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Fractals through Modified Iteration Scheme

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Abstract. In this paper we study the geometry of relative superior Mandelbrot sets through S-iteration scheme. Our results are quit significant from other Mandelbrot sets existing in the literature. Besides this, we also observe that S-iteration scheme converges faster than Ishikawa iteration scheme. We believe that the results of this paper can be inspired those who are interested in creating automatically aesthetic patterns.

1. Introduction

Complex graphics of nonlinear dynamicl systems is an exciting area of interest with diverse applications in sciences, art, textile industries, engineering and many other areas of human activity. The Mandelbrot set is the most popular infinitely complex object in the fractal theory given by Benoit Mandelbrot [11]. This object has been discussed extensively by researchers right from its advent [3, 11, 14]. Mandelbrot set and its and its various extensions have been studied in [4–10, 12, 13]. This object has been analyzed from different aspects, external and internal perturbation of Mandelbrot set has been discussed in [2]. Rani and Kumar [16] introduced the superior Mandelbrot sets using Mann iteration scheme. Rana et al. [15] introduced relative superior Mandelbrot sets using Ishikawa iteration seheme. They also explored relative superior Mandelbrot sets of quadraties, cubic and other complex values polynomials and discuss some related properties.

In this paper we generate the fractals using S-iteration scheme. Our results are entirely different from other Mandelbrot sets existing in the present literature.

2. Preliminaries

Let $\{z_n : n = 1, 2, 3, ...\}$, denoted by $\{z_n\}$ be a sequence of complex numbers. Then, we say $\lim_{n\to\infty} z_n = \infty$ if for given $M > 0$, there exists $N > 0$ such that for all $n > N$, we must have $|z_n| > M$. Thus all the values of z_n lies outside a circle of radius M for sufficiently large values of n. Let

$$Q(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z^1 + a_n z^0; \quad a_0 \neq 0$$

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be a polynomial of degree n, where $n \geqslant 2$. The coefficients are allowed to be complex numbers. In other words, it follows that $Q_c(z) = z^2 + c$.

Definition 2.1. *([9]) Let X be a nonempty set and $f : X \to X$. For any point $x_0 \in X$, the Picard's orbit is defined as the set of iterates of a point x_0, that is,*

$$O(f, x_0) = \{x_n; x_n = f(x_{n-1}),\ n = 1, 2, 3, ...\}.$$

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leavet he interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number. The collection of points that are bounded, i.e., there exists M, such that $|Q^n(z)| \leq M$ for all n, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Mandelbrot set for Q.

Definition 2.2. *([9]) The Mandelbrot set M for the quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is*

$$M = \{c \in C : \{Q_c^n(0)\};\ n = 0, 1, 2, ...\}$$

is bounded

An equivalent formulation is

$$M = \{c \in C : \{Q_c^n(0)\} \text{ does not tend to } \infty \text{ as } n \to \infty\}.$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

3. *S*-iteration Scheme for Relative Superior Mandelbrot sets

Let X be a subset of real or complex numbers and $f : X \to X$. For $x_0 \in X$, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_0 = (1 - s'_0)x_0 + s'_0 f(x_0),$$
$$y_1 = (1 - s'_1)x_1 + s'_1 f(x_1),$$
$$y_n = (1 - s'_n)x_n + s'_n f(x_n),$$

where $0 < s'_n \leq 1$ and s'_n is convergent to non-zero number, and

$$x_0 = (1 - s_0)f(x_0) + s_0 f(y_0),$$
$$x_1 = (1 - s_1)f(x_1) + s_1 f(y_1),$$
$$x_n = (1 - s_{n-1})f(x_{n-1}) + s_{n-1} f(y_{n-1}),$$

where $0 < s_n \leq 1$ and s_n is convergent to non-zero number [1].

Definition 3.1. *([15, 16]) The sequences $\{x_n\}$ and $\{y_n\}$ constructed above is called S-scheme sequences of iterations or relative superior sequences of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$.*

Now we define Mandelbrot sets for function with respect to *S*-scheme iterates.

Definition 3.2. *Relative superior Mandelbrot set (RSM) for the function of the form $Q_c(z) = z^n + c$, where $n = 1, 2, 3, ...$, is defined as the collection of $c \in \mathbb{C}$ for which the orbit of 0 is bounded, i.e.,*

$$RSM = \{c \in C : \{Q_c^k(0)\};\ k = 0, 1, 2, ...\}$$

is bounded.

We now define escape citerions for these sets.

3.1. Relative Superior Escape Criterions for Quadratics

The following result gives us escape criterions for the functions $Q_c(z) = z^2 + c = Q'_c(z)$ in respect to S-iteration scheme.

Theorem 3.3. *Let us assume that* $|z| \geq |c| > 2/s$ *and* $|z| \geq |c| > 2/s'$*, where* $0 < s \leq 1, 0 < s' \leq 1$ *and* c *is a complex number.*

Define

$$\begin{aligned} z_1 &= (1-s)(z^2+c) + sQ_c(z), \\ &\vdots \\ z_n &= (1-s)(z_{n-1}^2+c) + sQ_c(z_{n-1}), \end{aligned}$$

where $Q_c(z)$ *be a quadratic polynomial in terms of* s' *and* $n = 2, 3, 4, \ldots$*. Then* $|z_n| \to \infty$ *as* $n \to \infty$*.*

Proof. For $Q'_c(z) = z^2 + c$, consider

$$\begin{aligned} |Q_c(z)| &= \left|(1-s')z + s'Q'_c(z)\right| \\ &= \left|(1-s')z + s'(z^2+c)\right| \\ &= \left|(1-s')z + s'z^2 + s'c\right| \\ &\geq \left|s'z^2 + (1-s')z\right| - |s'c| \\ &\geq \left|s'z^2 + (1-s')z\right| - |s'z| \\ &\geq \left|s'z^2\right| - |(1-s')z| - s'\,|z| \\ &= \left|s'z^2\right| - |z| + |s'z| - s'\,|z| \\ &\geq |z|\,(|s'z| - 1). \end{aligned}$$

Also for

$$z_n = (1-s)f(z_{n-1}) + sQ_c(z),$$

we obtain

$$\begin{aligned} |z_1| &= \left|(1-s)(z^2+c) + s\,|z|\,(|s'z|-1)\right| \\ &= \left|(1-s)z^2 + (1-s)c + s\,|z|\,|s'z| - s\,|z|\right| \\ &= \left|ss'\,|z|\cdot|z| - s\,|z| + (1-s)z^2 + (1-s)c\right| \\ &\geq \left|ss'\,|z|\cdot|z| - s\,|z| - (s-1)z^2\right| - |(1-s)c| \\ &\geq \left|ss'\left|z^2\right| - s\,|z|\right| - \left|(s-1)z^2\right| - (1-s)\,|z| \\ &\geq ss'\left|z^2\right| - s\,|z| - (s-1)\left|z^2\right| - |z| + s\,|z| \\ &\geq (ss'-s+1)\left|z^2\right| - |z| \\ &\geq |z|\left((ss'-s+1)\,|z| - 1\right). \end{aligned}$$

Since $|z| > 2/s$ and $|z| > 2/s'$ it follows

$$\begin{aligned} |z| &> 2/(ss') \\ &> 2/(ss'-s+1), \end{aligned}$$

which implies that

$$(ss'-s+1)|z| - 1 > 1.$$

Hence there exists $\lambda > 0$ such that $(ss'-s+1)|z| - 1 > 1 + \lambda$. Consequently

$$\begin{aligned} |z_1| &> (1+\lambda)\,|z|, \\ &\vdots \\ |z_n| &> (1+\lambda)^n\,|z|. \end{aligned}$$

Hence $|z_n| \longrightarrow \infty$ as $n \to \infty$. □

Corollary 3.4. *Suppose that* $|c| > 2/s$ *and* $|c| > 2/s'$. *Then the relative superior orbit of S-iteration scheme* $RSO(Q_c, 0, s, s')$ *escape to infinity.*

In the proof of theorem we used the facts that $|z| \geq |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$. Hence the following corollary is the refinement of the escape criterion discussed in the above theorem.

Corollary 3.5. (Escape Criterion) *Suppose that* $|z| > \max(|c|, 2/s, 2/s')$. *Then* $|z_n| > (1+\lambda)^n |z|$ *and* $|z_n| \longrightarrow \infty$ *as* $n \to \infty$.

Corollary 3.6. *Suppose that* $|z_k| > \max(|c|, 2/s, 2/s')$ *for some* $k \geq 0$. *Then* $|z_{k+1}| > (1+\lambda)^n |z_k|$ *and* $|z_{k+1}| \longrightarrow \infty$ *as* $n \to \infty$.

This corollary gives us an algorithm for computing the relative superior Mandelbrot sets of Q_c for any c. Given any point $|z| \leq |c|$, we have computed the relative superior orbit of z. If for some n, $|z_n|$ lies outside the circle of radius $\max(|c|, 2/s, 2/s')$, we guarantee that the orbit escapes. Hence, z is not in the relative superior Mandelbrot sets. On the other hand, if $|z_n|$ never exceeds this bound, then by definition of the relative superior Mandelbrot sets, we can make extensive use of this algorithm in the next section.

3.2. Relative Superior Escape Criterion for Cubic Polynomials

We prove the following result for the function $Q_{a,b}(z) = z^3 + az + b = Q'_{a,b}(z)$ with respect to S-iteration scheme.

Theorem 3.7. *Suppose* $|z| > |b| > (|a| + 2/s)^{1/2}$ *and* $|z| > |b| > (|a| + 2/s')^{1/2}$ *exist, where* $0 < s \leq 1$, $0 < s' \leq 1$ *and* a, b *are in complex plane.*

Define

$$z_1 = (1-s)(z^3 + az + b) + sQ_{a,b}(z),$$

$$\vdots$$

$$z_n = (1-s)(z_{n-1}^3 + az_{n-1} + b) + sQ_{a,b}(z_{n-1}), \quad n = 2, 3, ...,$$

where $Q_{a,b}(z)$ *is the function of* s'. *Then* $|z_n| \to \infty$ *as* $n \to \infty$.

Proof. For $Q'_{a,b}(z) = z^3 + az + b$, consider

$$\begin{aligned} \left|Q_{a,b}(z)\right| &= \left|(1-s')z + s'Q'_{a,b}(z)\right| \\ &= \left|(1-s')z + s'(z^3 + az + b)\right| \\ &= \left|s'z^3 + s'az + (1-s')z + s'b\right| \\ &\geq \left|s'z^3 + s'az + (1-s')z\right| - |s'b| \\ &\geq \left|s'z^3 + s'az\right| - |(1-s')z| - s'|z| \\ &= |z|\,(s'(\left|z^2 + a\right|) - 1 + s' - s') \\ &\geq |z|\,(s'(\left|z^2 + a\right|) - 1). \end{aligned}$$

Also for

$$z_n = (1-s)f(z_{n-1}) + sQ_{a,b}(z),$$

we obtain

$$
\begin{aligned}
|z_1| = &\ \left|(1-s)(z^3+az+b)+sQ_{a,b}(z)\right| \\
&= \left|(1-s)(z^3+az+b)+s\,|z|\,(s'(\left|z^2+a\right|)-1)\right| \\
&= \left|(1-s)(z^3+az)+(1-s)b+ss'\,|z|\,(\left|z^2+a\right|)-s\,|z|\right| \\
&= \left|ss'\,|z|\,(\left|z^2+a\right|)-s\,|z|+(1-s)(z^3+az)+(1-s)b\right| \\
&\geq \left|ss'\,|z|\,(\left|z^2+a\right|)-s\,|z|+(1-s)(z^3+az)\right|-|(1-s)b| \\
&\geq \left|ss'\,|z|\,(\left|z^2+a\right|)-s\,|z|+(1-s)(z^3+az)\right|-(1-s)\,|z| \\
&\geq \left|ss'\,|z|\,(\left|z^2+a\right|)-s\,|z|\right|-\left|(s-1)(z^3+az)\right|-(1-s)\,|z| \\
&\geq ss'\,|z|\,(\left|z^2+a\right|)-s\,|z|-(s-1)\,|z|\,(\left|z^2+a\right|)-|z|+s\,|z| \\
&= (ss'-s+1)\,|z|\,(\left|z^2+a\right|)-|z| \\
&\geq |z|\,((ss'-s+1)(\left|z^2+a\right|)-1) \\
&\geq\ |z|\,((ss'-s+1)(\left|z^2\right|-|a|)-1) \\
&= |z|\,(ss'-s+1)(\left|z^2\right|-|a|-1/(ss'-s+1)) \\
&\geq |z|\,(ss'-s+1)(\left|z^2\right|-(|a|+1/(ss'-s+1))).
\end{aligned}
$$

Since $|z| > (|a|+2/s)^{1/2}$ and $|z| > (|a|+2/s')^{1/2}$ it follows

$$|z| > (|a|+2/(ss'))^{1/2}$$

and hence

$$\left|z^2\right|-|a| > 2/(ss'-s+1),$$

which implies that

$$(ss'-s+1)(\left|z^2\right|-(|a|+1/(ss'-s+1))) > 1.$$

Hence there exists $\gamma > 1$ such that $|z_1| > \gamma\,|z|$. Repeating the argument n times, we get $|z_n| > \gamma^n\,|z|$. Therefore, the relative superior orbit of z, under the cubic polynomial $Q_{a,b}(z)$ tends to infinity. This completes the proof. □

Corollary 3.8. *Suppose that* $|b| > (|a|+2/s)^{1/2}$ *and* $|b| > (|a|+2/s')^{1/2}$ *exists. Then the relative superior orbit* $RSO(Q_{a,b},0,s,s')$ *escapes to infinity.*

Corollary 3.9. (Escape criterion) *Suppose that* $|z| > \max(|b|,(|a|+2/s)^{1/2},(|a|+2/s')^{1/2})$. *Then* $|z_n| \to \infty$ *as* $n \to \infty$.

Corollary 3.10. *Assume that* $|z_k| > \max(|b|,(|a|+2/s)^{1/2},(|a|+2/s')^{1/2})$ *for some* $k \geq 0$. *Then* $|z_{k+1}| > \gamma^n\,|z_k|$ *and* $|z_n| \to \infty$ *as* $n \to \infty$.

3.3. A General Escape Criterion

We will obtain a general escape criterion for polynomials of the form $G_c(z) = z^n + c$.

Theorem 3.11. *For general function* $G_c(z) = z^n + c$, $n = 1,2,3,...,$ *where* $0 < s \leq 1$, $0 < s' \leq 1$ *and* c *is a complex number.*

Define

$$
\begin{aligned}
z_1 &= (1-s)(z^n+c)+sG_c(z), \\
&\ \ \vdots \\
z_n &= (1-s)(z_{n-1}^n+c)+sG_c(z_{n-1}).
\end{aligned}
$$

Thus the general escape criterion is $\max(|c|,(2/s)^{1/n-1},(2/s')^{1/n-1})$.

Proof. We shall proof the theorem by induction for $n = 1$, we get $G_c(z) = z + c$, so the escape criterion is $|c|$, which is obvious, i.e., $|z| > \max(|c|, 0, 0)$. For $n = 2$, we get $G_c(z) = z^2 + c$ so, the escape criterion is $|z| > \max(|c|, 2/s, 2/s')$ (Corollary 3.5). For $n = 3$, we get $G_c(z) = z^3 + c$ so, the result follows from Corollary 3.9 with $a = 0$ and $b = c$ such that the escape criterion is $|z| > \max(|c|, (2/s)^{1/2}, (2/s')^{1/2})$. Hence the theorem is true for $n = 1, 2, 3, \ldots$. Now suppose that theorem is true for any n. Let $G_c(z) = z^{n+1} + c$ and $|z| \geq |c| > (2/s)^{1/n}$ as well as $|z| \geq |c| > (2/s')^{1/n}$exists then for $G'_c(z) = z^{n+1} + c$, consider

$$\begin{aligned}
|G_c(z)| &= \left|(1-s')z + s'G'_c(z)\right| \\
&= \left|(1-s')z + s'(z^{n+1} + c)\right| \\
&= \left|(1-s')z + s'z^{n+1} + s'c\right| \\
&\geq \left|s'z^{n+1} + (1-s')z\right| - |s'c| \\
&\geq \left|s'z^{n+1} + (1-s')z\right| - |s'z| \\
&\geq \left|s'z^{n+1}\right| - |(1-s')z| - s'|z| \\
&= \left|s'z^{n+1}\right| - |z| + s'|z| - s'|z| \\
&\geq |z|\,(|s'z^n| - 1).
\end{aligned}$$

Also for

$$z_n = (1-s)f(z_{n-1}) + sG_c(z),$$

we obtain

$$\begin{aligned}
|z_1| &= \left|(1-s)(z^{n+1} + c) + s|z|\,(|s'z^n| - 1)\right| \\
&= \left|(1-s)z^{n+1} + (1-s)c + s|z|\,|s'z^n| - s|z|\right| \\
&= \left|ss'|z| \cdot |z^n| - s|z| + (1-s)z^{n+1} + (1-s)c\right| \\
&\geq \left|ss'|z| \cdot |z^n| - s|z| - (s-1)z^{n+1}\right| - |(1-s)c| \\
&\geq \left|ss'\left|z^{n+1}\right| - s|z|\right| - \left|(s-1)z^{n+1}\right| - (1-s)|z| \\
&\geq ss'\left|z^{n+1}\right| - s|z| - (s-1)\left|z^{n+1}\right| - |z| + s|z| \\
&\geq (ss' - s + 1)\left|z^{n+1}\right| - |z| \\
&\geq |z|\,((ss' - s + 1)\,|z^n| - 1).
\end{aligned}$$

Since $|z| > 2/s$ and $|z| > 2/s'$, it follows

$$\begin{aligned}
|z| &> 2/(ss') \\
&> 2/(ss' - s + 1),
\end{aligned}$$

which implies that

$$(ss' - s + 1)|z| - 1 > 1.$$

Hence there exists $\lambda > 0$ such that $(ss' - s + 1)|z| - 1 > 1 + \lambda$. Consequently

$$\begin{aligned}
|z_1| &> (1+\lambda)\,|z|, \\
&\vdots \\
|z_n| &> (1+\lambda)^n\,|z|.
\end{aligned}$$

Hence $|z_n| \longrightarrow \infty$ as $n \to \infty$. □

Corollary 3.12. *Suppose that $|c| > (2/s)^{1/n-1}$ and $|c| > (2/s')^{1/n-1}$. Then the relative superior orbit of S-iteration scheme $RSO(G_c, 0, s, s')$ escape to infinity.*

Corollary 3.13. *Assume that $|z_k| > \max(|c|, (2/s)^{1/k-1}, (2/s')^{1/k-1})$ for some $k \geq 0$. Then $|z_{k+1}| > \gamma^n |z_k|$ and $|z_n| \to \infty$ as $n \to \infty$.*

This corollary gives an algorithm for computing the relative superior Mandelbrot sets for the functions of the form $G_c(z) = z^n + c$, $n = 1, 2, 3, \ldots$.

4. Generation of Relative Superior Mandelbrot Sets

In this section we present some relative superior Mandelbrot sets for quadratic, cubic and biquadratic functions.

4.1. Relative Superior Mandelbrot Sets for Quadratic Function

Some relative superior Mandelbrot sets are presented for quadratic function in the following figures (Figures 1-6):

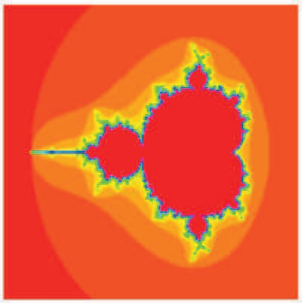

Figure 1: Relative superior Mandelbrot set for $s = 1.0$ and $s' = 1.0$

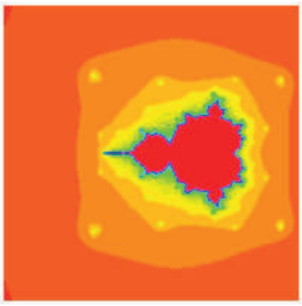

Figure 2: Relative superior Mandelbrot set for $s = 0.1$ and $s' = 0.6$

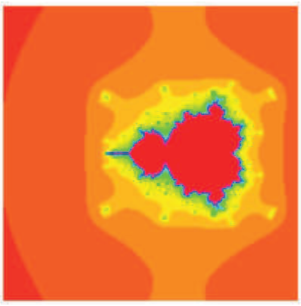

Figure 3: Relative superior Mandelbrot set for $s = 0.1$, $s' = 0.8$

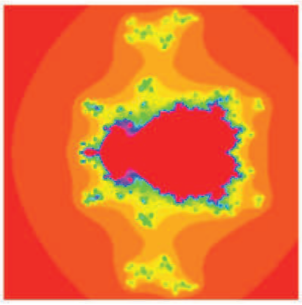

Figure 4: Relative superior Mandelbrot set for $s = 0.2$ and $s' = 0.7$

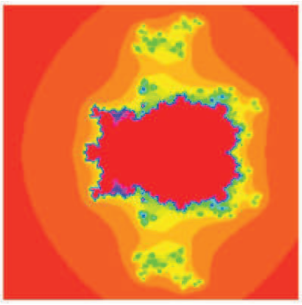

Figure 5: Relative superior Mandelbrot set for $s = 0.2$, $s' = 0.8$

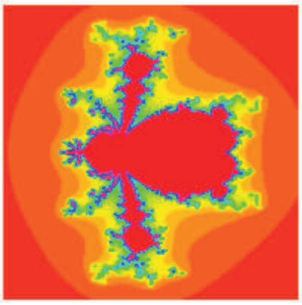

Figure 6: Relative superior Mandelbrot set for $s = 0.3$ and $s' = 0.5$

4.2. Relative Superior Mandelbrot Sets for Cubic Function

Some relative superior Mandelbrot sets are presented for cubic function in the following figures (Figures 7-12):

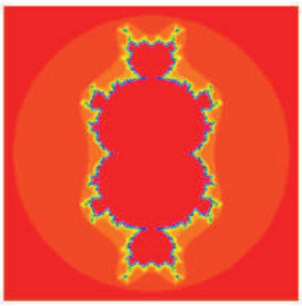

Figure 7: Relative superior Mandelbrot set for $s = 1.0$ and $s' = 1.0$

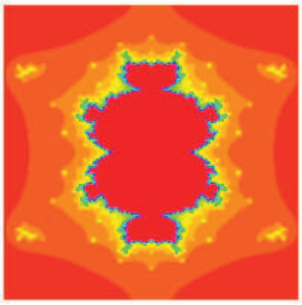

Figure 8: Relative superior Mandelbrot set for $s = 0.1$ and $s' = 0.6$

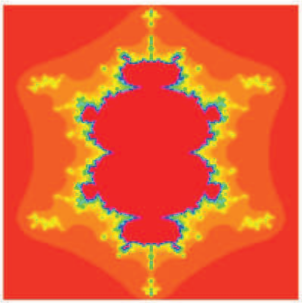

Figure 9: Relative superior Mandelbrot set for $s = 0.1$ and $s' = 0.8$

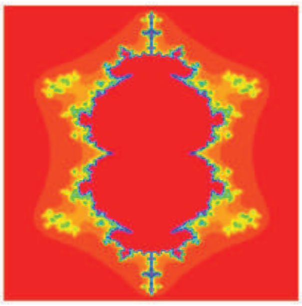

Figure 10: Relative superior Mandelbrot set for $s = 0.2$ and $s' = 0.7$

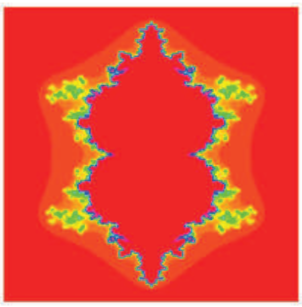

Figure 11: Relative superior Mandelbrot set for $s = 0.2$ and $s' = 0.8$

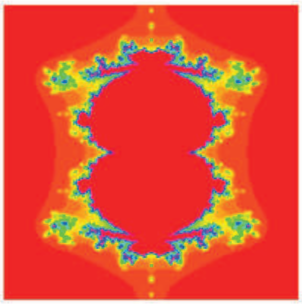

Figure 12: Relative superior Mandelbrot set for $s = 0.3$ and $s' = 0.5$

4.3. Relative Superior Mandelbrot Sets for Bi-quadratic Function

Some relative superior Mandelbrot sets are presented for bi-quadratic function in the following figures (Figures 13-18):

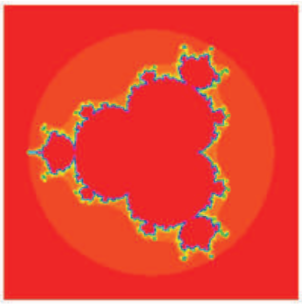

Figure 13: Relative superior Mandelbrot set for $s = 1.0$ and $s' = 1.0$

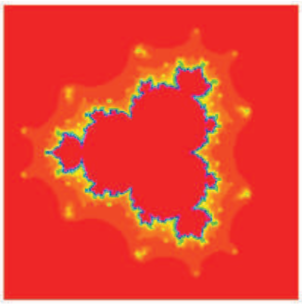

Figure 14: Relative superior Mandelbrot set for $s = 0.1$ and $s' = 0.6$

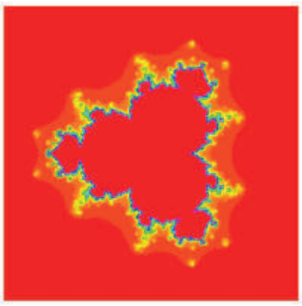

Figure 15: Relative superior Mandelbrot set for $s = 0.1$ and $s' = 0.8$

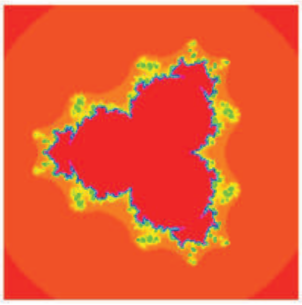

Figure 16: Relative superior Mandelbrot set for $s = 0.2$ and $s' = 0.7$

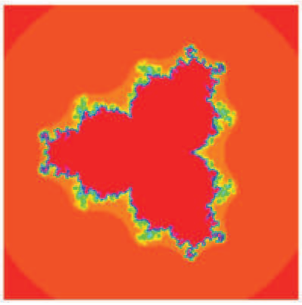

Figure 17: Relative superior Mandelbrot set for $s = 0.2$ and $s' = 0.8$

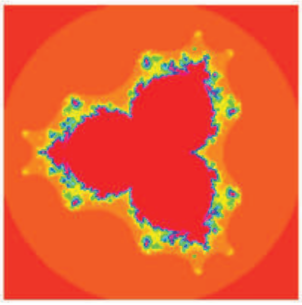

Figure 18: Relative superior Mandelbrot set for $s = 0.3$ and $s' = 0.5$

4.4. Generalization of Relative Superior Mandelbrot Set

Following figures represents the generalization of relative superior Mandelbrot set in the following figures (Figures 19-24):

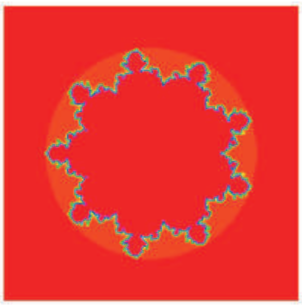

Figure 19: Relative superior Mandelbrot set for $s = 1.0$, $s' = 1.0$ and $n = 10$.

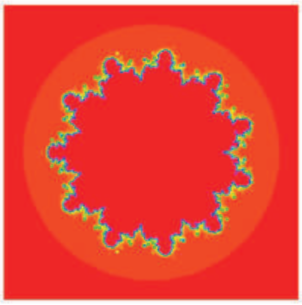

Figure 20: Relative superior Mandelbrot set for $s = 0.1$, $s' = 0.6$ and $n = 12$

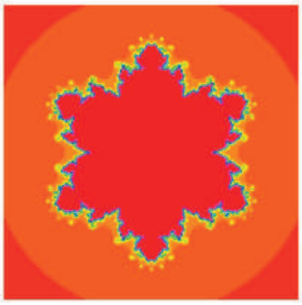

Figure 21: Relative superior Mandelbrot set for $s = 0.1$, $s' = 0.8$ and $n = 7$

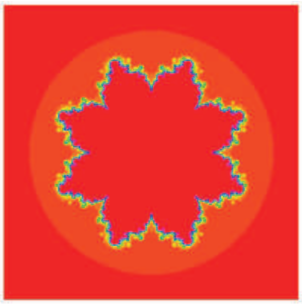

Figure 22: Relative superior Mandelbrot set for $s = 0.2$, $s' = 0.7$ and $n = 9$

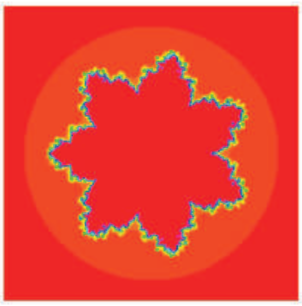

Figure 23: Relative superior Mandelbrot set for $s = 0.2$, $s' = 0.8$ and $n = 8$

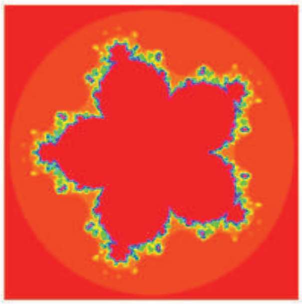

Figure 24: Relative superior Mandelbrot set for $s = 0.3$, $s' = 0.5$ and $n = 6$

5. Comparison of Ishikawa and *S*-iteration Schemes

Time for generating relative superior Mandelbrot set by using MAPLE softwere given in the following tables. Our results shows that convergence of *S*-iteration scheme relatively faster than Ishikawa iteration.

Table 1: Time for relative superior Mandelbrot sets for quadratic function

(α, β)	Time for Ishikawa iteration	Time for S-iteration scheme
$(0.1, 0.6)$	$42.21s$	$16.46s$
$(0.1, 0.8)$	$34.20s$	$16.89s$
$(0.2, 0.7)$	$35.90s$	$22.07s$
$(0.2, 0.8)$	$31.04s$	$23.03s$
$(0.3, 0.5)$	$34.00s$	$25.92s$

Table 2: Time for relative superior Mandelbrot sets for cubic function

(α, β)	Time for Ishikawa iteration	Time for S-iteration scheme
$(0.1, 0.6)$	$67.70s$	$36.12s$
$(0.1, 0.8)$	$55.40s$	$40.0s$
$(0.2, 0.7)$	$51.93s$	$41.81s$
$(0.2, 0.8)$	$46.70s$	$43.46s$
$(0.3, 0.5)$	$54.31s$	$43.43s$

Table 3: Time for relative superior Mandelbrot sets for bi-quadratic function

(α, β)	Time for Ishikawa iteration	Time for S-iteration scheme
$(0.1, 0.6)$	$56.51s$	$38.98s$
$(0.1, 0.8)$	$49.92s$	$40.31s$
$(0.2, 0.7)$	$49.37s$	$44.14s$
$(0.2, 0.8)$	$46.90s$	$45, 75s$
$(0.3, 0.5)$	$52.17s$	$44.93s$

Table 4: Time for generalization of relative superior Mandelbrot set

(α, β, n)	Time for Ishikawa iteration	Time for S-iteration scheme
$(0.1, 0.6, 12)$	$67.59s$	$66.12s$
$(0.1, 0.8, 7)$	$70.37s$	$68.15s$
$(0.2, 0.7, 9)$	$65.98s$	$59.39s$
$(0.2, 0.8, 8)$	$59.50s$	$54.59s$
$(0.3, 0.5, 6)$	$67.51s$	$64.51s$

6. Conclusions

In this paper we have generated many relative superior Mandelbrot figures for S-iteration scheme. It is noticed that there are remarkable changes found corresponding to the values of parameters s and s'. In the dynamics of complex polynomial $Q(z) = z^n + c$, where $n \geq 2$, we observe that there are several avoids or bulbs attached with the main body. The number of major secondary lobe is bifurcated into $(n-1)$ lobes. Besides this, we also observe that S-iteration scheme converges faster than Ishikawa iteration scheme. However for $s > 1/2$, the difference of timings taken by the two iteration schemes for the generation of figures is small. We believe that the results of this paper can be inspired those who are interested in creating automatically aesthetic patterns.

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