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The Small Inductive Dimension of Subsets of Alexandroff Spaces

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Abstract.
We describe the small inductive dimension *ind* in the class of Alexandroff spaces by the use of some standard spaces. Then for *ind* we suggest decomposition, sum and product theorems in the class. The sum and product theorems there we prove even for the small transfinite inductive dimension *trind*. As an application of that, for each positive integers k, n such that $k \leq n$ we get a simple description in terms of even and odd numbers of the family $\mathbb{S}(k, n) = \{S \subset K^n : |S| = k + 1 \text{ and } ind\, S = k\}$, where K is the Khalimsky line.

1. Introduction

Recall ([J]) that a topological space X is called *Alexandroff* if for each point $x \in X$ there is the minimal open set $V(x)$ containing x. We will keep the notation along the text. It is easy to see that for each point $y \in V(x)$ we have $V(y) \subset V(x)$. This implies that if X is a T_0-space and $x, y \in X$ then $V(x) = V(y)$ iff $x = y$. Moreover, if X is a T_1-space then $V(x) = \{x\}$ for each point $x \in X$, i.e. an Alexandroff space X is a T_1-space iff X is discrete. Alexandroff spaces appear by a natural way in studies of topological models of digital images. They are quotient spaces of the Euclidean spaces $\mathbb{R}^n$ defined by special decompositions (see [Kr]). Some studies of Alexandroff spaces from the general topology point of view can be found in [A] and [D].

We will follow the definition of the small inductive dimension *ind* suggested in [P]. Let X be a space and n an integer ≥ 0. Then

(a) $ind\, X = -1$ iff $X = \emptyset$;
(b) $ind\, X \leq n$ iff for each point $x \in X$ and each open set V containing x there is an open set W such that $x \in W \subset V$ and $ind\, Bd_X W < n$;
(c) $ind\, X = \infty$ iff $ind\, X \leq n$ does not valid for each integer $n \geq 0$.

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It is easy to see that if $ind\,X = n$ for some integer $n \geq 0$ then the cardinality of X is greater than n.

Let us also recall ([P]) that for a space X and any subspace Y of X we have $ind\,Y \leq ind\,X$.

Example 1.1. *Let E be the topological space $(\mathbb{R}, \tau)$, where $\mathbb{R}$ is the set of real numbers and τ is the topology on $\mathbb{R}$ defined by the base $\mathcal{B} = \{[x, \infty) : x \in \mathbb{R}\}$. It is easy to see that E is a connected Alexandroff T_0-space such that $ind\,B = \infty$ for each $B \in \mathcal{B}$. Moreover, for each integer $n \geq 0$ the subspace $E(n) = \{0, 1, \ldots, n\}$ of E has $ind\,E(n) = n$ and any subspace Y of E of cardinality $n + 1$ is homeomorphic to $E(n)$. Since the spaces $E(n), n = 0, 1, \ldots,$ will play some role in the paper, we will keep the notation along the text.*

In [WW1] P. Wiederhold and R. G. Wilson started to study the behavior of the small inductive dimension *ind* in the Alexandroff T_0-spaces. In particular (cf. [WW1] and [WW2]),

(A) they proved the product theorem (see Remark 2.11) for *ind*;

(B) they showed that if (X, τ) is an Alexandroff T_0-space and $\leq_\tau$ is its specialization partial order (i.e. $x \leq_\tau y$ iff $x \in Cl_X(\{y\})$) then the small inductive dimension of (X, τ) is equal to the partial order dimension of (X, τ) defined as the supremum of all lengths of chains in $(X, \leq_\tau)$; and

(C) they observed that the quotient spaces of the Euclidean spaces $\mathbb{R}^n$ defined by some standard decompositions based on the model of Kronheimer ([Kr]) have the dimension *ind* equal to n.

Let us also note that the coincidence of three kinds of dimension (one of them is *ind*) on partially ordered sets (close related to Alexandroff spaces) is established in [EKM].

In this paper we describe the dimension *ind* in the class of Alexandroff spaces by the use of spaces $E(n), n = 0, 1, \ldots$ (Proposition 2.1). Then for *ind* we suggest decomposition, sum and product theorems in the class (Propositions 2.2, 2.3 and 2.5, respectively). Let us note that the product theorem is written as an equality and thus it is stronger than the theorem from [WW1]. The sum and product theorems there we prove even for the small transfinite inductive dimension *trind* (Propositions 4.3 and 4.4).

As an application of these results, for each positive integers k, n such that $k \leq n$ we get a simple description in terms of even and odd numbers of the family $\mathbb{S}(k, n) = \{S \subset K^n : |S| = k + 1 \text{ and } ind\,S = k\}$, where K is the Khalimsky line (see Remarks 3.2 and 3.7). (Let us note that each element of the family $\mathbb{S}(k, n)$ is homeomorphic to the space $E(k)$.) Observe that for any subspace A of K^n we have

(D) $ind\,A = n$ iff A contains an element of $\mathbb{S}(n, n)$, and
(E) $ind\,A = k < n$ iff A contains an element of $\mathbb{S}(k, n)$ and it contains no element from $\mathbb{S}(k + 1, n)$.

Furthermore, we suggest some simple calculations of n-dimensional subsets of cardinality $n + 1$ in the closures of the minimal neighborhoods of points in K^n as follows (Remark 3.7). The closure $Cl_{K^n} V(x)$ in K^n of the minimal open neighborhood $V(x)$ of a point $x = (x_1, \ldots, x_n)$ with m odd coordinates contains exactly $2^{2n-m} \cdot n!$ n-dimensional in the sense of *ind* subsets of cardinality $n + 1$ (of course, each of theses sets is homeomorphic to the space $E(n)$).

We also discuss the behavior of the transfinite extension of *ind* in Alexandroff spaces (Section 4).

2. Properties of the Small Inductive Dimension in Alexandroff Spaces

The following trivial known facts about Alexandroff spaces will be useful in the paper.

(A) If a space X is Alexandroff and $Y \subset X$, then the subspace Y of X is also Alexandroff and for each point $y \in Y$ the set $V(x) \cap Y$ is the minimal open neighborhood of y in Y.
(B) If spaces X and Y are Alexandroff, then the topological product $X \times Y$ is also Alexandroff and for each point $(x, y) \in X \times Y$ the set $V(x) \times V(y)$ is the minimal neighborhood of (x, y) in $X \times Y$.
(C) If spaces X_α, $\alpha \in A$, are Alexandroff, then the topological union $\oplus_{\alpha \in A} X_\alpha$ is also Alexandroff and for each $\alpha \in A$ and each point $x \in X_\alpha$ the set $V(x)$ (defined in the Alexandroff space X_α) is the minimal open neighborhood of x in the space $\oplus_{\alpha \in A} X_\alpha$.

Let us also list some simple known facts about the dimension *ind* behavior in Alexandroff spaces. Let X be an Alexandroff space and n an integer ≥ 0. Then the following is valid.

(D) $ind\, X \leq n$ iff $\sup\{ind\, Bd_X V(x) : x \in X\} \leq n-1$.
In particular, $ind\, X = 0$ iff for every $x, y \in X$ we have either $V(x) = V(y)$ or $V(x) \cap V(y) = \emptyset$. Moreover, if X is a T_0-space then $ind\, X = 0$ iff X is discrete.

(E) If $ind\, X = n$, then there is a point x such that $ind\, Bd_X V(x) = n-1$ and $ind\, Cl_X V(x) = n$.

Proposition 2.1. *Let X be an Alexandroff space. Then $ind\, X \geq n \geq 0$ iff X contains a subspace which is homeomorphic to the space $E(n)$.*
In particular, if the cardinality of X is equal to $n+1$ then $ind\, X = n$ iff X is homeomorphic to $E(n)$.

Proof: The sufficiency follows from the monotonicity of *ind* and the fact that $ind\, E(n) = n$. For the necessity apply an induction on $n \geq 0$. Let $ind\, X \geq n = 0$. Hence, X contains a point which is homeomorphic to $E(0)$. Assume that the statement is valid for $n < k \geq 1$. Let $ind\, X \geq k$. Note that there is a point $x \in X$ such that $ind\, Bd_X V(x) \geq k-1$. By the inductive assumption there are points $x_0, \ldots, x_{k-1}$ of $Bd_X V(x)$ and a homeomorphism $f : Y = \{x_0, \ldots, x_{k-1}\} \to E(k-1)$ such that $f(x_i) = i$ for each $i \leq k-1$. It is easy to see that $V(x_{k-1}) \subsetneq \cdots \subsetneq V(x_0)$. Since $x_{k-1} \in Bd_X V(x)$, there is a point $x_k \in V(x) \cap V(x_{k-1})$. Note that $V(x_k) \subsetneq V(x_{k-1})$ and the mapping $g : Z = \{x_0, \ldots, x_k\} \to E(k)$, defined by $g(x_i) = i$ for each $i \leq k$, is a homeomorphism. □

Let X be an Alexandroff space and $0 < ind\, X = n < \infty$.

Put $\mathcal{F}(X) = \{Y \subseteq X :$ there is a homeomorphism $f_Y : E(n) \to Y\}$ and $X_0 = \cup\{V(f_Y(n)) : Y \in \mathcal{F}(X)\}$.

Proposition 2.2. *Let X be an Alexandroff space and $ind\, X = n$ for some integer $n > 0$. Then*

(i) for each $Y \in \mathcal{F}(X)$ either $|V(f_Y(n))| = 1$ or the subspace topology on the set $V(f_Y(n))$ is trivial; in particular, for any $Y_1, Y_2 \in \mathcal{F}(X)$ we have either $V(f_{Y_1}(n)) \cap V(f_{Y_2}(n)) = \emptyset$ or $V(f_{Y_1}(n)) = V(f_{Y_2}(n))$;

(ii) the set X_0 is open in X, $ind\, X_0 = 0$ and $ind\,(X \setminus X_0) = n-1$; moreover, $X_0 = \cup\{\{f_Y(n)\} : Y \in \mathcal{F}(X)\}$ and for each $Y \in \mathcal{F}(X)$ we have $Y \cap X_0 = \{f_Y(n)\}$;

(iii) there are disjoint subsets $X_0, \ldots, X_n$ of X such that $X = \cup_{j=0}^{n} X_j$ and for each $i \leq n$ we have $ind\, X_i = 0$ (the set X_i is discrete in itself, whenever $X \setminus \cup_{j<i} X_j$ is a T_0-space); moreover, $X_i \supseteq \cup\{\{f_Y(n-i)\} : Y \in \mathcal{F}(X)\}$ and the set $\cup_{j=0}^{i} X_j$ is open in X.

Proof: (i): Assume that $|V(f_Y(n))| > 1$ and the subspace topology on the set $V(f_Y(n))$ is not trivial. So there is a point $z \in V(f_Y(n))$ such that $V(z) \subsetneq V(f_Y(n))$. It is easy to see that the subspace $Z = Y \cup \{z\}$ of X is homeomorphic to the space $E(n+1)$. We have a contradiction.

(ii): It is easy to see that the set X_0 is open in X, $ind\, X_0 = 0$, $X_0 = \cup\{\{f_Y(n)\} : Y \in \mathcal{F}(X)\}$, and $ind\,(X \setminus X_0) \leq n-1$. Consider a $Y \in \mathcal{F}(X)$. Since $ind\, X_0 = 0$, we have $|Y \cap X_0| = 1$ and $|Y \cap (X \setminus X_0)| = n-1$. This implies that $Y \cap X_0 = \{f_Y(n)\}$ and $ind\,(X \setminus X_0) = n-1$.

(iii): Apply (ii). □

Proposition 2.3. *Let X be an Alexandroff space and $X = X_1 \cup X_2$, where X_i, $i = 1, 2$, is closed in X. Then $ind\, X = \max\{ind\, X_1, ind\, X_2\}$.*

Proof: Put $n = \max\{ind\, X_1, ind\, X_2\}$. It is enough to show that if $n < \infty$ then $n \geq ind\, X$. Assume that $n < ind\, X$. By Proposition 2.1 the space X contains a subspace Y which is homeomorphic to the space $E(n+1)$. Note that $Y = (Y \cap X_1) \cup (Y \cap X_2)$ and the sets $(Y \cap X_1), (Y \cap X_2)$ are closed in Y. Hence at least one of them is equal to Y. Let $(Y \cap X_1) = Y$. So $ind\, X_1 \geq n+1$. We have a contradiction. □

Corollary 2.4. *Let X be an Alexandroff space and $X = \cup_{i=1}^{k} X_i$, where k is a positive integer and for each $i \leq k$ the set X_i is closed in X. Then $ind\, X = \max\{ind\, X_i : i \leq k\}$.*

Proposition 2.5. *Let X and Y be non-empty Alexandroff spaces. Then we have $ind\,(X \times Y) = ind\, X + ind\, Y$.*

Proof: First, let us show that $ind\,(X \times Y) \leq ind\,X + ind\,Y$. Put $n = ind\,X + ind\,Y$. Apply induction on $n \geq 0$. Consider a point $(x, y) \in X \times Y$ and note that

$$Bd_{X\times Y}(V(x) \times V(y)) = (Bd_X V(x) \times Cl_Y V(y)) \cup (Cl_X V(x) \times Bd_Y V(y)).$$

So the case $n = 0$ is trivial. If $n > 0$ then by the inductive assumption we have

$$\max\{ind\,(Bd_X V(x) \times Cl_Y V(y)), ind\,(Cl_X V(x) \times Bd_Y V(y))\} \leq n - 1.$$

It follows from Proposition 2.3 that $ind\,(Bd_{X\times Y}(V(x) \times V(y))) \leq n - 1$. Hence, $ind\,(X \times Y) \leq n$.

Now let us show that $ind\,(X\times Y) \geq ind\,X + ind\,Y$. Apply again induction on $n \geq 0$. Note that the case $n = 0$ is trivial. Let $n > 0$. We consider a point $x \in X$ such that $ind\,Cl_X V(x) = ind\,X$ and $ind\,Bd_X V(x) = ind\,X - 1$, and a point $y \in Y$ such that $ind\,Cl_Y V(y) = ind\,Y$ and $ind\,Bd_Y V(y) = ind\,Y - 1$. By the inductive assumption we have

$$ind\,(Cl_X V(x) \times Bd_Y V(y)) = ind\,(Bd_X V(x) \times Cl_Y V(y)) = n - 1.$$

This implies that $ind\,(X \times Y) \geq n$. □

Remark 2.6. *Let us notice that the inequality $ind\,(X\times Y) \leq ind\,X + ind\,Y$ for non-empty Alexandroff T_0-spaces X, Y was announced in [WW1].*

Corollary 2.7. *Let X_i be an non-empty Alexandroff space for each $i \leq k$, where k is some positive integer. Then $ind\,(\prod_{i=1}^k X_i) = \Sigma_{i=1}^k ind\,X_i$. In particular, $ind\,(\prod_{i=j}^k E(i_j)) = \Sigma_{j=1}^k i_j$, where i_j is an integer ≥ 1 for each $j \leq k$.*

Corollary 2.8. *Let $X = \prod_{i=1}^m E(n_i)$, where n_i is a positive integer for each $i \leq m$. Then there is a subset Y of X such that Y is homeomorphic to the space $E(\sum_{i=1}^m n_i)$.*

Now, we will consider the finite powers $E(1)^n, n \geq 2$.

Let n and i be integers such that $1 \leq i \leq n$. We will use the following notations:

(a) Let $\pi_i^n : E(1)^n \to E(1)$ be the projection of $E(1)^n$ onto the i-th coordinate.

(b) Let $\iota_i^n : E(1)^{n-1} \to E(1)^n$ be the mapping of $E(1)^{n-1}$ into $E(1)^n$ defined by $\iota_i^n(x_1, \ldots, x_{n-1}) = (y_1, \ldots, y_n)$, where $y_i = 0$ and the ordered $(n-1)$-tuple $(x_1, \ldots, x_{n-1})$ coincides with the ordered $(n-1)$-tuple $(y_1, \ldots, \widehat{y_i} \ldots, y_n)$ with removed y_i.

Proposition 2.9. *We have $ind\,(\cup_{i=1}^n (\pi_i^n)^{-1}(0)) = n - 1$.*
(Note that $\cup_{i=1}^n (\pi_i^n)^{-1}(0) = E(1)^n \setminus \{(1, \ldots, 1)\}$.)

Proof: Note that for each $i \leq n$ the closed subset $(\pi_i^n)^{-1}(0)$ of $E(1)^n$ is homeomorphic to $E(1)^{n-1}$, and hence $ind\,(\pi_i^n)^{-1}(0) = n - 1$. Then one can apply Corollary 2.4. □

Proposition 2.10. *Let X be the disjoint union $Y \cup \{p\}$ of a closed subset Y with $ind\,Y \leq n \geq 0$ and a point p. Then $ind\,X \leq n + 1$.*

One can easily show Proposition 2.10 by a standard argument, so we omit the proof.

Let us consider the following subsets of $E(1)^2$:

$D(2) = \{(0, 1), (1, 0)\}$, $S_1 = \{(0, 0), (0, 1), (1, 1)\}$, $S_2 = \{(0, 0), (1, 0), (1, 1)\}$. $S_3 = \{(1, 1), (0, 1), (1, 0)\}$ and $S_4 = \{(1, 0), (0, 1), (0, 0)\}$.

Observe that the subspace $D(2)$ is discrete, the subspaces S_1, S_2 are homeomorphic to $E(2)$ and $ind\,S_3 = ind\,S_4 = 1$. Put $\mathbb{S}_2 = \{S_1, S_2\}$. Then for every integer $n > 2$ consider the subspace

$$D(n) = \{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$$

of $E(1)^n$ and define by induction the family

$$\mathbb{S}_n = \{\iota_m^n(S) \cup \{(1, \ldots, 1)\} : S \in \mathbb{S}_{n-1}, m \leq n\}$$

of subsets of $E(1)^n$.

Remark 2.11. *Note that each element S of $\mathbb{S}_n$ consists of $n+1$ points which can be ordered in a sequence $p_0, \ldots, p_n$ such that $p_0 = (0, \ldots, 0)$, $p_n = (1, \ldots, 1)$ and for each $i \leq n-1$ the point p_{i+1} obtained from the point p_i through replacing 0 by 1 in one of the coordinates.*

Proposition 2.12. *For each integer $n \geq 2$ we have the following.*

(a) The space $D(n)$ is discrete.
(b) $|\mathbb{S}_n| = n!$
(c) Every element S of $\mathbb{S}_n$ is homeomorphic to $E(n)$ and so $ind\, S = n$.
(d) For every subspace A of $E(1)^n$ which contains no element of $\mathbb{S}_n$ we have $ind\, A < n$.

Proof: (a), (b) and (c) are evident. Let us show (d). Apply induction on $n \geq 2$. For $n = 2$ the statement is evident. Let $n > 2$. Put $x = (1, \ldots, 1) \in E(1)^n$. We notice that x is an isolated point in $E(1)^n$. If $x \notin A$, then $ind\, A \leq n-1$ by Proposition 2.9. Assume that $x \in A$ and A does not contain any $S \in \mathbb{S}_n$. For each $i \leq n$ put $A_i = A \cap (\pi_i^n)^{-1}(0)$. Since $x \in A$, if we regard $(\pi_i^n)^{-1}(0)$ as $E(1)^{n-1}$ by a natural way, A_i does not contain any member of $\mathbb{S}_{n-1}$. Hence, by the inductive assumption, we have $ind\, A_i \leq n-2$. Note that A_i is a closed subset of A, and hence the union $\cup_{i=1}^n A_i = A \setminus \{x\}$ is a closed subset of A. Moreover, by Proposition 2.9, we have $ind\,(\cup_{i=1}^n A_i) \leq n-2$. Now it follows from Proposition 2.10 that $ind\, A \leq n-1$. □

Remark 2.13. *Since the space $E(1)^2$ contains the discrete subspace $D(2)$ of cardinality 2, there is no embedding of $E(1)^2$ into $E(n)$ for any integer $n \geq 1$.*

3. The Small Inductive Dimension in Khalimsky Spaces

In the present section, we shall consider the dimension properties in Khalimsky spaces. Let K be the Khalimsky line ([K]), *i.e.* the topological space $(\mathbb{Z}, \tau)$, where $\mathbb{Z}$ is the set of integers and τ is the topology of $\mathbb{Z}$ generated by the base $\mathcal{B} = \{\{2k+1\}, \{2k-1, 2k, 2k+1\} : k \in \mathbb{Z}\}$. Let us recall that K is a connected Alexandroff T_0-space with $ind\, K = 1$. Note that for each odd integer n the subset $R_n = \{n, n+1\}$ (resp. $L_n = \{n-1, n\}$) of K can be identified with the space $E(1)$. In addition, we notice some simple facts about K.

[Fact 3.1] For the minimal open neighborhoods of points in the Khalimsky line, we have the following.

(a) For each even integer n the set $V(n)$ (resp. $Cl_K V(n)$) is homeomorphic to $V(0) = \{-1, 0, 1\}$ (resp. $Cl_K V(0) = \{-2, -1, 0, 1, 2\}$).
(b) For each odd integer n the set $V(n)$ (resp. $Cl_K V(n)$) is homeomorphic to $V(1) = \{1\}$ (resp. $Cl_K V(1) = \{0, 1, 2\}$).
(c) The set $Cl_K V(0)$ is the union of its closed subsets $\{-2, -1, 0\}$ and $\{0, 1, 2\}$.

Lemma 3.1. *For each subset A of the Khalimsky line K with $ind\, A = 1$ there is an odd integer n such that either $R_n \subset A$ or $L_n \subset A$.*

Proof: The lemma is a base of induction for the proof of Theorem 3.3. Since $ind\, A = 1$ there is a point $x \in A$ such that $ind\, Cl_A V'(x) = 1$, where $V'(x)$ is the minimal open neighborhood of x in A. Since $V'(x) = V(x) \cap A$, where $V(x)$ is the minimal open neighborhood of x in K, we have $Cl_A V'(x) \subset A \cap Cl_K V(x)$. If x is an odd number, then $R_x \subset A \cap Cl_K V(x)$ or $L_x \subset A \cap Cl_K V(x)$, because $ind\, A \cap Cl_K V(x) = 1$. Now, we suppose that x is an even number. If $\{x-1, x+1\} \cap (A \cap Cl_K V(x)) = \emptyset$, then $A \cap Cl_K V(x) \subset \{x-2, x, x+2\}$. This implies that $A \cap Cl_K V(x)$ is discrete, and hence $ind\, A \cap Cl_K V(x) = 0$. This is a contradiction. Hence, $\{x-1, x+1\} \cap (A \cap Cl_K V(x)) \neq \emptyset$, and hence $\{x-1, x\} \subset A \cap Cl_K V(x)$ or $\{x, x+1\} \subset A \cap Cl_K V(x)$. This completes the proof. □

Put $\mathbb{S}(1) = \{R_{2n+1}, L_{2n+1} : n \in \mathbb{Z}\}$. Let k be any integer ≥ 2. For each positive integer $j \leq k$ consider a subspace Y_j of K which is either R_{n_j} or L_{n_j} for some odd integer n_j. The product $Y_1 \times \cdots \times Y_k$ can be identified with $E(1)^k$ and put $\mathbb{S}(Y_1 \times \cdots \times Y_k) = \mathbb{S}_k$. Set $\mathbb{S}(k) = \cup\{\mathbb{S}(Y_1 \times \cdots \times Y_k) : (Y_1, \ldots, Y_k) \in \mathbb{S}(1)^k\}$.

Remark 3.2. *Note that the family $\mathbb{S}(n), n \geq 2$, consists of subsets P of K^n of cardinality $n+1$ which can be defined as follows. For each $P \in \mathbb{S}(n)$ there exist a sequence $a_1, \ldots, a_n$ of n even integers, a sequence $b_1, \ldots, b_n$ of n odd integers and a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that*

(a) $|a_i - b_i| = 1$ for each $i \leq n$,

(b) $P = \{x_1, \ldots, x_{n+1}\}$, where $x_1 = (a_1, \ldots, a_n)$, $x_{n+1} = (b_1, \ldots, b_n)$ and for each $i \leq n$ the point x_{i+1} is obtained from the point x_i through replacing in the $\sigma(i)$-th coordinate the even number $a_{\sigma(i)}$ by the odd number $b_{\sigma(i)}$.

Let k be any positive integer with $k \leq n$. Put $\mathbb{S}(k,n) = \{P : P \subset S,\ S \in \mathbb{S}(n) \text{ and } |P| = k+1\}$. We notice that $\mathbb{S}(n,n) = \mathbb{S}(n)$. It follows from Proposition 2.12 and Example 1.1 that each $P \in \mathbb{S}(k,n)$ is homeomorphic to $E(k)$.

Theorem 3.3. *Let A be a subspace of K^n for some positive number n and k be a positive number such that $k \leq n$. Then $ind\, A \geq k$ iff A contains an element of the family $\mathbb{S}(k,n)$.*

Proof: The "if" part is obvious. Hence we shall show the "only if" part by the induction on n. For $n = 1$ the statement follows from Lemma 3.1. Let $n \geq 2$ and $k \leq n$. Consider a subset A of K^n with $ind\, A \geq k$. Let us notice that there is a point $x = (x_1, \ldots, x_n) \in A$ such that $ind\, Cl_A V'(x) \geq k$, where $V'(x)$ is the minimal neighborhood of x in A. Since $V'(x) = V(x) \cap A$, where $V(x)$ is the minimal neighborhood of x in K^n, we have $Cl_A V'(x) \subset A \cap Cl_{K^n} V(x)$. Recall that $V(x) = V(x_1) \times \cdots \times V(x_n)$, where $V(x_i)$ is the minimal neighborhood of x_i in K for each $i \leq n$. Without loss of generality, we can assume (by the use of Fact 3.1 and Corollary 2.4 if necessary) that $ind\,(A \cap Cl_{K^n} V((1, \ldots, 1))) \geq k$. Let us note that

$$Cl_{K^n} V((1, \ldots, 1)) = (Cl_K V(1))^n = \{(1, \ldots, 1)\} \cup \bigcup_{i=1}^{n} (p_i^n)^{-1}(\{0, 2\}),$$

where $p_i^n : (Cl_K V(1))^n \to Cl_K V(1)$ is the projection of $(Cl_K V(1))^n$ onto the i-th coordinate. First, we assume that $(1, \ldots, 1) \notin A$. Then, by Corollary 2.4, it follows that $k \leq ind\,(A \cap Cl_{K^n} V((1, \ldots, 1))) = ind\,(A \cap \cup_{i=1}^n ((p_i^n)^{-1}\{0, 2\})) \leq n-1$. By Corollary 2.4 again, there are $i \leq n$ and $j \in \{0, 2\}$ such that $ind\,(A \cap (p_i^n)^{-1}(j) \geq k$. Let $q_i^n : (Cl_K V(1))^n \to (Cl_K V(1))^{n-1}$ be the projection defined by $q_i^n(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, \hat{x}_i, \ldots, x_n)$, $i \leq n$. Then $A \cap (p_i^n)^{-1}(j)$ is homeomorphic to $q_i^n(A \cap ((p_i^n)^{-1}(j)) \subset K^{n-1}$. Since $ind\, q_i^n(A \cap ((p_i^n)^{-1}(j)) \geq k$, by the inductive assumption, there are $P' \in \mathbb{S}(k, n-1)$ and $S' \in \mathbb{S}(n-1)$ such that $P' \subset q_i^n(A \cap ((p_i^n)^{-1}(j))$ and $P' \subset S'$. Let $\kappa_i^n : Cl_K V(1)^{n-1} \to Cl_K V(1)^n$ be the mapping of $Cl_K V(1)^{n-1}$ into $Cl_K V(1)^n$ defined by $\kappa_i^n(y_1, \ldots, y_{n-1}) = (z_1, \ldots, z_n)$, where

$$z_k = \begin{cases} y_k, & \text{if } 1 \leq k \leq i-1, \\ j, & \text{if } k = i, \\ y_{k-1}, & \text{if } i+1 \leq k \leq n-1. \end{cases}$$

We put $P = \kappa_i^n(P')$ and $S = \{(1, \ldots, 1)\} \cup \kappa_i^n(S')$. Then $P \subset A \cap (p_i^n)^{-1}(j) \subset A$ and $P \subset S$. Furthermore, by the definition of $\mathbb{S}(n)$ and $\mathbb{S}(k,n)$, we have $S \in \mathbb{S}(n)$ and $P \in \mathbb{S}(k,n)$.

Next, we suppose that $(1, \ldots, 1) \in A$. Then, it follows from Proposition 2.10 that $ind\,(A \cap (\cup_{i=1}^n (p_i^n)^{-1}(\{0, 2\}))) \geq k-1$. Let i, j, p_i^n, q_i^n and κ_i^n be defined as in the above. By a similar argument as above, we can have $P' \in \mathbb{S}(k-1, n-1)$ and $S' \in \mathbb{S}(n-1)$ such that $P' \subset q_i^n(A \cap ((p_i^n)^{-1}(j)))$ and $P' \subset S'$. We put $P = \{(1, \ldots, 1)\} \cup \kappa_i^n(P')$ and $S = \{(1, \ldots, 1)\} \cup \kappa_i^n(S')$. Then $P \subset A$ and $P \subset S$. Furthermore, by the definition of $\mathbb{S}(n)$ and $\mathbb{S}(k,n)$, we have $S \in \mathbb{S}(n)$ and $P \in \mathbb{S}(k,n)$. This completes the proof. □

Remark 3.4. *Recall (cf. [E]) that a subset A of the Euclidean space $\mathbb{R}^n$ is n-dimensional iff A contains a non-empty open subset of $\mathbb{R}^n$. For an n-dimensional subset B of K^n there is an open set (one can always choose a one-point set, see Remark 3.2) which is contained in B but for every one-point open subset B (for example, $B = \{(1, \ldots, 1)\}$) of K^n we have $ind\, B = 0 \neq n$.*

Since the Euclidean topology is regular the equivalence above can be rewritten as follows: a subset A of the Euclidean space $\mathbb{R}^n$ is n-dimensional iff A contains the closure of a non-empty open subset of $\mathbb{R}^n$. Let us note that for the one-point open subset $B = \{(1, \dots, 1)\}$ of K^n, $|Cl_{K^n} B| = 3^n > n + 1$. Furthermore, $Cl_{K^n} B$ contains $2^n \cdot n!$ different n-dimensional in the sense of ind subsets of cardinality $n + 1$. More generally, the closure $Cl_{K^n} V(x)$ of the minimal open neighborhood $V(x)$ of $x = (x_1, \dots, x_n) \in K^n$ with m odd coordinates contains exactly $2^m \cdot 4^{n-m} \cdot n! = 2^{2n-m} \cdot n!$ different n-dimensional in the sense of ind subsets of cardinality $n + 1$. In fact, for $x = (x_1, \dots, x_n) \in K^n$ let $\mathcal{F}_n$ be the family of n-dimensional subsets of $Cl_{K^n} V(x)$ of cardinality $n + 1$. Without loss of generality we can assume that $x_1 = \cdots = x_m = 1$ and $x_{m+1} = \cdots = x_n = 0$. Then $\mathcal{F}_n = \cup\{\mathbb{S}(Y_1 \times \cdots \times Y_n) : (Y_1, \dots, Y_n) \in \{\{0, 1\}, \{1, 2\}\}^m \times \{\{-2, -1\}, \{-1, 0\}, \{0, 1\}, \{1, 2\}\}^{n-m}\}$, where $\mathbb{S}(Y_1 \times \cdots \times Y_n)$ is defined above. By Proposition 2.12 (b), we have $|\mathcal{F}_n| = 2^m \cdot 4^{n-m} \cdot n!$

Remark 3.5. *Let C be a class of subsets of the Euclidean space $\mathbb{R}^n$, where $n \geq 1$, such that for every set A in $\mathbb{R}^n$ we have ind $A = n$ iff A contains an element of C. Notice that each element E of C has ind $E = n$. Fix an element E of C and a point $p \in E$. Let us note that ind $(E \setminus \{p\}) = n$. By the property of the family C there is an element $F \in C$ such that $F \subset E \setminus \{p\} \subset E$. Put $C' = C \setminus \{E\}$ and note that for every set A in $\mathbb{R}^n$ we have ind $A = n$ iff A contains an element of C'. On the other hand, Theorem 3.3 does not hold if we replace the class $\mathbb{S}(n)$ by any its proper subclass.*

Denote by K_0 (respectively, K_1) the subspace of K consisting of even (respectively, odd) integers. It is clear that K_0 and K_1 are discrete, and hence *ind* K_0 = *ind* $K_1 = 0$. Taking into account Remark 3.2 we get the following.

Corollary 3.6. *Let A be a subset of K^n such that either $A \cap (K_0)^n = \emptyset$ or $A \cap (K_1)^n = \emptyset$. Then ind $A \leq n - 1$.*

Remark 3.7. *Theorem 3.3 implies that the family $\mathbb{S}(k, n)$ precisely consists of all subsets P of K^n with $|P| = k + 1$ and ind $P = k$.*

Put $Z_j = \{(x_1, \dots, x_n) \in K^n : |\{i \leq n : x_i \text{ is an even number}\}| = j\}$, $0 \leq j \leq n$. Note that the sets Z_j, $0 \leq j \leq n$, are disjoint, $K^n = \cup_{i=0}^n Z_i$. Furthermore, since each Z_i contains no elements of $\mathbb{S}(1, n)$, it follows from Theorem 3.3 that *ind* $Z_i = 0$ for each $j \leq n$. Hence, we get a decomposition theorem for the Khalimsky spaces K^n into zero-dimensional (*i.e.* discrete) subsets.

4. The Small Transfinite Inductive Dimension in Alexandroff Spaces

Let us note that the the small inductive dimension *ind* can be extended to infinite ordinals. The extension we will call the small transfinite dimension *trind* (cf. [E]). Observe that *trind* is also monotone *w.r.t.* subsets, *i.e.* for any $Y \subseteq X$ we have *trind* $Y \leq$ *trind* X.

It is easy to see that the space E from Example 1.1 has *trind* $E = \infty$.

Let X be a topological space and p a point such that $p \notin X$. Recall ([M]) that *the join $p \vee X$ of p and X* is the topological space (Y, τ), where $Y = \{p\} \cup X$ and $\tau = \{\emptyset, Y\} \cup \{\{p\} \cup A : A \text{ is an open subset of } X\}$. Let us notice that the point p is an open subset of $p \vee X$ and the subspace $Bd_{p\vee X}\{p\}$ of $p \vee X$ is homeomorphic to the space X. Moreover, if the space X is Alexandroff then the space $p \vee X$ is also Alexandroff. Furthermore, the set $\{p\}$ (resp. $\{p\} \cup V(x)$) is the minimal open subset of $p \vee X$ containing p (resp. $x \in X$, where the set $V(x)$ is the minimal open subset of X containing x).

Below we will use disjoint copies of the corresponding spaces when it is necessary. For each ordinal $\alpha \geq 0$ choose a point p_α and set $Y(0) = \{p_0\}$. Then define by transfinite induction the space $Y(\alpha), \alpha > 0$, as follows.

(a) If α is limit $\geq \omega_0$, then the space $Y(\alpha)$ is the topological union $\oplus_{\beta<\alpha} Y(\beta)$ of $Y(\beta), \beta < \alpha$;
(b) If α is non-limit, then $Y(\alpha) = p_\alpha \vee Y(\alpha - 1)$.

Let us note that for each positive integer n the space $Y(n)$ is homeomorphic to the space $E(n)$.

Proposition 4.1. *For each ordinal $\alpha \geq 0$ we have trind $Y(\alpha) = \alpha$.*

Proof: It is clear that $trind\, Y(\alpha) = \alpha$ for each $0 \le \alpha < \omega_0$. Then apply induction. For limit $\alpha \ge \omega_0$ the equality $trind\, Y(\alpha) = \alpha$ is evident. Assume that α is not limit and $\ge \omega_0$. So $\alpha = (\alpha - 1) + 1$ and $Y(\alpha) = p_\alpha \vee Y(\alpha - 1)$. Note that for each point $y \in Y(\alpha)$ we have $BdV_y \subseteq Y(\alpha - 1)$. By inductive assumption and monotonicity of *trind* we have $trind\, BdV_y \le trind\ Y(\alpha - 1) = \alpha - 1$. So $trind\, Y(\alpha) \le \alpha$. However, $BdV_p = Y(\alpha - 1)$ and hence $trind\, BdV_y = trind\, Y(\alpha - 1) = \alpha - 1$. This implies that $trind\, Y(\alpha) \ge \alpha$. □

Example 4.2. *Let $E(1)^\omega_B$ be the Cartesian product of countably many copies of the space $E(1)$ endowed with the box topology. Note that $E(1)^\omega_B$ is a connected Alexandroff T_0-space with $trind\, E(1)^\omega_B = \infty$ containing for each integer $n \ge 1$ a copy of $E(1)^n$ as a closed subset. Hence $E(1)^\omega_B$ contains discrete subspaces of any finite cardinality.*

Proposition 4.3. *Let X be an Alexandroff space and $X = X_1 \cup X_2$, where X_i is closed in X for each $i = 1, 2$. Then $trind\, X = \max\{trind\, X_1, trind\, X_2\}$.*

Proof: Put $\alpha = \max\{indX_1, indX_2\}$. Apply induction on $\alpha \ge -1$. It is trivial for $n = -1$. Consider the case $n \ge 0$. Let $x \in X$. First, suppose that $x \in X \setminus X_2$. Then $V(x) \subset X \setminus X_2 \subset X_1$ and $Bd_X V(x) = Bd_{X_1} V(x)$. Hence $indBd_X V(x) < indX_1 \le \alpha$. It is similar when $x \in X \setminus X_1$. Next, we suppose that $x \in X_1 \cap X_2$. Then $Bd_X V(x) = Bd_{X_1}(V(x) \cap X_1) \cup Bd_{X_2}(V(x) \cap X_2)$ and the set $V(x) \cap X_i$ is the minimal open neighborhood of x in X_i for each i. Note that $indBd_{X_i}(V(x) \cap X_i) < \alpha$, $i = 1, 2$. Hence, by the inductive assumption, we have $indBd_X V(x) \le \max\{indBd_{X_1}(V(x) \cap X_1), indBd_{X_2}(V(x) \cap X_2)\} < \alpha$. □

Recall (cf. [KM]) that every ordinal number $\alpha > 0$ can be uniquely represented as $\alpha = \omega_0^{\xi_1} \cdot n_1 + \cdots + \omega_0^{\xi_k} \cdot n_k$, where n_i are positive integers and ξ_i are ordinals such that $\xi_1 > \cdots > \xi_k \ge 0$.

Let α, β be ordinal numbers and $\alpha = \omega_0^{\xi_1} \cdot n_1 + \cdots + \omega_0^{\xi_k} \cdot n_k$ and $\beta = \omega_0^{\xi_1} \cdot m_1 + \cdots + \omega_0^{\xi_k} \cdot m_k$, where n_i, m_i are non-negative integers and ξ_i are ordinals such that $\xi_1 > \cdots > \xi_k \ge 0$.

The ordinal $\alpha \oplus \beta = \omega_0^{\xi_1} \cdot (n_1 + m_1) + \cdots + \omega_0^{\xi_k} \cdot (n_k + m_k)$ is called the natural sum of α, β or the sum of ordinals in the sense of Hessenberg.

The following statement is evident.

Proposition 4.4. *Let X and Y be non-empty Alexandroff spaces. Then*

$$trind\, X \times Y \le trind\, X \oplus trind\, Y.$$

Remark 4.5. *The equality such as in Proposition 2.5 for the small transfinite inductive dimension does not hold. In fact, let us choose for each non-negative integer i a space Z_i which is homeomorphic to the space $E(i)$ such that the chosen spaces are pairwise disjoint. Consider the topological union $Z = \oplus_{i=0}^\infty Z_i$ of $Z_i, i = 0, 1, \ldots$. Note that Z is an Alexandroff T_0-space, and $trind\, Z = \omega_0$. However, $trind\, (Z \times E(n)) = \omega_0 < \omega_0 + n$ for each positive integer n.*

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