G$. The composition of n-ary and m-ary operations $\mu_n \circ \mu_m$ gives a $(n + m - 1)$-ary operation by the iteration $\mu_{n+m-1}[\mathbf{g}, \mathbf{h}] = \mu_n[\mathbf{g}, \mu_m[\mathbf{h}]]$. If we compose μ_n with the 0-ary operation $\mu_0^{(c)}$, then we obtain the arity "collapsing" $\mu_{n-1}^{(c)}[\mathbf{g}] = \mu_n[\mathbf{g}, c]$, because $\mathbf{g}$ is a polyad of length $(n-1)$. A universal algebra is a set which is closed under several polyadic operations [18]. If a concrete universal algebra has one fundamental n-ary operation, called a *polyadic multiplication* (or *n-ary multiplication*) μ_n, we name it a "polyadic system"[1)].

Definition 2.1. *A polyadic system* $\mathrm{G} = \langle set | one\ fundamental\ operation \rangle$ *is a set G which is closed under polyadic multiplication.*

More specifically, a *n-ary system* $\mathrm{G}_n = \langle G \mid \mu_n \rangle$ is a set G closed under one n-ary operation μ_n (without any other additional structure).

For a given n-ary system $\langle G \mid \mu_n \rangle$ one can construct another polyadic system $\left\langle G \mid \mu'_{n'} \right\rangle$ over the same set G, but with another multiplication $\mu'_{n'}$ of different arity n'. In general, there are three ways of changing the arity:

1. *Iterating.* Composition of the operation μ_n with itself increases the arity from n to $n' = n_{iter} > n$. We denote the number of iterating multiplications by ℓ_μ and call the resulting composition an *iterated product*[2)] $\boldsymbol{\mu}_n^{\ell_\mu}$ (using the bold Greek letters) as (or $\boldsymbol{\mu}_n^{\bullet}$ if ℓ_μ is obvious or not important)

$$\mu'_{n'} = \boldsymbol{\mu}_n^{\ell_\mu} \stackrel{def}{=} \overbrace{\mu_n \circ \left(\mu_n \circ \ldots \left(\mu_n \times \mathrm{id}^{\times(n-1)}\right) \ldots \times \mathrm{id}^{\times(n-1)}\right)}^{\ell_\mu}, \tag{2.1}$$

[1)]A set with one closed binary operation without any other relations was called a groupoid by Hausmann and Ore [19] (see, also [20]). Nowadays the term "groupoid" is widely used in the category theory and homotopy theory for a different construction, the so-called Brandt groupoid [21]. Bourbaki [22] introduced the term "magma". To avoid misreading we will use the neutral notation "polyadic system".

[2)]Sometimes $\boldsymbol{\mu}_n^{\ell_\mu}$ is named a long product [3].

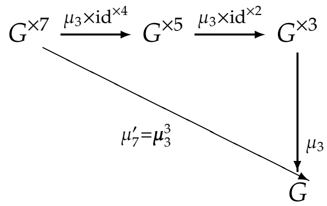
where the final arity is

$$n' = n_{iter} = \ell_\mu (n-1) + 1. \tag{2.2}$$

There are many variants of placing μ_n's among id's in the r.h.s. of (2.1), if no associativity is assumed. An example of the iterated product can be given for a ternary operation μ_3 ($n = 3$), where we can construct a 7-ary operation ($n' = 7$) by $\ell_\mu = 3$ compositions

$$\mu'_7 [g_1, \dots, g_7] = \boldsymbol{\mu}_3^3 [g_1, \dots, g_7] = \mu_3 [\mu_3 [\mu_3 [g_1, g_2, g_3], g_4, g_5], g_6, g_7], \tag{2.3}$$

and the corresponding commutative diagram is



(2.4)

In the general case, the horizontal part of the (iterating) diagram (2.4) consists of ℓ_μ terms.

2. *Reducing (Collapsing).* To decrease arity from n to $n' = n_{red} < n$ one can use n_c distinguished elements ("constants") as additional 0-ary operations $\mu_0^{(c_i)}$, $i = 1, \dots n_c$, such that[3)] the reduced product is defined by

$$\mu'_{n'} = \mu_{n'}^{(c_1 \dots c_{n_c})} \stackrel{def}{=} \mu_n \circ \left(\overbrace{\mu_0^{(c_1)} \times \dots \times \mu_0^{(c_{n_c})}}^{n_c} \times \mathrm{id}^{\times (n - n_c)} \right), \tag{2.5}$$

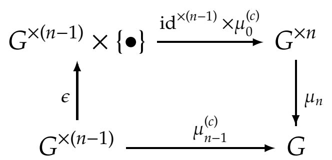
where

$$n' = n_{red} = n - n_c, \tag{2.6}$$

and the 0-ary operations $\mu_0^{(c_i)}$ can be on any places in (2.5). For instance, if we compose μ_n with the 0-ary operation $\mu_0^{(c)}$, we obtain

$$\mu_{n-1}^{(c)} [g] = \mu_n [g, c], \tag{2.7}$$

and this reduced product is described by the commutative diagram



(2.8)

which can be treated as a definition of a new $(n-1)$-ary operation $\mu_{n-1}^{(c)} = \mu_n \circ \mu_0^{(c)}$.

3. *Mixing.* Changing (increasing or decreasing) arity by combining the iterating and reducing (collapsing) methods.

[3)]In [23] $\mu_n^{(c_1 \dots c_{n_c})}$ is called a retract, which is already a busy and widely used term in category theory for another construction.

Example 2.2. *If the initial multiplication is binary $\mu_2 = (\cdot)$, and there is one 0-ary operation $\mu_0^{(c)}$, we can construct the following mixing operation*

$$\mu_n^{(c)}[g_1, \ldots, g_n] = g_1 \cdot g_2 \cdot \ldots \cdot g_n \cdot c, \tag{2.9}$$

which in our notation can be called a c-iterated multiplication[4)].

Let us recall some special elements of polyadic systems. A positive power of an element (according to Post [4]) coincides with the number of multiplications ℓ_μ in the iteration (2.1).

Definition 2.3. *A (positive) polyadic power of an element is*

$$g^{\langle \ell_\mu \rangle} = \boldsymbol{\mu}_n^{\ell_\mu}\left[g^{\ell_\mu(n-1)+1}\right]. \tag{2.10}$$

Example 2.4. *Let us consider a polyadic version of the binary q-addition which appears in study of nonextensive statistics (see, e.g., [25, 26])*

$$\mu_n[\boldsymbol{g}] = \sum_{i=1}^{n} g_i + \hbar \prod_{i=1}^{n} g_i, \tag{2.11}$$

where $g_i \in \mathbb{C}$ and $\hbar = 1 - q_0$, q_0 is a real constant (we put here $q_0 \neq 1$ or $\hbar \neq 0$). It is obvious that $g^{\langle 0 \rangle} = g$, and

$$g^{\langle 1 \rangle} = \mu_n\left[g^{n-1}, g^{\langle 0 \rangle}\right] = ng + \hbar g^n. \tag{2.12}$$

So we have the following recurrence formula

$$g^{\langle k \rangle} = \mu_n\left[g^{n-1}, g^{\langle k-1 \rangle}\right] = (n-1)\, g + \left(1 + \hbar g^{n-1}\right) g^{\langle k-1 \rangle}. \tag{2.13}$$

Solving this for an arbitrary polyadic power we get

$$g^{\langle k \rangle} = g\left(1 + \frac{n-1}{\hbar} g^{1-n}\right)\left(1 + \hbar g^{n-1}\right)^k - \frac{n-1}{\hbar} g^{2-n}. \tag{2.14}$$

Definition 2.5. *A polyadic (n-ary) identity (or neutral element) of a polyadic system is a distinguished element ε (and the corresponding 0-ary operation $\mu_0^{(\varepsilon)}$) such that for any element $g \in G$ we have [27]*

$$\mu_n\left[g, \varepsilon^{n-1}\right] = g, \tag{2.15}$$

where g can be on any place in the l.h.s. of (2.15)*.*

In polyadic systems, for an element g there can exist many *neutral polyads* $\boldsymbol{n} \in G^{\times(n-1)}$ satisfying

$$\mu_n[g, \boldsymbol{n}] = g, \tag{2.16}$$

where g may be on any place. The neutral polyads are not determined uniquely. It follows from (2.15) and (2.16) that ε^{n-1} is a neutral polyad.

Definition 2.6. *An element of a polyadic system g is called ℓ_μ-idempotent, if there exist such ℓ_μ that*

$$g^{\langle \ell_\mu \rangle} = g. \tag{2.17}$$

[4)] According to [24] the operation (2.9) can be called c-derived.

It is obvious that an identity is ℓ_μ-idempotent with arbitrary ℓ_μ. We define *(total) associativity* as invariance of the composition of two n-ary multiplications

$$\boldsymbol{\mu}_n^2 [\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u}] = invariant \tag{2.18}$$

under placement of the internal multiplication in the r.h.s. with a fixed order of elements in the whole polyad of $(2n-1)$ elements $\boldsymbol{t}^{(2n-1)} = (\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u})$. Informally, "internal brackets/multiplication can be moved on any place", which gives

$$\mu_n \circ \left(\overset{i=1}{\mu_n} \times \mathrm{id}^{\times(n-1)}\right) = \mu_n \circ \left(\mathrm{id} \times \overset{i=2}{\mu_n} \times \mathrm{id}^{\times(n-2)}\right) = \ldots = \mu_n \circ \left(\mathrm{id}^{\times(n-1)} \times \overset{i=n}{\mu_n}\right), \tag{2.19}$$

where the internal μ_n can be on any place $i = 1, \ldots, n$. There are many other particular kinds of associativity which were introduced in [4, 28] and studied in [29, 30] (see, also [31]). Here we will confine ourselves to the most general, total associativity (2.18).

Definition 2.7. *A polyadic semigroup (n-ary semigroup) is a n-ary system whose operation is associative, or* $\mathrm{G}_n^{semigrp} = \langle G \mid \mu_n \mid associativity\ (2.18) \rangle$.

In general, it is very important to find the *associativity preserving conditions*, when an associative initial operation μ_n leads to an associative final operation $\mu'_{n'}$, while changing the arity (by iterating (2.1) or reducing (2.5)).

Example 2.8. *An associativity preserving reduction can be given by the construction of a binary associative operation using a* $(n-2)$*-tuple* $\boldsymbol{c}$ *as*

$$\mu_2^{(c)} [g, h] = \mu_n [g, \boldsymbol{c}, h] . \tag{2.20}$$

The associativity preserving mixing constructions with different arities and places were considered in [23, 30, 32].

In polyadic systems, there are several analogs of binary commutativity. The most straightforward one comes from commutation of the multiplication with permutations.

Definition 2.9. *A polyadic system is σ-commutative, if* $\mu_n = \mu_n \circ \sigma$*, where σ is a fixed element of* S_n*, the permutation group on n elements. If this holds for all* $\sigma \in S_n$*, then a polyadic system is commutative.*

A special type of the σ-commutativity

$$\mu_n [g, \boldsymbol{t}, h] = \mu_n [h, \boldsymbol{t}, g] \tag{2.21}$$

is called *semicommutativity*. So for a n-ary semicommutative system we have

$$\mu_n \left[g, h^{n-1}\right] = \mu_n \left[h^{n-1}, g\right]. \tag{2.22}$$

If a n-ary semigroup $\mathrm{G}_n^{semigrp}$ is iterated from a commutative binary semigroup with identity, then $\mathrm{G}_n^{semigrp}$ is semicommutative. Another possibility is to generalize the binary mediality in semigroups

$$(g_{11} \cdot g_{12}) \cdot (g_{21} \cdot g_{22}) = (g_{11} \cdot g_{21}) \cdot (g_{12} \cdot g_{22}) , \tag{2.23}$$

which follows from the binary commutativity. For n-ary systems, it is seen that the mediality should contain $(n+1)$ multiplications, that it is a relation between $n \times n$ elements, and therefore that it can be presented in a matrix from.

Definition 2.10. *A polyadic system is medial (or entropic), if [33, 34]*

$$\mu_n \begin{bmatrix} \mu_n [g_{11}, \ldots, g_{1n}] \\ \vdots \\ \mu_n [g_{n1}, \ldots, g_{nn}] \end{bmatrix} = \mu_n \begin{bmatrix} \mu_n [g_{11}, \ldots, g_{n1}] \\ \vdots \\ \mu_n [g_{1n}, \ldots, g_{nn}] \end{bmatrix}. \tag{2.24}$$

In the case of polyadic semigroups we use the notation (2.1) and can present the mediality as follows

$$\boldsymbol{\mu}_n^n [\boldsymbol{G}] = \boldsymbol{\mu}_n^n \left[\boldsymbol{G}^T\right], \tag{2.25}$$

where $\boldsymbol{G} = \left\|g_{ij}\right\|$ is the $n \times n$ matrix of elements and $\boldsymbol{G}^T$ is its transpose.

The semicommutative polyadic semigroups are medial, as in the binary case, but, in general (except $n = 3$) not vice versa [35].

Definition 2.11. *A polyadic system is cancellative, if*

$$\mu_n [g, \boldsymbol{t}] = \mu_n [h, \boldsymbol{t}] \Longrightarrow g = h, \tag{2.26}$$

where g, h can be on any place. This means that the mapping μ_n is one-to-one in each variable.

If g, h are on the same i-th place on both sides of (2.26), the polyadic system is called *i-cancellative*. The *left* and *right* cancellativity are 1-cancellativity and n-cancellativity respectively. A right and left cancellative n-ary semigroup is cancellative (with respect to the same subset).

Definition 2.12. *A polyadic system is called (uniquely) i-solvable, if for all polyads $\boldsymbol{t}$, $\boldsymbol{u}$ and element h, one can (uniquely) resolve the equation (with respect to h) for the fundamental operation*

$$\mu_n [\boldsymbol{u}, h, \boldsymbol{t}] = g \tag{2.27}$$

where h can be on any i-th place.

Definition 2.13. *A polyadic system which is uniquely i-solvable for all places $i = 1, \ldots, n$ in* (2.27) *is called a n-ary (or polyadic) quasigroup.*

It follows, that, if (2.27) uniquely i-solvable for all places, then

$$\boldsymbol{\mu}_n^{\ell_\mu} [\boldsymbol{u}, h, \boldsymbol{t}] = g \tag{2.28}$$

can be (uniquely) resolved with respect to h being on any place.

Definition 2.14. *An associative polyadic quasigroup is called a n-ary (or polyadic) group.*

In a polyadic group the only solution of (2.27) is called a *querelement*[5)] of g and is denoted by $\bar{g}$ [3], such that

$$\mu_n [\boldsymbol{h}, \bar{g}] = g, \tag{2.29}$$

where $\bar{g}$ can be on any place. Obviously, any idempotent g coincides with its querelement $\bar{g} = g$.

Example 2.15. *For the q-addition* (2.11) *from Example* 2.4*, using* (2.29) *with $\boldsymbol{h} = g^{n-1}$ we obtain*

$$\bar{g} = -\frac{(n-2)\, g}{1 + \hbar g^{n-1}}. \tag{2.30}$$

[5)] We use the original notation after [3] and do not use "skew element", because it can be confused with the wide usage of "skew" in other, different senses.

It follows from (2.29) and (2.16), that the polyad

$$\boldsymbol{n}_{(\bar{g})} = \left(g^{n-2}, \bar{g}\right) \tag{2.31}$$

is neutral for any element g, where $\bar{g}$ can be on any place. If this i-th place is important, then we write $\boldsymbol{n}_{(g),i}$. More generally, because any neutral polyad plays a role of identity (see (2.16)), for any element g we define its *polyadic inverse* (the sequence of length $(n-2)$ denoted by the same letter $\boldsymbol{g}^{-1}$ in bold) as (see [4] and by modified analogy with [15, 36])

$$\boldsymbol{n}_{(g)} = \left(\boldsymbol{g}^{-1}, g\right) = \left(g, \boldsymbol{g}^{-1}\right), \tag{2.32}$$

which can be written in terms of the multiplication as

$$\mu_n\left[g, \boldsymbol{g}^{-1}, h\right] = \mu_n\left[h, \boldsymbol{g}^{-1}, g\right] = h \tag{2.33}$$

for all h in G. It is obvious that the polyads

$$\boldsymbol{n}_{(g^k)} = \left(\left(\boldsymbol{g}^{-1}\right)^k, g^k\right) = \left(g^k, \left(\boldsymbol{g}^{-1}\right)^k\right) \tag{2.34}$$

are neutral as well for any $k \geq 1$. It follows from (2.31) that the polyadic inverse of g is $\left(g^{n-3}, \bar{g}\right)$, and one of $\bar{g}$ is $\left(g^{n-2}\right)$, and in this case g is called *querable*. In a polyadic group all elements are querable [37, 38].

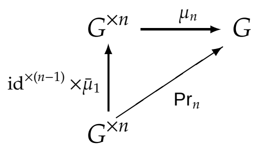
The number of relations in (2.29) can be reduced from n (the number of possible places) to only 2 (when g is on the first and last places [3, 39]), such that in a polyadic group the *Dörnte relations*

$$\mu_n\left[g, \boldsymbol{n}_{(g),i}\right] = \mu_n\left[\boldsymbol{n}_{(g),j}, g\right] = g \tag{2.35}$$

hold valid for any allowable i, j, and (2.35) are analogs of $g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g$ in binary groups. The relation (2.29) can be treated as a definition of the (unary) *queroperation* $\bar{\mu}_1 : G \to G$ by

$$\bar{\mu}_1[g] = \bar{g}, \tag{2.36}$$

such that the diagram



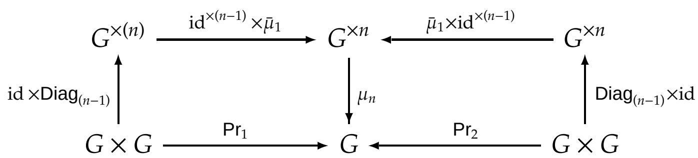
(2.37)

commutes. Then, using the queroperation (2.36) one can give a diagrammatic definition of a polyadic group (cf. [40]).

Definition 2.16. *A polyadic group is a universal algebra*

$$\mathrm{G}_n^{grp} = \langle G \mid \mu_n, \bar{\mu}_1 \mid \textit{associativity, Dörnte relations} \rangle, \tag{2.38}$$

where μ_n is n-ary associative operation and $\bar{\mu}_1$ is the queroperation (2.36), such that the following diagram



(2.39)

commutes, where $\bar{\mu}_1$ can be only on the first and second places from the right (resp. left) on the left (resp. right) part of the diagram.

A straightforward generalization of the queroperation concept and corresponding definitions can be made by substituting in the above formulas (2.29)–(2.36) the n-ary multiplication μ_n by the iterating multiplication $\boldsymbol{\mu}_n^{\ell_\mu}$ (2.1) (cf. [41] for $\ell_\mu = 2$ and [42]).

Let us define the *querpower* k of g recursively by [43, 44]

$$\bar{g}^{\langle\langle k\rangle\rangle} = \overline{(\bar{g}^{\langle\langle k-1\rangle\rangle})}, \tag{2.40}$$

where $\bar{g}^{\langle\langle 0\rangle\rangle} = g$, $\bar{g}^{\langle\langle 1\rangle\rangle} = \bar{g}$, $\bar{g}^{\langle\langle 2\rangle\rangle} = \bar{\bar{g}}$,... or as the k composition $\bar{\mu}_1^{\circ k} = \overbrace{\bar{\mu}_1 \circ \bar{\mu}_1 \circ \ldots \circ \bar{\mu}_1}^{k}$ of the unary queroperation (2.36). We can define the *negative polyadic power* of an element g by the recursive relationship

$$\mu_n \left[g^{\langle \ell_\mu - 1\rangle}, g^{n-2}, g^{\langle -\ell_\mu \rangle} \right] = g, \tag{2.41}$$

or (after the use of the positive polyadic power (2.10)) as a solution of the equation

$$\boldsymbol{\mu}_n^{\ell_\mu} \left[g^{\ell_\mu(n-1)}, g^{\langle -\ell_\mu \rangle} \right] = g. \tag{2.42}$$

The querpower (2.40) and the polyadic power (2.42) are connected [45]. We reformulate this connection using the so called Heine numbers [46] or q-deformed numbers [47]

$$[[k]]_q = \frac{q^k - 1}{q - 1}, \tag{2.43}$$

which have the "nondeformed" limit $q \to 1$ as $[[k]]_q \to k$ and $[[0]]_q = 0$. If $[[k]]_q = 0$, then q is a k-th root of unity. From (2.40) and (2.42) we obtain

$$\bar{g}^{\langle\langle k\rangle\rangle} = g^{\langle -[[k]]_{2-n} \rangle}, \tag{2.44}$$

which can be treated as the following "deformation" statement:

Assertion 2.17. *The querpower coincides with the negative polyadic deformed power with the "deformation" parameter q which is equal to the "deviation" $(2 - n)$ from the binary group.*

Example 2.18. *Let us consider a binary group $\mathrm{G}_2 = \langle G \mid \mu_2 \rangle$, we denote $\mu_2 = (\cdot)$, and construct (using* (2.1) *and* (2.5)*) the reduced* 4*-ary product by $\mu_4' [\mathbf{g}] = g_1 \cdot g_2 \cdot g_3 \cdot g_4 \cdot c$, where $g_i \in G$ and c is in the center of the group G_2. In the 4-ary group $\mathrm{G}_4' = \langle G, \mu_4' \rangle$ we derive the following positive and negative polyadic powers (obviously $g^{\langle 0\rangle} = \bar{g}^{\langle\langle 0\rangle\rangle} = g$)*

$$g^{\langle 1\rangle} = g^4 \cdot c,\ \ g^{\langle 2\rangle} = g^7 \cdot c^2, \ldots, g^{\langle k\rangle} = g^{3k+1} \cdot c^k, \tag{2.45}$$

$$g^{\langle -1\rangle} = g^{-2} \cdot c^{-1},\ \ g^{\langle -2\rangle} = g^{-5} \cdot c^{-2}, \ldots, g^{\langle -k\rangle} = g^{-3k+1} \cdot c^{-k}, \tag{2.46}$$

and the querpowers

$$\bar{g}^{\langle\langle 1\rangle\rangle} = g^{-2} \cdot c^{-1},\ \ \bar{g}^{\langle\langle 2\rangle\rangle} = g^{-4} \cdot c, \ldots,\ \ \bar{g}^{\langle\langle k\rangle\rangle} = g^{(-2)^k} \cdot c^{[[k]]_{-2}}. \tag{2.47}$$

Let $\mathrm{G}_n = \langle G \mid \mu_n \rangle$ and $\mathrm{G}'_{n'} = \langle G' \mid \mu'_{n'} \rangle$ be two polyadic systems of any kind. If their multiplications are of the same arity $n = n'$, then one can define the following *one-place* mappings from G_n to G'_n (for *many-place* mappings, which *change* arity $n \neq n'$ and corresonding *heteromorphisms*, see [17]).

Suppose we have $n + 1$ mappings $\Phi_i : G \to G'$, $i = 1, \ldots, n + 1$. An ordered system of mappings $\{\Phi_i\}$ is called a *homotopy* from G_n to G'_n, if (see, e.g., [34])

$$\Phi_{n+1} \left(\mu_n \left[g_1, \ldots, g_n \right] \right) = \mu'_n \left[\Phi_1 \left(g_1 \right), \ldots, \Phi_n \left(g_n \right) \right], \quad g_i \in G. \tag{2.48}$$

A *homomorphism* from G_n to G'_n is given, if there exists a (one-place) mapping $\Phi : G \to G'$ satisfying

$$\Phi\left(\mu_n\left[g_1, \ldots, g_n\right]\right) = \mu'_n\left[\Phi\left(g_1\right), \ldots, \Phi\left(g_n\right)\right], \quad g_i \in G, \tag{2.49}$$

which means that the corresponding (*equiary*[6)]) diagram is commutative

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ {\scriptstyle \mu_n}\uparrow & & \uparrow{\scriptstyle \mu'_n} \\ G^{\times n} & \xrightarrow{(\varphi)^{\times n}} & (G')^{\times n} \end{array} \tag{2.50}$$

It is obvious that, if a polyadic system contains distinguished elements (identities, querelements, etc.), they are also mapped by φ correspondingly (for details and a review, see, e.g., [42, 48]). The most important application of one-place mappings is in establishing a general structure for n-ary multiplication.

3. The Hosszú-Gluskin Theorem

Let us consider possible concrete forms of polyadic multiplication in terms of lesser arity operations. Obviously, the simplest way of constructing a n-ary product μ'_n from the binary one $\mu_2 = (*)$ is $\ell_\mu = n$ iteration (2.1) [8, 49]

$$\mu'_n[\mathbf{g}] = g_1 * g_2 * \ldots * g_n, \quad g_i \in G. \tag{3.1}$$

In [3] it was noted that not all n-ary groups have a product of this special form. The binary group $\mathrm{G}^*_2 = \langle G \mid \mu_2 = *, e\rangle$ was called a *covering group* of the n-ary group $\mathrm{G}'_n = \langle G \mid \mu'_n\rangle$ in [4] (see, also, [50]), where a theorem establishing a more general (than (3.1)) structure of $\mu'_n[\mathbf{g}]$ in terms of subgroup structure of the covering group was given. A manifest form of the n-ary group product $\mu'_n[\mathbf{g}]$ in terms of the binary one and a special mapping was found in [11, 13] and is called the Hosszú-Gluskin theorem, despite the same formulas having appeared much earlier in [4, 51] (for the relationship between the formulations, see [52]). A simple construction of $\mu'_n[\mathbf{g}]$ which is present in the Hosszú-Gluskin theorem was given in [16]. Here we follow this scheme in the opposite direction, by just deriving the final formula step by step (without writing it immediately) with clear examples. Then we introduce a "deformation" to it in such a way that a generalized "q-deformed" Hosszú-Gluskin theorem can be formulated.

First, let us rewrite (3.1) in its equivalent form

$$\mu'_n[\mathbf{g}] = g_1 * g_2 * \ldots * g_n * e, \quad g_i, e \in G, \tag{3.2}$$

where e is a distinguished element of the binary group $\langle G \mid *, e\rangle$, that is the identity. Now we apply to (3.2) an "extended" version of the homotopy relation (2.48) with $\Phi_i = \psi_i$, $i = 1, \ldots n$, and the l.h.s. mapping $\Phi_{n+1} = \mathrm{id}$, but add an action ψ_{n+1} on the identity e of the binary group $\langle G \mid *, e\rangle$. Then we get (see (2.7) and (2.9))

$$\mu_n[\mathbf{g}] = \mu_n^{(e)}[\mathbf{g}] = \psi_1(g_1) * \psi_2(g_2) * \ldots * \psi_n(g_n) * \psi_{n+1}(e) = \left(*\prod_{i=1}^{n} \psi_i(g_i)\right) * \psi_{n+1}(e). \tag{3.3}$$

In this way we have obtained the most general form of polyadic multiplication in terms of $(n+1)$ "extended" homotopy maps ψ_i, $i = 1, \ldots n+1$, such that the diagram

$$\begin{array}{ccccc} G^{\times(n)} \times \{\bullet\} & \xrightarrow{\mathrm{id}^{\times n} \times \mu_0^{(e)}} & G^{\times(n+1)} & \xrightarrow{\psi_1 \times \ldots \times \psi_{n+1}} & G^{\times(n+1)} \\ {\scriptstyle \epsilon}\uparrow & & & & \downarrow{\scriptstyle \mu_2^{\times n}} \\ G^{\times(n)} & & \xrightarrow{\qquad\quad \mu_n^{(e)} \qquad\quad} & & G \end{array} \tag{3.4}$$

[6)]The map is equiary, if it does not change the arity of operations i.e. $n = n'$, for nonequiary maps see [17] and refs. therein.

commutes. A natural question arises, whether all associative polyadic systems have this form of multiplication or do we have others? In general, we can correspondingly classify polyadic systems as:

1) *Homotopic* polyadic systems which can be presented in the form (3.3). (3.5)

2) *Nonhomotopic* polyadic systems with multiplication of other than (3.3) shapes. (3.6)

If the second class is nonempty, it would be interesting to find examples of nonhomotopic polyadic systems. The Hosszú-Gluskin theorem considers the homotopic polyadic systems and gives one of the possible choices for the "extended" homotopy maps ψ_i in (3.3). We will show that this choice can be extended ("deformed") to the infinite "q-series".

The main idea in constructing the "automatically" associative n-ary operation μ_n in (3.3) is to express the binary multiplication $(*)$ and the "extended" homotopy maps ψ_i in terms of μ_n itself [16]. A simplest binary multiplication which can be built from μ_n is (see (2.20))

$$g *_{\boldsymbol{t}} h = \mu_n \left[g, \boldsymbol{t}, h\right], \tag{3.7}$$

where $\boldsymbol{t}$ is any fixed polyad of length $(n-2)$. If we apply here the equations for the identity e in a binary group

$$g *_{\boldsymbol{t}} e = g, \quad e *_{\boldsymbol{t}} h = h, \tag{3.8}$$

then we obtain

$$\mu_n \left[g, \boldsymbol{t}, e\right] = g, \quad \mu_n \left[e, \boldsymbol{t}, h\right] = h. \tag{3.9}$$

We observe from (3.9) that $(\boldsymbol{t}, e)$ and $(e, \boldsymbol{t})$ are neutral sequences of length $(n-1)$, and therefore using (2.32) we can take $\boldsymbol{t}$ as a polyadic inverse of e (the identity of the binary group) considered as an element (but not an identity) of the polyadic system $\langle G \mid \mu_n \rangle$, that is $\boldsymbol{t} = \boldsymbol{e}^{-1}$. Then, the binary multiplication constructed from μ_n and which has the standard identity properties (3.8) can be chosen as

$$g * h = g *_e h = \mu_n \left[g, \boldsymbol{e}^{-1}, h\right]. \tag{3.10}$$

Using this construction any element of the polyadic system $\langle G \mid \mu_n \rangle$ can be distinguished and may serve as the identity of the binary group, and is then denoted by e (for clarity and convenience).

We recognize in (3.10) a version of the Maltsev term (see, e.g., [18]), which can be called a *polyadic Maltsev term* and is defined as

$$p\left(g, e, h\right) \overset{def}{=} \mu_n \left[g, \boldsymbol{e}^{-1}, h\right] \tag{3.11}$$

having the standard term properties [18]

$$p\left(g, e, e\right) = g, \quad p\left(e, e, h\right) = h, \tag{3.12}$$

which now follow from (3.9), i.e. the polyads $\left(e, \boldsymbol{e}^{-1}\right)$ and $\left(\boldsymbol{e}^{-1}, e\right)$ are neutral, as they should be (2.32). Denote by g^{-1} the inverse element of g in the binary group $(g * g^{-1} = g^{-1} * g = e)$ and $\boldsymbol{g}^{-1}$ its polyadic inverse in a n-ary group (2.32), then it follows from (3.10) that $\mu_n \left[g, \boldsymbol{e}^{-1}, g^{-1}\right] = e$. Thus, we get

$$g^{-1} = \mu_n \left[e, \boldsymbol{g}^{-1}, e\right], \tag{3.13}$$

which can be considered as a connection between the inverse g^{-1} in the binary group and the polyadic inverse in the polyadic system related to the same element g. For n-ary group we can write $\boldsymbol{g}^{-1} = \left(g^{n-3}, \bar{g}\right)$ and the binary group inverse g^{-1} becomes

$$g^{-1} = \mu_n \left[e, g^{n-3}, \bar{g}, e\right]. \tag{3.14}$$

If $\langle G \mid \mu_n \rangle$ is a n-ary group, then the element e is querable (2.33), for the polyadic inverse $\boldsymbol{e}^{-1}$ one can choose $\left(e^{n-3}, \bar{e}\right)$ with $\bar{e}$ being on any place, and the polyadic Maltsev term becomes [53] $p\,(g, e, h) = \mu_n\left[g, e^{n-3}, \bar{e}, h\right]$ (together with the multiplication (3.10)). For instance, if $n = 3$, we have

$$g * h = \mu_3\,[g, \bar{e}, h]\,, \quad g^{-1} = \mu_3\,[e, \bar{g}, e]\,, \tag{3.15}$$

and the neutral polyads are $(e, \bar{e})$ and $(\bar{e}, e)$.

Now let us turn to build the main construction, that of the "extended" homotopy maps ψ_i (3.3) in terms of μ_n, which will lead to the Hosszú-Gluskin theorem. We start with a simple example of a ternary system (3.15), derive the Hosszú-Gluskin "chain formula", and then it will be clear how to proceed for generic n. Instead of (3.3) we write

$$\mu_3\,[g, h, u] = \psi_1\,(g) * \psi_2\,(h) * \psi_3\,(u) * \psi_4\,(e) \tag{3.16}$$

and try to construct ψ_i in terms of the ternary product μ_3 and the binary identity e. We already know the structure of the binary multiplication (3.15): it contains $\bar{e}$, and therefore we can insert between g, h and u in the l.h.s. of (3.16) a neutral ternary polyad $(\bar{e}, e)$ or its powers $\left(\bar{e}^k, e^k\right)$. Thus, taking for all insertions the *minimal number* of neutral polyads, we get

$$\mu_3\,[g, h, u] = \boldsymbol{\mu}_3^2\left[g, \overset{\overset{*}{\downarrow}}{\bar{e}}, e, h, u\right] = \boldsymbol{\mu}_3^4\left[g, \overset{\overset{*}{\downarrow}}{\bar{e}}, e, h, \bar{e}, \overset{\overset{*}{\downarrow}}{\bar{e}}, e, e, u\right]$$

$$= \boldsymbol{\mu}_3^7\left[g, \overset{\overset{*}{\downarrow}}{\bar{e}}, e, h, \bar{e}, \overset{\overset{*}{\downarrow}}{\bar{e}}, e, e, u, \bar{e}, \bar{e}, \overset{\overset{*}{\downarrow}}{\bar{e}}, e, e, e\right]. \tag{3.17}$$

We show by arrows the binary products in special places: there should be $1, 3, 5, \ldots (2k-1)$ elements in between them to form inner ternary products. Then we rewrite (3.17) as

$$\mu_3\,[g, h, u] = \boldsymbol{\mu}_3^3\left[g, \overset{\overset{*}{\downarrow}}{\bar{e}}, \mu_3\,[e, h, \bar{e}]\,, \overset{\overset{*}{\downarrow}}{\bar{e}}, \boldsymbol{\mu}_3^2\,[e, e, u, \bar{e}, \bar{e}]\,, \overset{\overset{*}{\downarrow}}{\bar{e}}, \mu_3\,[e, e, e]\right]. \tag{3.18}$$

Comparing this with (3.16), we can exactly identify the "extended" homotopy maps ψ_i as

$$\psi_1\,(g) = g, \tag{3.19}$$

$$\psi_2\,(g) = \varphi\,(g)\,, \tag{3.20}$$

$$\psi_3\,(g) = \varphi\,(\varphi\,(g)) = \varphi^2\,(g)\,, \tag{3.21}$$

$$\psi_4\,(e) = \mu_3\,[e, e, e]\,, \tag{3.22}$$

where

$$\varphi\,(g) = \mu_3\,[e, g, \bar{e}]\,, \tag{3.23}$$

which can be described by the commutative diagram

$$\begin{array}{ccccc}
\{\bullet\} \times G \times \{\bullet\} & \xrightarrow{\mu_0^{(e)} \times \mathrm{id} \times \mu_0^{(e)}} & G^{\times 3} & \xrightarrow{\mathrm{id}^{\times 2} \times \bar{\mu}_1} & G^{\times 3} \\
\uparrow{\scriptstyle \epsilon} & & & & \downarrow{\scriptstyle \mu_3} \\
G & & \xrightarrow{\qquad \varphi \qquad} & & G
\end{array} \tag{3.24}$$

The mapping ψ_4 is the first polyadic power (2.10) of the binary identity e in the ternary system

$$\psi_4(e) = e^{\langle 1 \rangle}. \tag{3.25}$$

Thus, combining (3.18)–(3.25) we obtain the Hosszú-Gluskin "chain formula" for $n = 3$

$$\mu_3[g, h, u] = g * \varphi(h) * \varphi^2(u) * b, \tag{3.26}$$

$$b = e^{\langle 1 \rangle}, \tag{3.27}$$

which depends on one mapping φ (taken in the chain of powers) only, and the first polyadic power $e^{\langle 1 \rangle}$ of the binary identity e. The corresponding Hosszú-Gluskin diagram

$$\begin{array}{ccccc} G^{\times 3} \times \{\bullet\}^3 & \xrightarrow{\mathrm{id} \times \varphi \times \varphi^2 \times \left(\mu_0^{(e)}\right)^{\times 3}} & G^{\times 6} & \xrightarrow{\mathrm{id}^{\times 3} \times \mu_3} & G^{\times 4} \\ \uparrow \epsilon & & & & \downarrow \mu_2^{\times 3} \\ G \times G \times G & & \xrightarrow{\mu_3} & & G \end{array} \tag{3.28}$$

commutes.

The mapping φ is an automorphism of the binary group $\langle G \mid *, e \rangle$, because it follows from (3.15) and (3.23) that

$$\varphi(g) * \varphi(h) = \mu_3[\mu_3[e, g, \bar{e}], \bar{e}, \mu_3[e, h, \bar{e}]] = \boldsymbol{\mu}_3^3\left[e, g, \bar{e}, \overset{\text{neutral}}{(\bar{e}, e)}, h, \bar{e}\right]$$

$$= \boldsymbol{\mu}_3^2[e, g, \bar{e}, h, \bar{e}] = \mu_3[e, g * h, \bar{e}] = \varphi(g * h), \tag{3.29}$$

$$\varphi(e) = \mu_3[e, e, \bar{e}] = \mu_3\left[e, \overset{\text{neutral}}{(e, \bar{e})}\right] = e. \tag{3.30}$$

It is important to note that not only the binary identity e, but also its first polyadic power $e^{\langle 1 \rangle}$ is a fixed point of the automorphism φ, because

$$\varphi\left(e^{\langle 1 \rangle}\right) = \mu_3\left[e, e^{\langle 1 \rangle}, \bar{e}\right] = \boldsymbol{\mu}_3^2\left[e, e, e, \overset{\text{neutral}}{(e, \bar{e})}\right] = \mu_3[e, e, e] = e^{\langle 1 \rangle}. \tag{3.31}$$

Moreover, taking into account that in the binary group (see (3.15))

$$\left(e^{\langle 1 \rangle}\right)^{-1} = \mu_3\left[e, \overline{e^{\langle 1 \rangle}}, e\right] = \boldsymbol{\mu}_3^2[e, \bar{e}, \bar{e}, \bar{e}, e] = \bar{e}, \tag{3.32}$$

we get

$$\varphi^2(g) = \boldsymbol{\mu}_3^2[e, e, g, \bar{e}, \bar{e}] = \boldsymbol{\mu}_3^2\left[e, e, \overset{\text{neutral}}{(e, \bar{e})}\, g, \bar{e}, \bar{e}\right] = e^{\langle 1 \rangle} * g * \left(e^{\langle 1 \rangle}\right)^{-1}. \tag{3.33}$$

The higher polyadic powers $e^{\langle k \rangle} = \boldsymbol{\mu}_3^k\left[e^{2k+1}\right]$ of the binary identity e are obviously also fixed points

$$\varphi\left(e^{\langle k \rangle}\right) = e^{\langle k \rangle}. \tag{3.34}$$

The elements $e^{\langle k \rangle}$ form a subgroup H of the binary group $\langle G \mid *, e \rangle$, because

$$e^{\langle k \rangle} * e^{\langle l \rangle} = e^{\langle k+l \rangle}, \tag{3.35}$$

$$e^{\langle k \rangle} * e = e * e^{\langle k \rangle} = e^{\langle k \rangle}. \tag{3.36}$$

We can express the even powers of the automorphism φ through the polyadic powers $e^{\langle k\rangle}$ in the following way

$$\varphi^{2k}(g) = e^{\langle k\rangle} * g * \left(e^{\langle k\rangle}\right)^{-1}. \tag{3.37}$$

This gives a manifest connection between the Hosszú-Gluskin "chain formula" and the sequence of cosets (see, [4]) for the particular case $n = 3$.

Example 3.1. *Let us consider the ternary copula associative multiplication [54, 55]*

$$\mu_3[g,h,u] = \frac{g(1-h)u}{g(1-h)u + (1-g)h(1-u)}, \tag{3.38}$$

where $g_i \in G = [0,1]$ and $0/0 = 0$ is assumed[7]*. It is associative and cannot be iterated from any binary group. Obviously, $\mu_3\left[g^3\right] = g$, and therefore this polyadic system is ℓ_μ-idempotent (2.17) $g^{\langle \ell_\mu \rangle} = g$. The querelement is $\bar{g} = \bar{\mu}_1[g] = g$. Because each element is querable, then $\langle G \mid \mu_3, \bar{\mu}_1\rangle$ is a ternary group. Take a fixed element $e \in [0,1]$. We define the binary multiplication as $g * h = \mu_3[g,e,h]$ and the automorphism*

$$\varphi(g) = \mu_3[e,g,e] = e^2 \frac{1-g}{e^2 - 2ge + g} \tag{3.39}$$

which has the property $\varphi^{2k} = \mathrm{id}$ and $\varphi^{2k+1} = \varphi$, where $k \in \mathbb{N}$. Obviously, in (3.39) g can be on any place in the product $\mu_3[e,g,e] = \mu_3[e,e,g] = \mu_3[e,e,g]$. Now we can check the Hosszú-Gluskin "chain formula" (3.26) for the ternary copula

$$\mu_3[g,h,u] = (((g * \varphi(h)) * u) * e) = \boldsymbol{\mu}_3^{\bullet}\left[g,e,e^2\frac{1-h}{e^2-2he+g},e,(u,e,e)\right]$$
$$= \boldsymbol{\mu}_3^{\bullet}\left[g,\left(e,e^2\frac{1-h}{e^2-2he+g},e\right),u\right] = \mu_3\left[g,\varphi^2(h),u\right] = \mu_3[g,h,u]. \tag{3.40}$$

The language of polyadic inverses allows us to generalize the Hosszú-Gluskin "chain formula" from $n = 3$ (3.26) to arbitrary n in a clear way. The derivation coincides with (3.18) using the multiplication (3.10) (with substitution $\bar{e} \to \boldsymbol{e}^{-1}$), neutral polyads $\left(\boldsymbol{e}^{-1}, e\right)$ or their powers $\left(\left(\boldsymbol{e}^{-1}\right)^k, e^k\right)$, but contains n terms

$$\mu_n[g_1,\ldots,g_n] = \boldsymbol{\mu}_n^{\bullet}\left[g_1, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, e, g_2, \ldots, g_n\right] = \boldsymbol{\mu}_n^{\bullet}\left[g_1, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, e, g_2, \boldsymbol{e}^{-1}, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, e, e, g_3, \ldots, g_n\right] = \ldots$$
$$= \boldsymbol{\mu}_n^{\bullet}\left[g_1, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, e, g_2, \boldsymbol{e}^{-1}, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, e, e, g_3, \ldots, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, \overbrace{e,\ldots,e}^{n-1}, g_n, \overbrace{\boldsymbol{e}^{-1},\ldots,\boldsymbol{e}^{-1}}^{n-1}, \overset{*}{\downarrow}{\boldsymbol{e}^{-1}}, \overbrace{e,\ldots,e}^{n}\right]. \tag{3.41}$$

We observe from (3.41) that the mapping φ in the n-ary case is

$$\varphi(g) = \mu_n\left[e,g,\boldsymbol{e}^{-1}\right], \tag{3.42}$$

[7] In this example all denominators are supposed nonzero.

and the last product of the binary identities $\mu_n[e,\ldots,e]$ is also the first n-ary power $e^{\langle 1\rangle}$ (2.10). It follows from (3.42) and (3.10), that

$$\varphi^{n-1}(g) = e^{\langle 1\rangle} * g * \left(e^{\langle 1\rangle}\right)^{-1}. \tag{3.43}$$

In this way, we obtain the Hosszú-Gluskin "chain formula" for arbitrary n

$$\mu_n[g_1,\ldots,g_n] = g_1 * \varphi(g_2) * \varphi^2(g_3) * \ldots * \varphi^{n-2}(g_{n-1}) * \varphi^{n-1}(g_n) * e^{\langle 1\rangle} = \left(* \prod_{i=1}^{n} \varphi^{i-1}(g_i)\right) * e^{\langle 1\rangle}. \tag{3.44}$$

Thus, we have found the "extended" homotopy maps ψ_i from (3.3) as

$$\psi_i(g) = \varphi^{i-1}(g), \quad i = 1,\ldots,n, \tag{3.45}$$
$$\psi_{n+1}(g) = g^{\langle 1\rangle}, \tag{3.46}$$

where we put by definition $\varphi^0(g) = g$. Using (3.31) and (3.44) we can formulate the Hosszú-Gluskin theorem in the language of polyadic powers.

Theorem 3.2. *On a polyadic group $\mathrm{G}_n = \langle G \mid \mu_n, \bar{\mu}_1\rangle$ one can define a binary group $\mathrm{G}_2^* = \langle G \mid \mu_2 = *, e\rangle$ and its automorphism φ such that the Hosszú-Gluskin "chain formula" (3.44) is valid, where the polyadic powers of the identity e are fixed points of φ (3.34), form a subgroup H of G_2^*, and the $(n-1)$ power of φ is a conjugation (3.43) with respect to H.*

The following reverse Hosszú-Gluskin theorem holds.

Theorem 3.3. *If in a binary group $\mathrm{G}_2^* = \langle G \mid \mu_2 = *, e\rangle$ one can define an automorphism φ such that*

$$\varphi^{n-1}(g) = b * g * b^{-1}, \tag{3.47}$$
$$\varphi(b) = b, \tag{3.48}$$

where $b \in G$ is a distinguished element, then the "chain formula"

$$\mu_n[g_1,\ldots,g_n] = \left(* \prod_{i=1}^{n} \varphi^{i-1}(g_i)\right) * b \tag{3.49}$$

determines a n-ary group, in which the distinguished element is the first polyadic power of the binary identity

$$b = e^{\langle 1\rangle}. \tag{3.50}$$

4. "Deformation" of Hosszú-Gluskin Chain Formula

Let us raise the question: can the choice (3.45)-(3.46) of the "extended" homotopy maps (3.3) be generalized? Before answering this question *positively* we consider some preliminary statements.

First, we note that we keep the general idea of inserting neutral sequences into a polyadic product (see (3.17) and (3.41)), because this is the only way to obtain "automatic" associativity. Second, the number of the inserted neutral polyads can be chosen *arbitrarily*, not only minimally, as in (3.17) and (3.41) (as they are neutral). Nevertheless, we can show that this arbitrariness is somewhat restricted.

Indeed, let us consider a polyadic group $\langle G \mid \mu_n, \bar{\mu}_1\rangle$ in the particular case $n = 3$, where for any $e_0 \in G$ and natural k the sequence $\left(\bar{e}_0^k, e_0^k\right)$ is neutral, then we can write

$$\mu_3[g,h,u] = \boldsymbol{\mu}_3^{\bullet}\left[g, \bar{e}_0^k, e_0^k, h, \bar{e}_0^{lk}, e_0^{lk}, u, \bar{e}_0^{mk}, e_0^{mk}\right]. \tag{4.1}$$

If we make the change of variables $e_0^k = e$, then we obtain

$$\mu_3[g,h,u] = \boldsymbol{\mu}_3^{\bullet}\left[g,\bar{e},e,h,\bar{e}^l,e^l,u,\bar{e}^m,e^m\right]. \tag{4.2}$$

Because this should reproduce the formula (3.16), we immediately conclude that $\psi_1(g) = \mathrm{id}$, and the multiplication is the same as in (3.15), and e is again the identity of the binary group $\mathrm{G}^* = \langle G, *, e\rangle$. Moreover, if we put $\psi_2(g) = \varphi(g)$, as in the standard case, then we have a first "half" of the mapping φ, that is $\varphi(g) = \mu_3[e,h,\text{something}]$. Now we are in a position to find this "something" and other "extended" homotopy maps ψ_i from (3.16), but *without* the requirement of a minimal number of inserted neutral polyads, as it was in (3.17). By analogy, we rewrite (4.2) as

$$\mu_3[g,h,u] = \boldsymbol{\mu}_3^{\bullet}\left[g,\bar{e},(e,h,\bar{e}^q),\bar{e},e^{q+1},u,\bar{e}^m,e^m\right], \tag{4.3}$$

where we put $l = q+1$. So we have found the "something", and the map φ is

$$\varphi_q(g) = \boldsymbol{\mu}_3^{\ell_\varphi(q)}\left[e,g,\bar{e}^q\right], \tag{4.4}$$

where the number of multiplications

$$\ell_\varphi(q) = \frac{q+1}{2} \tag{4.5}$$

is an integer $\ell_\varphi(q) = 1,2,3\ldots$, while $q = 1,3,5,7\ldots$. The diagram defined φ_q (e.g., for $q = 3$ and $\ell_\varphi(q) = 2$)

$$\begin{array}{ccccc}
\{\bullet\}\times G\times\{\bullet\}^3 & \xrightarrow{\mu_0^{(e)}\times\mathrm{id}\times\left(\mu_0^{(e)}\right)^3} & G^{\times 5} & \xrightarrow{\mathrm{id}^{\times 2}\times\left(\bar{\mu}_1\right)^3} & G^{\times 5} \\
\Big\uparrow{\scriptstyle \epsilon} & & & & \Big\downarrow{\scriptstyle \mu_3\times\mu_3} \\
G & & \xrightarrow{\qquad\qquad\varphi_q\qquad\qquad} & & G
\end{array} \tag{4.6}$$

commutes (cf. (3.24)). Then, we can find power m in (4.3)

$$\mu_3[g,h,u] = \boldsymbol{\mu}_3^{\bullet}\left[g,\bar{e},(e,h,\bar{e}^q),\bar{e},(e,u,\bar{e}^q)^{q+1},\bar{e},e^{q(q+1)+1}\right], \tag{4.7}$$

and therefore $m = q(q+1)+1$. Thus, we have obtained the "q-deformed" maps ψ_i (cf. (3.19)–(3.22))

$$\psi_1(g) = \varphi_q^{[[0]]_q}(g) = \varphi_q^0(g) = g, \tag{4.8}$$

$$\psi_2(g) = \varphi_q(g) = \varphi_q^{[[1]]_q}(g), \tag{4.9}$$

$$\psi_3(g) = \varphi_q^{q+1}(g) = \varphi_q^{[[2]]_q}(g), \tag{4.10}$$

$$\psi_4(g) = \boldsymbol{\mu}_3^{\bullet}\left[g^{q(q+1)+1}\right] = \boldsymbol{\mu}_3^{\bullet}\left[g^{[[3]]_q}\right], \tag{4.11}$$

where φ is defined by (4.4) and $[[k]]_q$ is the q-deformed number (2.43), and we put $\varphi_q^0 = \mathrm{id}$. The corresponding "$q$-deformed" chain formula (for $n = 3$) can be written as (cf. (3.26)–(3.27) for "nondeformed" case)

$$\mu_3[g,h,u] = g * \varphi_q^{[[1]]_q}(h) * \varphi_q^{[[2]]_q}(u) * b_q, \tag{4.12}$$

$$b_q = e^{\langle \ell_e(q)\rangle}, \tag{4.13}$$

where the degree of the binary identity polyadic power

$$\ell_e(q) = q\frac{[[2]]_q}{2} = \ell_\varphi(q)\left(2\ell_\varphi(q)+1\right) \tag{4.14}$$

is an integer. The corresponding "deformed" chain diagram (e.g., for $q = 3$)

$$\begin{array}{ccccc} G^{\times 3}\times\{\bullet\}^{13} & \xrightarrow{\mathrm{id}\times\varphi_q\times\varphi_q^4\times\left(\mu_0^{(e)}\right)^{\times 13}} & G^{\times 16} & \xrightarrow{\mathrm{id}^{\times 3}\times\mu_3^6} & G^{\times 4} \\ \uparrow{\scriptstyle \epsilon} & & & & \downarrow{\scriptstyle \mu_2^{\times 3}} \\ G\times G\times G & & \xrightarrow{\quad\mu_3\quad} & & G \end{array} \tag{4.15}$$

commutes (cf. the Hosszú-Gluskin diagram (3.28)). In the "deformed" case the polyadic power $e^{\langle \ell_e(q)\rangle}$ is not a fixed point of φ_q and satisfies

$$\varphi_q\left(e^{\langle \ell_e(q)\rangle}\right) = \varphi_q\left(\boldsymbol{\mu}_3^{\bullet}\left[e^{q^2+q+1}\right]\right) = \boldsymbol{\mu}_3^{\bullet}\left[e^{q^2+2}\right] = e^{\langle \ell_e(q)\rangle} * \varphi_q(e) \tag{4.16}$$

or

$$\varphi_q\left(b_q\right) = b_q * \varphi_q(e). \tag{4.17}$$

Instead of (3.33) we have

$$\varphi_q^{q+1}(g) * e^{\langle \ell_e(q)\rangle} = \boldsymbol{\mu}_3^{\bullet}\left[e^{q+1}, g\right] = \boldsymbol{\mu}_3^{\bullet}\left[e^{q+2}\right] * g = e^{\langle \ell_e(q)\rangle} * \varphi_q^{q+1}(e) * g \tag{4.18}$$

or

$$\varphi_q^{q+1}(g) * b_q = b_q * \varphi_q^{q+1}(e) * g. \tag{4.19}$$

The "nondeformed" limit $q \to 1$ of (4.12) gives the Hosszú-Gluskin chain formula (3.26) for $n = 3$. Now let us turn to arbitrary n and write the n-ary multiplication using neutral polyads analogously to (4.3). By the same arguments, as in (4.2), we insert only one neutral polyad $\left(\boldsymbol{e}^{-1}, e\right)$ between the first and second elements in the multiplication, but in other places we insert powers $\left(\left(\boldsymbol{e}^{-1}\right)^k, e^k\right)$ (*allowed* by the chain properties), and obtain

$$\mu_n\left[g_1,\ldots,g_n\right] = \boldsymbol{\mu}_n^{\bullet}\left[g_1, \boldsymbol{e}^{-1}, e, g_2, \ldots, g_n\right] = \boldsymbol{\mu}_n^{\bullet}\left[g_1, \boldsymbol{e}^{-1}, \left(e, g_2, \left(\boldsymbol{e}^{-1}\right)^q\right), \boldsymbol{e}^{-1}, e^{q+1}, g_3, \ldots, g_n\right] = \ldots$$

$$= \boldsymbol{\mu}_n^{\bullet}\left[g_1, \boldsymbol{e}^{-1}, \left(e, g_2, \left(\boldsymbol{e}^{-1}\right)^q\right), \boldsymbol{e}^{-1}, \left(e^{q+1}, g_3, \overbrace{\boldsymbol{e}^{-1},\ldots,\boldsymbol{e}^{-1}}^{q(q+1)}\right) \boldsymbol{e}^{-1}, e^{q(q+1)+1}, g_3, \ldots\right.$$

$$\left.\ldots, \left(\overbrace{e,\ldots,e}^{q^{n-2}+\ldots+q+1}, g_{n-1}, \overbrace{\boldsymbol{e}^{-1},\ldots,\boldsymbol{e}^{-1}}^{q\left(q^{n-2}+\ldots+q+1\right)}\right), \boldsymbol{e}^{-1}, \left(\overbrace{e,\ldots,e}^{q^{n-1}+\ldots+q+1}, g_n, \overbrace{\boldsymbol{e}^{-1},\ldots,\boldsymbol{e}^{-1}}^{q\left(q^{n-1}+\ldots+q+1\right)}\right), \boldsymbol{e}^{-1}, \overbrace{e,\ldots,e}^{q^n+\ldots+q+1}\right]. \tag{4.20}$$

So we observe that the binary product is now the same as in the "nondeformed" case (3.10), while the map φ is

$$\varphi_q(g) = \boldsymbol{\mu}_n^{\ell_\varphi(q)}\left[e, g, \left(\boldsymbol{e}^{-1}\right)^q\right], \tag{4.21}$$

where the number of multiplications

$$\ell_\varphi(q) = \frac{q(n-2)+1}{n-1} \tag{4.22}$$

is an integer and $\ell_\varphi(q) \to q$, as $n \to \infty$, in the nondeformed case $\ell_\varphi(1) = 1$, as in (3.42). Note that the "deformed" map φ_q is the a-quasi-endomorphism [56] of the binary group G_2^*, because from (4.21) we get

$$\begin{aligned}\varphi_q(g) * \varphi_q(h) &= \boldsymbol{\mu}_n^{\bullet}\left[e, g, \left(e^{-1}\right)^q, e^{-1}, e, h, \left(e^{-1}\right)^q\right] \\ &= \boldsymbol{\mu}_n^{\bullet}\left[e, g, e^{-1}, \left(e, e, \left(e^{-1}\right)^q\right), e^{-1}, h, \left(e^{-1}\right)^q\right] = \varphi_q(g * a * h),\end{aligned} \tag{4.23}$$

where

$$a = \boldsymbol{\mu}_n^{\ell_\varphi(q)}\left[e, e, \left(e^{-1}\right)^q\right] = \varphi_q(e). \tag{4.24}$$

In general, a *quasi-endomorphism* can be defined by

$$\varphi_q(g) * \varphi_q(h) = \varphi_q\left(g * \varphi_q(e) * h\right). \tag{4.25}$$

The corresponding diagram

$$\begin{array}{ccccc} G \times G & \xrightarrow{\mu_2} & G & \xleftarrow{\varphi_q} & G \\ \uparrow{\scriptstyle \varphi_q \times \varphi_q} & & & & \uparrow{\scriptstyle \mu_2 \times \mu_2} \\ G \times G & \xrightarrow{\epsilon} & G \times \{\bullet\} \times G & \xrightarrow{\mathrm{id} \times \mu_0^{(e)} \times \mathrm{id}} & G \times G \times G \end{array} \tag{4.26}$$

commutes. If $q = 1$, then $\varphi_q(e) = e$, and the distinguished element a turns to the binary identity $a = e$, such that the a-quasi-endomorphism φ_q becomes an automorphism of G_2^*.

Remark 4.1. *The choice* (4.21) *of the a-quasi-endomorphism* φ_q *is different from [56], the latter (in our notation) is* $\varphi_k(g) = \mu_n\left[a^{k-1}, g, a^{n-k}\right]$, $k = 1, \ldots, n-1$, *it has only one multiplication and leads to the "nondeformed" chain formula* (3.44) *(for semigroup case).*

It follows from (4.20), that the "extended" homotopy maps ψ_i (3.3) are (cf. (4.8)–(4.11))

$$\psi_1(g) = \varphi_q^{[[0]]_q}(g) = \varphi_q^0(g) = g, \tag{4.27}$$

$$\psi_2(g) = \varphi_q(g) = \varphi_q^{[[1]]_q}(g), \tag{4.28}$$

$$\psi_3(g) = \varphi_q^{q+1}(g) = \varphi_q^{[[2]]_q}(g), \tag{4.29}$$

$$\vdots$$

$$\psi_{n-1}(g) = \varphi_q^{q^{n-3}+\ldots+q+1}(g) = \varphi_q^{[[n-2]]_q}(g), \tag{4.30}$$

$$\psi_n(g) = \varphi_q^{q^{n-2}+\ldots+q+1}(g) = \varphi_q^{[[n-1]]_q}(g), \tag{4.31}$$

$$\psi_{n+1}(g) = \boldsymbol{\mu}_n^{\bullet}\left[g^{q^{n-1}+\ldots+q+1}\right] = \boldsymbol{\mu}_n^{\bullet}\left[g^{[[n]]_q}\right]. \tag{4.32}$$

In terms of the polyadic power (2.10), the last map is

$$\psi_{n+1}(g) = g^{\langle \ell_e \rangle}, \tag{4.33}$$

where (cf. (4.22))

$$\ell_e(q) = q\frac{[[n-1]]_q}{n-1} \tag{4.34}$$

is an integer. Thus the "q-deformed" n-ary chain formula is (cf. (3.44))

$$\mu_n[g_1, \ldots, g_n] = g_1 * \varphi_q^{[[1]]_q}(g_2) * \varphi_q^{[[2]]_q}(g_3) * \ldots * \varphi_q^{[[n-2]]_q}(g_{n-1}) * \varphi_q^{[[n-1]]_q}(g_n) * e^{\langle \ell_e(q) \rangle}. \tag{4.35}$$

In the "nondeformed" limit $q \to 1$ (4.35) reproduces the Hosszú-Gluskin chain formula (3.44). Let us obtain the "deformed" analogs of the distinguished element relations (3.47)–(3.48) for arbitrary n (the case $n = 3$ is in (4.16)–(4.18)). Instead of the fixed point relation (3.48) we now have from (4.21), (4.34) and (4.32) the *quasi-fixed point*

$$\varphi_q\left(b_q\right) = b_q * \varphi_q(e), \tag{4.36}$$

where the "deformed" distinguished element b_q is (cf. (3.50))

$$b_q = \boldsymbol{\mu}_n^{\bullet}\left[e^{[[n]]_q}\right] = e^{\left\langle \ell_e(q)\right\rangle}. \tag{4.37}$$

The conjugation relation (3.47) in the "deformed" case becomes the *quasi-conjugation*

$$\varphi_q^{[[n-1]]_q}(g) * b_q = b_q * \varphi_q^{[[n-1]]_q}(e) * g. \tag{4.38}$$

This allows us to rewrite the "deformed" chain formula (4.35) as

$$\mu_n[g_1, \ldots, g_n] = g_1 * \varphi_q^{[[1]]_q}(g_2) * \varphi_q^{[[2]]_q}(g_3) * \ldots * \varphi_q^{[[n-2]]_q}(g_{n-1}) * b_q * \varphi_q^{[[n-1]]_q}(e) * g_n. \tag{4.39}$$

Using the above proof sketch, we formulate the following "q-deformed" analog of the Hosszú-Gluskin theorem:

Theorem 4.2. *On a polyadic group* $\mathrm{G}_n = \langle G \mid \mu_n, \bar{\mu}_1\rangle$ *one can define a binary group* $\mathrm{G}_2^* = \langle G \mid \mu_2 = *, e\rangle$ *and (the infinite "q-series" of) its automorphism* φ_q *such that the "deformed" chain formula* (4.35) *is valid*

$$\mu_n[g_1, \ldots, g_n] = \left(* \prod_{i=1}^{n} \varphi^{[[i-1]]_q}(g_i)\right) * b_q, \tag{4.40}$$

where (the infinite "q-series" of) the "deformed" distinguished element b_q *(being a polyadic power of the binary identity* (4.37)*) is the quasi-fixed point of* φ_q (4.36) *and satisfies the quasi-conjugation* (4.38) *in the form*

$$\varphi_q^{[[n-1]]_q}(g) = b_q * \varphi_q^{[[n-1]]_q}(e) * g * b_q^{-1}. \tag{4.41}$$

In the "nondeformed" case $q = 1$ we obtain the Hosszú-Gluskin chain formula (3.44) and the corresponding **Theorem 3.2**.

Example 4.3. *Let us have a binary group* $\langle G \mid (\cdot), 1\rangle$ *and a distinguished element* $e \in G$, $e \neq 1$, *then we can define a binary group* $\mathrm{G}_2^* = \langle G \mid (*), e\rangle$ *by the product*

$$g * h = g \cdot e^{-1} \cdot h. \tag{4.42}$$

The quasi-endomorphism

$$\varphi_q(g) = e \cdot g \cdot e^{-q} \tag{4.43}$$

satisfies (4.25) *with* $\varphi_q(e) = e^{2-q}$, *and we take*

$$b_q = e^{[[n]]_q}. \tag{4.44}$$

Then we can obtain the "q-deformed" chain formula (4.40) *(for* $q = 1$ *see, e.g., [52]).*

We observe that the chain formula is the "q-series" of equivalence relations (4.40), which can be formulated as an invariance. Indeed, let us denote the r.h.s. of (4.40) by $\mathcal{M}_q(g_1, \ldots, g_n)$, and the l.h.s. as $\mathcal{M}_0(g_1, \ldots, g_n)$, then the chain formula can be written as some invariance (cf. associativity as an invariance (2.18)).

Theorem 4.4. *On a polyadic group* $\mathrm{G}_n = \langle G \mid \mu_n, \bar{\mu}_1 \rangle$ *we can define a binary group* $\mathrm{G}^* = \langle G \mid \mu_2 = *, e \rangle$ *such that the following invariance is valid*

$$\mathcal{M}_q(g_1, \dots, g_n) = \text{invariant}, \quad q = 0, 1, \dots , \tag{4.45}$$

where

$$\mathcal{M}_q(g_1, \dots, g_n) = \begin{cases} \mu_n[g_1, \dots, g_n], & q = 0, \\ \left(* \prod_{i=1}^{n} \varphi^{[[i-1]]_q}(g_i) \right) * b_q, & q > 0, \end{cases} \tag{4.46}$$

and the distinguished element $b_q \in G$ *and the quasi-endomorphism* φ_q *of* G_2^* *are defined in* (4.37) *and* (4.21) *respectively.*

Example 4.5. *Let us consider the ternary q-product used in the nonextensive statistics [26]*

$$\mu_3[g, t, u] = \left(g^{\hbar} + t^{\hbar} + u^{\hbar} - 3 \right)^{\frac{1}{\hbar}}, \tag{4.47}$$

where $\hbar = 1 - q_0$*, and* $g, t, u \in G = \mathbb{R}_+$*,* $0 < q_0 < 1$*, and also* $g^{\hbar} + t^{\hbar} + u^{\hbar} - 3 > 0$ *(as for other terms inside brackets with power* $\frac{1}{\hbar}$ *below). In case* $\hbar \to 0$ *the q-product becomes an iterated product in* $\mathbb{R}_+$ *as* $\mu_3[g, t, u] \to gtu$*. The quermap* $\bar{\mu}_1$ *is given by*

$$\bar{g} = \left(3 - g^{\hbar} \right)^{\frac{1}{\hbar}}. \tag{4.48}$$

The polyadic system $\mathrm{G}_n = \langle G \mid \mu_3, \bar{\mu}_1 \rangle$ *is a ternary group, because each element is querable. Take a distinguished element* $e \in G$ *and use* (3.15)*,* (4.47) *and* (4.48) *to define the product*

$$g * t = \left(g^{\hbar} - e^{\hbar} + t^{\hbar} \right)^{\frac{1}{\hbar}} \tag{4.49}$$

of the binary group $\mathrm{G}_2^* = \langle G \mid \mu_2 = (*), e \rangle$*.*

1) The Hosszú-Gluskin chain formula $(q = 1)$*. The automorphism* (3.23) *of* G^* *is now the identity map* $\varphi = \mathrm{id}$*. The first polyadic power of the distinguished element e is*

$$b = e^{\langle 1 \rangle} = \mu_3\left[e^3\right] = \left(3e^{\hbar} - 3 \right)^{\frac{1}{\hbar}}. \tag{4.50}$$

The chain formula (3.26) *can be checked as follows*

$$\mu_3[g, t, u] = (((g * t) * u) * b) = \left(\left(\left(g^{\hbar} - e^{\hbar} + t^{\hbar} \right) - e^{\hbar} + u^{\hbar} \right) - e^{\hbar} + b^{\hbar} \right)^{\frac{1}{\hbar}}$$
$$= \left(g^{\hbar} - e^{\hbar} + t^{\hbar} - e^{\hbar} + u^{\hbar} - e^{\hbar} + 3e^{\hbar} - 3 \right)^{\frac{1}{\hbar}} = \left(g^{\hbar} + t^{\hbar} + u^{\hbar} - 3 \right)^{\frac{1}{\hbar}}. \tag{4.51}$$

2) The "q-deformed" chain formula (for conciseness we consider only the case $q = 3$*). Now the quasi-endomorphism* φ_q (4.4) *is not the identity, but is*

$$\varphi_{q=3}(g) = \left(g^{\hbar} - 2e^{\hbar} + 3 \right)^{\frac{1}{\hbar}}. \tag{4.52}$$

In case $q = 3$ *we need its* 4*th* $(= q + 1)$ *power* (4.12)

$$\varphi_{q=3}^4(g) = \left(g^{\hbar} - 8e^{\hbar} + 12 \right)^{\frac{1}{\hbar}}. \tag{4.53}$$

The deformed polyadic power $e^{\langle \ell_e \rangle}$ from (4.12) is (see, also, (4.11))

$$b_{q=3} = e^{\langle 5 \rangle} = \boldsymbol{\mu}_3^5 \left[e^{13} \right] = \left(13 e^{\hbar} - 18 \right)^{\frac{1}{\hbar}} . \tag{4.54}$$

Now we check the "q-deformed" chain formula (4.12) as

$$\mu_3 [g, t, u] = g * \varphi_{q=3}(t) * \varphi_{q=3}^4 (u) * b_{q=3} = \left(\left(\left(g * \varphi_{q=3}(t) \right) * \varphi_{q=3}^4 (u) \right) * b_{q=3} \right) \tag{4.55}$$

$$= \left(g^{\hbar} - e^{\hbar} + \left(t^{\hbar} - 2e^{\hbar} + 3 \right) - e^{\hbar} + \left(u^{\hbar} - 8e^{\hbar} + 12 \right) - e^{\hbar} + \left(13e^{\hbar} - 18 \right) \right)^{\frac{1}{\hbar}} \tag{4.56}$$

$$= \left(g^{\hbar} + t^{\hbar} + u^{\hbar} - 3 \right)^{\frac{1}{\hbar}} . \tag{4.57}$$

In a similar way, one can check the "q-deformed" chain formula for any allowed q (determined by (4.22) and (4.34) to obtain an infinite q-series of the chain representation of the same n-ary multiplication.

5. Generalized "Deformed" Version of the Homomorphism Theorem

Let us consider a homomorphism of the binary groups entering into the "deformed" chain formula (4.40) as $\Phi^* : \mathrm{G}_2^* \to \mathrm{G}_2^{*\prime}$, where $\mathrm{G}_2^{*\prime} = \langle G' \mid *', e' \rangle$. We observe that, because Φ^* commutes with the binary multiplication, we need its commutation also with the automorphisms φ_q in each term of (4.40) (which fixes equality of the "deformation" parameters $q = q'$) and its homomorphic action on b_q. Indeed, if

$$\Phi^* \left(\varphi_q (g) \right) = \varphi'_q \left(\Phi^* (g) \right), \tag{5.1}$$

$$\Phi^* \left(b_q \right) = b'_q, \tag{5.2}$$

then we get from (4.40)

$$\begin{aligned} \Phi^* \left(\mu_n [g_1, \ldots, g_n] \right) &= \Phi^* (g_1) *' \Phi^* \left(\varphi_q^{[[1]]_q} (g_2) \right) *' \ldots *' \Phi^* \left(\varphi_q^{[[n-1]]_q} (g_n) \right) *' \Phi^* \left(b_q \right) \\ &= \Phi^* (g_1) *' \varphi_q'^{[[1]]_q} \left(\Phi^* (g_2) \right) *' \ldots *' \varphi_q^{[[n-1]]_q} \left(\Phi^* (g_n) \right) *' b'_q \\ &= \mu'_n \left[\Phi^* (g_1), \ldots, \Phi^* (g_n) \right], \end{aligned} \tag{5.3}$$

where $g' *' h' = \mu'_n \left[g', e'^{-1}, h' \right]$, $\varphi'_q (g') = \boldsymbol{\mu}_n'^{\,\ell_\varphi (q)} \left[e', g', \left(e'^{-1} \right)^q \right]$, $b'_q = \boldsymbol{\mu}_n'^{\bullet} \left[e'^{\,[[n]]_q} \right]$. Comparison of (5.3) and (2.49) leads to

Theorem 5.1. *A homomorphism Φ^* of the binary group G_2^* gives rise to a homomorphism Φ of the corresponding n-ary group G_n, if Φ^* satisfies the "deformed" compatibility conditions (5.1)–(5.2).*

The "nondeformed" version ($q = 1$) of this theorem and the case of Φ^* being an isomorphism was considered in [23].

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