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Properties of Concircular Curvature Tensors on Riemann Spaces

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Abstract. This paper studies conditions of pseudo-symmetric and semi-symmetric type on geodesic and subgeodesic related Riemann spaces.

Properties of concircular transformations of metrics are characterized, using certain concircular-Riemann type flows. Also concircular-Riemann solitons are introduced, as natural extensions of Ricci solitons. Some special gradient concircular-Riemann solitons on Riemannian spaces are considered.

Introduction

Pseudo-symmetric manifolds [11], [13] constitute a generalization of spaces of constant sectional curvature, along the line of locally symmetric and semi-symmetric spaces, [16], [18], consecutively. Both the study of an intrinsic and the study of an extrinsic aspect let to the concept of pseudo-symmetry. For example, every manifold M which can be mapped geodesically onto a semi-symmetric manifold is pseudo-symmetric [7].

Also, every totally umbilical submanifold, with parallel mean curvature vector field, of a semi-symmetric manifold is pseudo-symmetric.

In the present paper we extend the approach, considering pseudo-symmetric spaces subgeodesically related, using certain concircular transformations. Concircular semi-symmetric spaces geodesically related are studied. The last section is devoted to a special class of concircular related metrics produced by concircular-Riemann flows. Also concircular-Riemann solitons are considered.

1. Concircular Transformations on Pseudo-Symmetric Spaces

Let (M, g) be a Riemannian space and $T \in \mathcal{T}^{0,k}M$.

We define $R \cdot T, Q(g, T) \in \mathcal{T}^{0,k+2}M$, by

$$\begin{aligned}(R \cdot T)(X_1, \ldots, X_k; X, Y) &= (R(X, Y) \cdot T)(X_1, \ldots, X_k) = \\ &= -T(R(X, Y)X_1, \ldots, X_k) - \cdots - T(X_1, \ldots, R(X, Y)X_k).\end{aligned}$$

$$\begin{aligned}Q(g, T)(X_1, \ldots, X_k; X, Y) &= -((X \wedge Y) \cdot T)(X_1, \ldots, X_k) = \\ &= T((X \wedge Y)X_1, \ldots, X_k) + \cdots + T(X_1, \ldots, (X \wedge Y)X_k),\end{aligned}$$

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where $(X \wedge_g Y)U = g(U, Y)X - g(U, X)Y$.

A Riemannian manifold is called pseudo-symmetric if at every point of M the following condition is satisfied:

the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The notion arose during the consideration of geodesic mappings.

Let $\xi \in \mathcal{X}(M)$. A diffeomorphism $f : V_n = (M, g) \mapsto \overline{V}_n = (M, \bar{g})$ is called $\xi-$ subgeodesic mapping if maps $\xi-$ subgeodesics into $\xi-$ subgeodesics, where $\xi-$ subgeodesics on M are given by the following equations:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^k}{dt}\frac{dx^j}{dt} = a\frac{dx^i}{dt} + b\xi^i,$$

where $a(t), b(t)$ are differentiable function on M.

There exists a $\xi-$ subgeodesic mapping f if and only if the Yano formulae are satisfied

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X - \varphi(X, Y)\xi,$$

where $\psi \in \wedge^1(M)$ and $\varphi \in \mathcal{T}^{0,2}(M)$. In the sequel φ coincides with the Riemannian metric g.

f is called nontrivial if $\psi_i - \xi_i \neq 0, \forall i \in \{1, \dots n\}$.

There exists f geodesic mapping (i.e. $\xi = 0$) [15] if and only if the Weyl formulae are satisfied

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X.$$

The projective curvature tensor [12], P, defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X, -S(X, Z)Y],$$

where S is the Ricci tensor, is invariant under the geodesic mappings, i.e. $\overline{P} = P$.

The geodesic correspondence is special if $\psi_{ij} = f g_{ij}$, where f is a differentiable function and

$$\psi_{ij} = \psi_{i,j} - \psi_i\psi_j.$$

One has

$$(n+1)\psi_k - \xi_k = \frac{\partial}{\partial x^k} ln\sqrt{\left|\frac{det(\overline{g}_{ij})}{det(g_{ij})}\right|},$$

where $\xi_k = g_{ks}\xi^r$.

There exists the functions $v(x^1, \dots, x^n), u(x^1, \dots, x^n)$ such that $\psi_k = \frac{\partial v}{\partial x^k}$ and $\xi_k = \frac{\partial u}{\partial x^k}$.

Let (M, g) be a Riemann space and $B \in \mathcal{T}^{0,2}(M)$, where $B_{rs} = \xi_{r,s} - \xi_s\xi_r$.

If $B = \frac{1}{n}Tr(B)g$, then the conformal transformation

$$g \mapsto \tilde{g} = e^{2\xi}g,$$

is called concircular transformation.

A concircular transformation carries all the circles of the manifold into circles (a curve in a Riemannian manifold is called circle when the first curvature is constant and all the other curvatures are identically zero).

A space of constant sectional curvature remains a space of constant sectional curvature by a concircular transformation.

The concircular curvature tensor [3], [14]

$$Z(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{\rho}{n(n-1)}(X_1 \wedge_g X_2)X_3,$$

is invariant under concircular transformations, where ρ is the scalar curvature.

Proposition 1.1. *Let $V_n = (M, g)$ be a pseudo-symmetric Riemann space and $g \mapsto \tilde{g}$ be a concircular transformation. Then $\tilde{V}_n = (M, \tilde{g})$ is a pseudo-symmetric Riemann space.*

Theorem 1.1. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be ξ-subgeodesically related Riemann spaces. If V_n is pseudo-symmetric and $g \mapsto \tilde{g} = e^{2\xi} g$ is a concircular transformation, then $\overline{V}_n$ is also pseudo-symmetric.*

Proof. V_n and $\overline{V_n}$ being subgeodesically related, we have

$\overline{\left| \begin{matrix} i \\ j\,k \end{matrix} \right|} = \left| \begin{matrix} i \\ j\,k \end{matrix} \right| + \delta^i_j \psi_k + \delta^i_k \psi_j - g_{jk} \xi^i.$

Because V_n and $\widetilde{V_n} = (M, \tilde{g})$ are conformally related, the Christoffel symbols are transformed by

$\widetilde{\left| \begin{matrix} i \\ j\,k \end{matrix} \right|} = \left| \begin{matrix} i \\ j\,k \end{matrix} \right| + \delta^i_j \xi_k + \delta^i_k \xi_j - g_{jk} \xi^i.$

Then we have $\overline{\left| \begin{matrix} i \\ j\,k \end{matrix} \right|} = \widetilde{\left| \begin{matrix} i \\ j\,k \end{matrix} \right|} + \delta^i_j \omega_k + \delta^i_k \omega_j$, where $\omega_k = \psi_k - \xi_k$.

So, $\overline{V_n}$ and $\widetilde{V_n}$ are geodesically related.

If V_n is pseudo-symmetric then $\tilde{V}_n$ is pseudo-symmetric. The last two properties imply that $\overline{V_n}$ is pseudo-symmetric. □

This theorem generalizes the following result [7]:

Let (M, g) be a pseudo-symmetric manifold admitting a nontrivial geodesic mapping f on $(M, \bar{g})$. Then $(M, \bar{g})$ is also a pseudo-symmetric space.

2. On Concircular Semi-Symmetric Spaces

Our aim is to characterize concircular semi-symmetric ($R \cdot Z = 0$) spaces geodesically related.

Theorem 2.1. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g}), n \geq 3$, be two nontrivial geodesically related Riemann spaces.*

If $\overline{V}_n$ is concircular semi-symmetric, then V_n and $\overline{V}_n$ are spaces with constant sectional curvature or are special geodesically related.

Proof. $\overline{V_n}$ is $\overline{Z}$-semi-symmetric.

Then $(\overline{R} \cdot \overline{Z})^h_{ijkrm} = \overline{Z}^i_{jkh;sm} - \overline{Z}^i_{jkh;ms} = 0$.

Contracting this relation with g^{kr}, one gets

$$\begin{gathered} g^{kr}(R^s_{ikj} R_{hsmr} + R^s_{imr} R_{hsjk} + R^s_{jmr} R_{hisk} + \\ + R^s_{kmr} R_{hijs}) + R^s_{ihj} \psi_{sm} - g_{hm} g^{kr} R^s_{ikj} \psi_{sr} + \\ + \psi_{im} S_{jh} - \psi_{is} R^s_{jmh} + \psi_{js} R^s_{imh} - g_{jh} g^{kr} \psi_{sk} R^s_{imr} - \\ - \psi_{js} R^s_{mih} + \psi_{is} R^s_{jmh} + \psi_{ms} R^s_{jih} - f R_{hijm} - g_{jh} \psi_{is} g^{sr} S_{rm} = 0, \end{gathered} \tag{1}$$

where $f = g^{ij} \psi_{ij}$.

Summing the above equation with the same obtained interchanging the indices h and i, we obtain

$$\begin{gathered} \psi_{sm} R^s_{ihj} + \psi_{sm} R^s_{hij} - g_{hm} g^{kr} \psi_{sr} R^s_{ikj} - \psi_{sr} g_{im} g^{kr} R^s_{hkj} + \\ + S_{jh} \psi_{im} + S_{ij} \psi_{mh} + \psi_{js} R^s_{imh} + \psi_{js} R^s_{hmi} - \\ - g_{jh} g^{kr} \psi_{ks} R^s_{imr} - g_{ij} g^{kr} \psi_{ks} R^s_{hmr} - g_{jh} \psi_{is} g^{sr} S_{rm} - \\ - g_{ij} \psi_{hs} S_{rm} = 0. \end{gathered} \tag{2}$$

Summing the relation (2) with the same equation obtained permuting the indices j with m, we have

$$\begin{aligned} S_{jh}\psi_{im} + S_{ij}\psi_{hm} - g_{jh}\psi_{is}S^s_m + S_{hm}\psi_{ij} - g_{ij}\psi_{sh}S^s_m + \\ +S_{im}\psi_{hj} - g_{mh}\psi_{is}S^s_j - g_{im}\psi_{sh}S^s_j = 0, \end{aligned} \tag{3}$$

where $S_{ij} = S^r_i g_{rj}$. After a contraction of (3) with g^{ij}, we get the equation

$$(n+1)\psi_{hs}S^s_m - \rho\psi_{hm} - fS_{hm} + g_{hm}\psi_{sr}S^{sr} - \psi_{sm}S^s_h = 0, \tag{4}$$

where $S^{ij} = g^{ir}S^j_r,\ \rho = g^{ij}S_{ij}$. From (4) we obtain

$$\rho f = nS^{ij}\psi_{ij}. \tag{5}$$

The relations (5) and (3) lead to

$$\psi_{sh}S^s_m = \frac{f}{n}S_{mh} - \frac{f\rho}{n^2}g_{mh} + \frac{\rho}{n}\psi_{mh} = \psi_{sm}S^s_h. \tag{6}$$

Using (6), the relation (3) becomes

$$\begin{aligned} (fg_{hm} - n\psi_{hm})(nS_{ij} - \rho g_{ij}) + (fg_{ij} - n\psi_{ij})(nS_{hm} - \rho g_{hm}) + \\ +(fg_{jm} - n\psi_{jm})(nS_{ih} - \rho g_{ih}) + (fg_{ih} - n\psi_{ih})(nS_{jm} - \rho g_{jm}) = 0. \end{aligned}$$

We obtain $(\psi_{ij} - \frac{f}{n}g_{ij})(S_{hm} - \frac{\rho}{n}g_{hm}) = 0$. Hence the correspondence is special or the space V_n is Einstein. In the second case one has

$$\psi_{ir} - \frac{f}{n}g_{ir} = 0 \qquad \text{or} \qquad P_{ijkh} = 0,$$

where P is the projective Weyl curvature tensor. V_n being an Einstein space, if $P = 0$, then V_n becomes a space with constant curvature. Hence, V_n and $\overline{V_n}$ are spaces with constant curvature, using the Beltrami theorem. □

Theorem 2.2. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g})$, $n \geq 3$, be two nontrivial geodesically related Riemann spaces. If $\overline{V}_n$ is $\overline{Z}$-semi-symmetric, with irreducible curvature tensor, then V_n and $\overline{V}_n$ are spaces with constant sectional curvature.*

Proof. If V_n and $\overline{V_n}$ are two special geodesically related Riemannian spaces then

$$\overline{R}^i_{jkh} = R^i_{jkh} + f(\delta^i_h g_{jk} - \delta^i_k g_{jh}),\ \text{where } \psi_{ij} = fg_{ij}.$$

The above relation leads to

$$g_{is}\overline{R}^s_{jkh} + g_{js}\overline{R}^s_{ikh} = 0.$$

The space $\overline{V_n}$ being with irreducible curvature tensor, then the system

$$x_{is}\overline{R}^s_{jkh} + x_{js}\overline{R}^s_{ikh} = 0 \tag{7}$$

has an unique solution, abstraction a factor. Because g_{ij} and $\overline{g}_{ij}$ are solutions of the system (7), we obtain $\overline{g}_{ij} = e^{2u}g_{ij}$, where u is a function with variables $x^1, ..., x^n$. V_n and $\overline{V_n}$ being geodesically related, we have $u =$ ct. and we obtain $\overline{\left| \begin{matrix} i \\ j\,k \end{matrix} \right|} = \left| \begin{matrix} i \\ j\,k \end{matrix} \right|$. Then $\delta^i_j\psi_k + \delta^i_k\psi_j = 0$ and $\psi_k = 0$. Using the previous result, the theorem is proved. □

The relation between the subgeodesic correspondence and the conformal related spaces leads to:

Theorem 2.3. *Let $V_n = (M, g)$ and $\overline{V}_n = (M, \overline{g})$, $n \geq 3$, be two nontrivial ξ- subgeodesically related Riemann spaces. If $\overline{V}_n$ is $\overline{Z}$-semi-symmetric, with irreducible curvature tensor, then $\overline{V}_n$ and $\tilde{V}_n = (M, \tilde{g} = e^{2u}g)$ are spaces with constant sectional curvature.*

3. On Concircular-Riemann Flows

Hamilton [8] introduced the concept of the Ricci flow. This means to control geometric quantities associated to the metric as it evolves. For a Riemannian manifold (M, g_0) the Ricci flow is the PDE:

$$\frac{\partial}{\partial t} g(t) = -2S(g(t)), g(0) = g_0,$$

where $S(g(t))$ denotes the Ricci curvature tensor associated to the metric $g(t)$. The idea is to evolve the metric in some way that will make the manifold "rounder and rounder". The hope is one may draw topological properties from the existence of such round metrics.

Along this line, the notion of Riemann flow [10] generalizes the Ricci flow. Let $(\Lambda^2(M), G(t)) = g(t) \odot g(t))$ be the Riemann manifold of skew symmetric 2-forms,where

$$G_{ijkl} = g_{ik} g_{jl} - g_{il} g_{jk}.$$

The Riemann flow is a PDE that evolves the metric tensor G:

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)),$$

where $R(g(t))$ is the Riemann curvature tensor associated to the metric $g(t)$.

These extensions are natural, since some results in the Riemann flow resemble the case of Ricci flow. For instance, the Riemann flow satisfies the short time existence and the uniqueness [10]. Also [10]:

Theorem A. *If (M, g_0) is a Riemann manifold $(n \geq 2)$ of constant sectional curvature -1, then the evolution metric of the Riemann flow is $g(t) = \sqrt{1 + 2(n-1)t} g_0$. The manifold expands homothetically for all time.*

Theorem B. *For the round unit sphere (S^n, g_0), $n \geq 2$, the evolution metric of the Riemann flow is $g(t) = \sqrt{1 - 2(n-1)t} g_0$ and the sphere collapses to a point in finite time.*

In order to generalize these notions, we introduce a special Riemann type flow. Let M be a smooth manifold endowed with a Riemann metric $g(t)$. For $(0, 2)$-tensors A and B, their Kulkarni-Nomizu product $A \wedge B$ is given by

$$\begin{aligned}(A \wedge B)(X_1, X_2; X, Y) &= A(X_1, Y)B(X_2, X) + A(X_2, X)B(X_1, Y) - \\ &- A(X_1, X)B(X_2, Y) - A(X_2, Y)B(X_1, X).\end{aligned}$$

Let $G = \frac{1}{2} g \wedge g$ be the Riemann metric induced on 2-forms.

A concircular-Riemann flow or a Z-Riemann type flow is a means of processing the Riemann metric $g(t)$ by allowing it to evolve under the PDE's system

$$\frac{\partial G}{\partial t}(t) = -2Z(g(t)), G(0) = G_0.$$

Theorem 3.1. *On a Riemannian manifold $(M, g(t))$ $(n \geq 3)$ the concircular-Ricci type flow*

$$\frac{\partial g_{ij}(t)}{\partial t} = \alpha g_{ij}(t) + \beta Z_{ij}(g(t))$$

determines the following concircular-Riemann type flow:

$$\frac{\partial G_{ijkl}(t)}{\partial t} = 2\alpha G_{ijkl}(t) + \beta E_{ijkl}(g(t)),$$

where α and β are functions on M, $Z_{ij}(g(t)) = Z^k_{ikj}(g(t))$ and $E_{ijkl}(g(t))$ is the semi-traceless part of the concircular curvature tensor.

Proof. Since

$$\frac{\partial G_{ijkl}}{\partial t} = -\frac{\partial g_{il}}{\partial t} g_{kj} - \frac{\partial g_{kj}}{\partial t} g_{il}$$

one gets

$$\frac{\partial G_{ijkl}(t)}{\partial t} = 2\alpha G_{ijkl}(t) + \beta E_{ijkl}(g(t)).$$

□

The following result gives a new family of concircular related metrics:

Theorem 3.2. *Let (M, g_0) be a Riemann manifold. The class g_t of concircular related metrics with g_0, given by the Z-Riemann type flow satisfies*

$$G(t) = -2Z(g_0)t + G_0.$$

Proof. Implicit solution of a Cauchy problem associated to the concircular-Riemann flow. □

As a consequence of the previous theorem, one has

Proposition 3.1. *Let $t \in (0, \epsilon), \epsilon > 0$, $f_t : (M, g_0) \mapsto (M, g_t)$ be a concircular mapping.*
If (M, g_0) is an Einstein manifold, then (M, g_t) is an Einstein manifold.

Ricci solitons are natural extensions of Einstein metrics and also correspond to self-similar solutions of Hamilton's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow. Ricci solitons are called quasi-Einstein metrics in physics literature.

We generalize this notion to the concept of concircular-Riemann soliton, which exibits rich geometric properties and are of interests to physicists as well, since the concircular curvature represents the deviation of the space time manifold from the space of constant curvature.

Suppose the family of diffeomorphisms $\varphi_t(x)$ is generated by the vector field $X(x)$. The evolutive metric $g(x, t) = \sigma(t)\varphi_t^*(x)g(x, 0)$ is a concircular-Riemann soliton iff the profile metric $g(x, 0) = g(x)$ is a solution of the nonlinear stationary PDE

$$Z(g) + \lambda G + \frac{1}{2} g \wedge \mathcal{L}_X g = 0,$$

where $\mathcal{L}_X$ is the Lie-derivative of the metric g with respect to X, λ is a constant and $\wedge$ is the Kulkarni-Nomizu product.

If X is a gradient, i.e., $X = \nabla f$, then we get the notion of gradient concircular-Riemann solitons, whose profile $g(x)$ satisfies the PDE

$$Z(g) + \lambda G + g \wedge h_f = 0,$$

for some smooth potential function f on M, where h_f is the Hessian.

Theorem 3.3. *A concircular-Riemann soliton on a compact Riemannian manifold is a gradient concircular-Riemann soliton.*

Theorem 3.4. *Let (M, g, λ, f) be an n-dimensional $(n > 2)$ gradient concircular-Riemann soliton.*
If the potential function f is harmonic, then the manifold has constant sectional curvature.

Proof. The gradient concircular-Riemann soliton verifies

$$Z_{ijkl} + \lambda G_{ijkl} + g_{il}h_{f\,jk} + g_{jk}h_{f\,il} - g_{ik}h_{f\,jl} - g_{jl}h_{f\,ik} = 0$$

and

$$Z_{il} + g_{il}[\lambda(n-1) + \Delta f] + (n-2)h_{f\,il} = 0,$$

where Δ is the Laplace operator. Therefore $\Delta f = -\frac{\lambda n}{2}$.

If f is harmonic, then the manifold is concircular flat and hence a space of constant sectional curvature. □

Open problem. Classification of gradient concircular-Riemann soliton for arbitrary potential function.

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