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# Multiobjective Programming under Nondifferentiable $G$-$V$-Invexity

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**Abstract.** In the paper, new Fritz John type necessary optimality conditions and new Karush-Kuhn-Tucker type necessary opimality conditions are established for the considered nondifferentiable multiobjective programming problem involving locally Lipschitz functions. Proofs of them avoid the alternative theorem usually applied in such a case. The sufficiency of the introduced Karush-Kuhn-Tucker type necessary optimality conditions are proved under assumptions that the functions constituting the considered nondifferentiable multiobjective programming problem are $G$-$V$-invex with respect to the same function $\eta$. Further, the so-called nondifferentiable vector $G$-Mond-Weir dual problem is defined for the considered nonsmooth multiobjective programming problem. Under nondifferentiable $G$-$V$-invexity hypotheses, several duality results are established between the primal vector optimization problem and its $G$-dual problem in the sense of Mond-Weir.

## 1. Introduction

During the last few decades, multiobjective programming (also referred to as vector optimization) has received much attention. This is a consequence of the fact that multiobjective optimization is known as a useful mathematical model in order to investigate some real world problems with conflicting objectives, arising from economics, engineering and human decision making.

Optimality conditions for nonsmooth multiobjective programming problems have been studied extensively in the literature (see, for example, [1], [3], [5], [9], [10], [11], [12], [13], [14], [22], [23], [24], [25], [26], [27], [28], and others). The theory and applications of multiobjective programming problems have been closely tied with convex analysis. In [17], Kanniappan established necessary optimality conditions of Fritz-John and Karush-Kuhn-Tucker type for nondifferentiable convex multiobjective programming problems.

However, not all practical problems, when formulated as multi-objective programming problems, fulfill the requirements of convexity. Therefore, generalizations of convexity related to the sufficiency of the necessary optimality conditions and various duality results for nonsmooth nonlinear multiobjective optimization problems have been of much interest in the recent past. Jeyakumar and Mond [16] introduced a new class of nonconvex differentiable vector-valued functions, namely $V$-invex functions, in order to resolve the difficulty of demanding the same function $\eta$ for objective and constraint functions in problems dealing with the concept of invexity introduced by Hanson [15] for scalar optimization problems. They established sufficient optimality criteria and duality results in the multiobjective static case for weak

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minima solutions under $V$-invexity. Kuk et al. [20] defined the concept of $V$-$\rho$-invexity for vector-valued functions, which is a generalization of the definition of a $V$-invex function [16], and they proved the generalized Karush-Kuhn-Tucker sufficient optimality theorem, and weak and strong duality for nonsmooth multiobjective programs under the $V$-$\rho$-invexity assumptions. In [5], Antczak introduced the concept of nondifferentiable $V$-$r$-invexity for the considered nondifferentiable multiobjective programming problem. Under $V$-$r$-invexity hypotheses, he proved optimality conditions and duality results for a new class of nonconvex nondifferentiable vector optimization problems with locally Lipschitz functions. Later, Mishra et al. [25] introduced generalized $V$-$r$-invexity notions for a nonsmooth multiobjective programming problem and, under generalized $V$-$r$-invexity hypotheses, they proved sufficient optimality conditions and duality results. Kuk and Tanino [21] obtained Karush-Kuhn-Tucker type necessary and sufficient optimality conditions and duality theorems for nonsmooth multiobjective programming problems involving vector-valued generalized type I functions. Tong and Zheng [29] defined the concept of nondifferentiable generalized $(F, \alpha, \rho, \theta)$-$d$-$V$-univexity and established some alternatives theorems and saddle-point necessary optimality conditions for nonsmooth multiobjective programming problems with locally Lipschitz functions.

In [4], Antczak introduced the concept of $G$-invexity for scalar differentiable functions as a generalization Hanson's definition of invexity (see [15]). He applied the introduced $G$-invexity notion to develop optimality conditions of F.John type and Karush-Kuhn-Tucker type for constrained differentiable mathematical programming problems and in proving new duality results. In a natural way, Antczak [6] and [7] extended the definition of $G$-invexity to the case of differentiable vector-valued functions. He [6] applied the vector $G$-invexity notion to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints and established the so-called $G$-Karush-Kuhn-Tucker necessary optimality conditions for this kind of vector optimization problems under the Kuhn-Tucker constraint qualification. Also under vector $G$-invexity hypotheses, Antczak [7] proved a number of new duality results between a nonlinear differentiable multiobjective programming problem and defined for it new vector dual problems.

Fritz John necessary optimality conditions and Karush-Kuhn-Tucker necessary optimality ones for nonsmooth vector optimization problems with inequality constraints are among the most important directions of investigation in optimization theory. Our aim in this paper is, therefore, to get the necessary optimality conditions for the considered nondifferentiable multiobjective programming problem with inequality constraints in which every component of the functions involved is a locally Lipschitz function. In this paper, therefore, new versions of Fritz John and Karush-Kuhn-Tucker necessary optimality conditions are established for the considered nonsmooth multiobjective programming problem of such a type. The so-called $G$-Fritz John type necessary optimality conditions and the $G$-Karush-Kuhn-Tucker type necessary optimality conditions are generalizations of the necessary optimality conditions of such type [6] to the nondifferentiable vectorial case. In many proofs of Fritz John necessary optimality conditions, an alternative theorem is used (see, for example, [18]). Whereas we don't use any alternative theorem in proving the necessary optimality conditions mentioned above. Further, along the lines Jeyakumar and Mond [16] and Antczak [6], we introduce the concept of nonsmooth $G$-$V$-invexity which is defined in terms of a Clarke generalized gradient of a locally Lipschitz function. By utilizing this concept of nondifferentiable generalized invexity, we prove the sufficiency of the G-Karush-Kuhn-Tucker type necessary optimality conditions established in this paper for the considered nonconvex nonsmooth multiobjective programming problem. An example of a nonconvex nonsmooth multiojective programming problem illustrates the fact that the new optimality results are more useful for some class of nonconvex nonsmooth vector optimization problems than similarly optimality results established in the literature under other nondifferentiable generalized convexity hypotheses.

Further, for the considered nonsmooth multiobjective programming problem, we also define its vector dual problem in the sense of Mond-Weir. The so-called vector $G$-dual problem in the sense of Mond-Weir is a generalization of a vector dual problem of such a type defined for differentiable vector dual problems by Antczak [7]. Under $G$-$V$-invexity hypotheses, we establish duality results for this kind of nonsmooth multiobjective programming problems.

## 2. Preliminaries

The following convention for equalities and inequalities will be used in the paper.
For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, we define:
(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

**Definition 2.1.** *[11] A function $f : R^n \to R$ is locally Lipschitz at a point $x \in R^n$ if there exist scalars $K > 0$ and $\varepsilon > 0$ such that, the following inequality*

$$\left| f(y) - f(z) \right| \leqq K \left\| y - z \right\|$$

*holds for all $y, z \in x + \varepsilon B$, where $B$ signifies the open unit ball in $R^n$, so that $x + \varepsilon B$ is the open ball of radius $\varepsilon$ about $x$.*

Let $X$ be a nonempty open subset of $R^n$.

**Definition 2.2.** *[11] The Clarke generalized directional derivative of a locally Lipschitz function $f : X \to R$ at $x \in X$ in the direction $v \in R^n$, denoted $f^0(x; v)$, is given by*

$$f^0(x; v) = \limsup_{\substack{y \to x \\ \theta \downarrow 0}} \frac{f(y + \theta v) - f(y)}{\theta}.$$

**Definition 2.3.** *[11] The Clarke generalized gradient of a locally Lipschitz function $f : X \to R$ at $x \in X$, denoted $\partial f(x)$, is defined as follows:*

$$\partial f(x) = \left\{ \xi \in R^n : f^0(x; d) \geq \langle \xi, d \rangle \text{ for all } d \in R^n \right\}.$$

**Lemma 2.4.** *[11] Let $f : X \to R$ be a locally Lipschitz function on $X$, $u$ is an arbitrary point of $X$ and $\lambda \in R$. Then*

$$\partial (\lambda f)(u) \subseteq \lambda \partial f(u).$$

**Proposition 2.5.** *[11] Let $f_i : X \to R$, $i = 1, ..., k$, be locally Lipschitz functions on a nonempty set $X \subset R^n$, $u$ is an arbitrary point of $X \subset R^n$. Then*

$$\partial \left( \sum_{i=1}^{k} f_i \right)(u) \subseteq \sum_{i=1}^{k} \partial f_i(u).$$

*Equality holds in the above relation if all but at most one of the functions $f_i$ is strictly differentiable at $u$.*

**Corollary 2.6.** *[11] For any scalars $\beta_i$, one has*

$$\partial \left( \sum_{i=1}^{k} \beta_i f_i \right)(u) \subseteq \sum_{i=1}^{k} \beta_i \partial f_i(u),$$

*and equality holds if all but at most one of the $f_i$ is strictly differentiable at $u$.*

**Theorem 2.7.** *[11] Let the function $f : R^n \to R$ be locally Lipschitz at a point $\overline{x} \in R^n$ and attain its (local) minimum at $\overline{x}$. Then*

$$0 \in \partial f(\overline{x}).$$

**Proposition 2.8.** *[11] Let the functions $f_i : R^n \to R$, $i \in I = \{1, ..., k\}$, be locally Lipschitz at a point $\overline{x} \in R^n$. Then the function $f : R^n \to R$ defined by $f(x) := \max_{i=1,..,k} f_i(x)$ is also locally Lipschitz at $\overline{x}$. In addition,*

$$\partial f(\overline{x}) \subset conv\{\partial f_i(\overline{x}) : i \in I(\overline{x})\},$$

*where $I(\overline{x}) := \{i \in I : f(\overline{x}) = f_i(\overline{x})\}$.*

Many generalizations of the definition of a convex function have been introduced in optimization theory in order to weak the assumption of convexity for establishing optimality and duality results for new classes of nonconvex optimization problems, including vector optimization problems. One of such a generalization of convexity in the vectorial case is the $G$-invexity notion introduced by Antczak for differentiable scalar and vector optimization problems (see [4], [6], respectively). We now generalize and extend it to the nondifferentiable vectorial case. Namely, motivated also by Jeyakumar and Mond [16] and Antczak [6], we introduce the concept of nondifferentiable $G$-$V$-invexity. To do this, we give some helpful denotations.

Let $X$ be a nonempty open subset of $R^n$ and $u \in X$. Further, let $f = (f_1, ..., f_k) : X \to R^k$, where each $f_i$ is a locally Lipschitz function on $X$ and $I_{f_i}(X)$, $i \in I = \{1, ..., k\}$, be the range of $f_i$, that is, the image of $X$ under $f_i$.

**Definition 2.9.** *If there exist $G_f = \left(G_{f_1}, ..., G_{f_k}\right) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$, $i = 1, ..., k$, is a strictly increasing differentiable real-valued function on its domain $I_{f_i}(X)$, a vector-valued function $\alpha_f = \left(\alpha_{f_1}, ..., \alpha_{f_k}\right) : X \times X \to R^k$, where $\alpha_{f_i} : X \times X \to R\backslash\{0\}$, $i = 1, ..., k$, and a vector-valued function $\eta : X \times X \to R^n$ such that the inequalities*

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geqq \alpha_{f_i}(x, u) G'_{f_i}(f_i(u)) \langle \xi_i, \eta(x, u) \rangle, \ i \in I \tag{1}$$

*hold for all $x \in X$ and each $\xi_i \in \partial f_i(u)$, $i = 1, ..., k$, then $f$ is said to be a nondifferentiable $G$-$V$-invex function at $u \in X$ on $X$ with respect to $\eta$, $G_f$ and $\alpha_f$.*
*We say that $f$ is nondifferentiable $G$-$V$-invex on $X$ with respect to $\eta$, $G_f$ and $\alpha_f$ if the inequalities (1) are satisfied at each $u$.*
*If the inequalities (1) are strict for all $x \in X$, $x \neq u$, then $f$ is said to be a nondifferentiable strictly $G$-$V$-invex function at $u \in X$ with respect to $\eta$, $G_f$ and $\alpha_f$.*
*Each function $f_i$ satisfying (1) is said to be a nondifferentiable $G$-$\alpha_i$-invex function at $u \in X$ on $X$ with respect to $\eta$ and $G_{f_i}$.*

**Remark 2.10.** *In the case when $G_{f_i}(a) \equiv a$, $i \in I$, for any $a \in I_{f_i}(X)$, we obtain the definition of a nondifferentiable vector-valued invex function (see [19], [22]).*

**Remark 2.11.** *In the case when $\alpha_{f_i}(x, u) = 1$, $i \in I$, for all $x, u \in X$, we obtain the definition of a nondifferentiable vector-valued $G$-invex function.*

**Remark 2.12.** *In the case when $\alpha_{f_i}(x, u) = 1$, $i \in I$, for all $x, u \in X$, and, moreover, $f$ is a differentiable function, then we obtain the definition of a differentiable vector-valued $G$-invex function (see [6], [7]).*

## 3. Optimality

In the paper, consider the following nonsmooth multiobjective programming problem

$$\begin{aligned} &V\text{-minimize } f(x) = (f_1(x), ..., f_k(x)) \\ &\text{subject to} \quad g_j(x) \leqq 0, \ j \in J, \qquad \text{(MOP)} \\ &\qquad\qquad x \in X, \end{aligned}$$

where $f_i : X \to R$, $i \in I = \{1, ..., k\}$ and $g_j : X \to R$, $j \in J = \{1, ...m\}$, are locally Lipschitz functions defined on a nonempty open set $X \subset R^n$.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. Let

$$D := \left\{x \in X : g_j(x) \leqq 0,\ j \in J\right\}$$

be the set of all feasible solutions in the considered vector optimization problem (MOP).

Further, we denote the set of active inequality constraints at point $\overline{x} \in D$ by

$$J(\overline{x}) = \left\{j \in J : g_j(\overline{x}) = 0\right\}.$$

For such multicriterion optimization problems as the considered vector optimization one, the optimal solution is defined in terms of a (weak) Pareto solution ((weakly) efficient solution) in the following sense:

**Definition 3.1.** *A feasible point $\overline{x}$ is said to be a weak Pareto solution (weakly efficient solution, weak minimum) for (MOP) if and only if there exists no $x \in D$ such that*

$$f(x) < f(\overline{x}).$$

**Definition 3.2.** *A feasible point $\overline{x}$ is said to be a Pareto solution (efficient solution) for (MOP) if and only if there exists no $x \in D$ such that*

$$f(x) \leq f(\overline{x}).$$

In [4], Antczak established new necessary optimality conditions for a feasible solution to be optimal in differentiable scalar optimization problems. He named them the $G$-F.John necessary optimality conditions and $G$-Karush-Kuhn-Tucker necessary optimality conditions. Later, Antczak [6] established the $G$-Karush-Kuhn-Tucker necessary conditions also for differentiable multiobjective programming problems. In [8], Antczak corrected the $G$-Karush-Kuhn-Tucker necessary conditions proved in [4] to assure that the Lagrange multiplier associated to the objective function is not equal to 0. Now, we extend the necessary optimality conditions mentioned above to the nonsmooth vectorial case. Namely, we prove the $G$-Fritz John type necessary optimality conditions and the $G$-Karush-Kuhn-Tucker type necessary optimality conditions for the considered nondifferentiable multiobjective programming problem in which each component of the involved functions is locally Lipschitz.

**Theorem 3.3.** *(G-Fritz John Type Necessary Optimality Conditions). Let $\overline{x}$ be a weak Pareto solution in the considered nondifferentiable multiobjective programming problem (MOP). Further, assume that there exist strictly increasing differentiable real-valued functions $G_{f_i}$, $i \in I$, defined on $I_{f_i}(D)$ and strictly increasing differentiable real-valued functions $G_{g_j}$, $j \in J$, defined on $I_{g_j}(D)$ with $G_{g_j}(0) = 0$, $j \in J$. Then there exist $\overline{\lambda} \in R^k$ and $\overline{\mu} \in R^m$ such that the following conditions*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x}))\, \partial f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right) \partial g_j(\overline{x}) \tag{2}$$

$$\overline{\mu}_j \left(G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(\overline{x})\right)\right) \leqq 0, \quad j \in J,\ \forall\, x \in D, \tag{3}$$

$$\left(\overline{\lambda}, \overline{\mu}\right) \geq 0 \tag{4}$$

*hold.*

*Proof.* Let $\overline{x}$ be a weak Pareto solution in the considered multiobjective programming problem (MOP). We define a function $H$ as follows:

$$H(x) = \max\left\{G_{f_i}(f_i(x)) - G_{f_i}(f_i(\overline{x})),\, G_{g_j}\left(g_j(x)\right) : i = 1, ..., k,\ j = 1, ..., m\right\}. \tag{5}$$

Now, we prove that the following inequality

$$H(x) \geqq 0 \tag{6}$$

holds for all $x \in X$. Suppose, contrary to the result, that there exists $\widetilde{x} \in X$ such that

$$H(\widetilde{x}) < 0. \tag{7}$$

Hence, by (7), it follows that $G_{g_j}\left(g_j(\widetilde{x})\right) < 0,\ j \in J$. By assumption, $G_{g_j}(0) = 0,\ j \in J$. Thus, two relations above yield

$$G_{g_j}\left(g_j(\widetilde{x})\right) < G_{g_j}(0),\ j \in J. \tag{8}$$

Since each $G_{g_j}$, $j \in J$, is a strictly increasing function on its domain, therefore, inequalities (8) imply

$$g_j(\widetilde{x}) < 0,\ j \in J. \tag{9}$$

This means that $\widetilde{x}$ is feasible in the considered multiobjective programming problem. Again using (7), we have that the inequalities

$$G_{f_i}(f_i(\widetilde{x})) < G_{f_i}(f_i(\overline{x})),\ \ i = 1, ..., k \tag{10}$$

hold. By assumption, each $G_{f_i}$, $i = 1, ..., k$, is a strictly increasing function on its domain. Thus, (10) implies that the following inequalities

$$f_i(\widetilde{x}) < f_i(\overline{x}),\ \ i = 1, ..., k \tag{11}$$

hold. Since we have shown above that $\widetilde{x} \in D$, inequalities (11) mean that $\overline{x}$ is not a weak Pareto solution in problem (MOP), which is a contradiction. Hence, the relation (6) is satisfied.

By assumption, $\overline{x}$ is a weak Pareto solution in the considered multiobjective programming problem (MOP). Therefore, it is feasible in problem (MOP). This means that $g_j(\overline{x}) \leqq 0,\ j \in J$. Since $G_{g_j}(0) = 0,\ j \in J$, and each $G_{g_j}$, $j \in J$, is a strictly increasing function on its domain, the inequalities

$$G_{g_j}\left(g_j(\overline{x})\right) \leqq G_{g_j}(0) = 0,\ j \in J$$

hold. Using the inequalities above together with (5), we get

$$H(\overline{x}) = 0. \tag{12}$$

Taking into account (6) and (12), we conclude that $\overline{x}$ is a global minimizer of $H$. Hence, by Theorem 2.7, it follows that

$$0 \in \partial H(\overline{x}). \tag{13}$$

Then, using Chain Rule [11] together with Corollary 2.6, we have

$$\partial\left(G_{f_i}(f_i(x)) - G_{f_i}(f_i(\overline{x}))\right) = \partial\left(G_{f_i}(f_i(x))\right) = G'_{f_i}(f_i(x))\,\partial f_i(x),\ i = 1, ..., k, \tag{14}$$

$$\partial\left(G_{g_j}\left(g_j(x)\right)\right) = G'_{g_j}\left(g_j(x)\right)\partial g_j(x),\ j \in J. \tag{15}$$

We denote by $J_H(\overline{x})$ the index set of inequality constraints indices $j$ for which $H(\overline{x}) = G_{g_j}\left(g_j(\overline{x})\right)$. Thus, by Proposition 2.8, the relations (13), (14) and (15) imply

$$0 \in conv\left\{G'_{f_i}(f_i(\overline{x}))\,\partial f_i(\overline{x}),\ G'_{g_j}\left(g_j(\overline{x})\right)\partial g_j(\overline{x}) : i = 1, ..., k,\ j \in J_H(\overline{x})\right\}.$$

Hence, by the definition of a convex hull, there exist $\overline{\lambda}_i \geqq 0$, $i = 1, ..., k$, $\overline{\mu}_j \geqq 0$, $j \in J_H(\overline{x})$ with $\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{j \in J_H(\overline{x})} \overline{\mu}_j = 1$ such that

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x}))\,\partial f_i(\overline{x}) + \sum_{j \in J_H(\overline{x})} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\partial g_j(\overline{x}).$$

If we set $\overline{\mu}_j = 0$ for $j \notin J_H(\overline{x})$, then we get (2).

The proof of the condition (3.8) is trivial. Indeed, if $g_j(\overline{x}) < 0$ for some $j \notin J_H(\overline{x})$, then, using the assumption that $G_{g_j}$ is a strictly increasing function on its domain together with $G_{g_j}(0) = 0$, we have $G_{g_j}\left(g_j(\overline{x})\right) < G_{g_j}(0) = 0$. Thus, if we set $\overline{\mu}_j = 0$, then $\overline{\mu}_j\left(G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(\overline{x})\right)\right) \leqq 0$ for all $x \in D$. If $j \in J_H(\overline{x})$, then $G_{g_j}\left(g_j(\overline{x})\right) = G_{g_j}(0) = 0$. Hence, it is not difficult to note that (3.8) holds for all $x \in D$ and $\overline{\mu}_j \geqq 0$, $j \in J_H(\overline{x})$. Thus, the proof of this theorem is completed. □

**Remark 3.4.** *Note that, in order to prove the G-Fritz John type necessary optimality conditions, we don't use any alternative theorem. Further, the G-Fritz John type necessary optimality conditions established in this paper differ from those ones proved in [18] and its proof is simpler than in [18].*

It is well-known (see, for example, [13], [19]) that, under a suitable constraint qualification, if $\overline{x} \in D$ is a (weak) Pareto solution in the considered multiobjective programming problem (MOP), then the necessary optimality conditions, known as Karush-Kuhn-Tucker conditions, are satisfied.

Now, for the considered nonsmooth multiobjective programming problem (MOP), we prove the so-called $G$-Karush-Kuhn-Tucker type necessary optimality conditions. They are generalization of the $G$-Karush-Kuhn-Tucker type necessary optimality conditions introduced by Antczak in [6] for the differentiable vector optimization problems. In order to prove them, we need the following constraint qualification.

**Definition 3.5.** *The multiobjective programming problem (MOP) is said to satisfy the G-V-constraint qualification (G-V-CQ) at $\overline{x} \in D$ if $g_j$, $j \in J(\overline{x})$, are $G_{g_j}$-$\alpha_{g_j}$-invex at $\overline{x}$ on D with respect to the same function $\eta : D \times D \to R^n$ with $\sum_{j\in J(\overline{x})}\left[G'_{g_j}\left(g_j(\overline{x})\right)\right]^2 \neq 0$ and, moreover, there exists $\widetilde{x} \in D$ such that $G_{g_j}\left(g_j(\widetilde{x})\right) < G_{g_j}\left(g_j(\overline{x})\right)$, $j \in J(\overline{x})$.*

**Theorem 3.6.** *(G-Karush-Kuhn-Tucker Type Necessary Optimality Conditions). Let $\overline{x}$ be a weak Pareto solution in the considered nondifferentiable multiobjective programming problem (MOP). Further, assume that there exist strictly increasing differentiable real-valued functions $G_{f_i}$, $i \in I$, defined on $I_{f_i}(D)$ and strictly increasing differentiable real-valued functions $G_{g_j}$, $j \in J$, defined on $I_{g_j}(D)$ with $G_{g_j}(0) = 0$, $j \in J$. If the G-V-constraint qualification (G-V-CQ) is satisfied at $\overline{x}$ for problem (MOP), then there exist $\overline{\lambda} \in R^k$ and $\overline{\mu} \in R^m$ such that the following conditions*

$$0 \in \sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right)\partial f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\partial g_j(\overline{x}) \tag{16}$$

$$\overline{\mu}_j\left(G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(\overline{x})\right)\right) \leqq 0, \quad j \in J, \forall x \in D, \tag{17}$$

$$\overline{\lambda} \geq 0, \ \overline{\mu} \geqq 0 \tag{18}$$

*hold.*

*Proof.* Since $\overline{x}$ is a weak Pareto solution in the considered multiobjective programming problem (MOP), the $G$-Fritz John Type necessary optimality conditions (2)-(4) are satisfied. In order to prove the $G$-Karush-Kuhn-Tucker necessary optimality conditions (16)-(18), therefore, it is sufficient to prove that $\overline{\lambda} \neq 0$. Suppose, contrary to the result, that $\overline{\lambda} = 0$. Then, by the $G$-Fritz John Type necessary optimality condition (2), it follows that

$$0 \in \sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\partial g_j(\overline{x}). \tag{19}$$

Since the $G$-$V$-constraint qualification ($G$-$V$-CQ) is satisfied at $\overline{x} \in D$, the constraint functions $g_j$, $j \in J(\overline{x})$, are $G_{g_j}$-$\alpha_{g_j}$-invex at $\overline{x}$ on $D$ with respect to the same function $\eta : D \times D \to R^n$. Hence, by Definition 2.9, the inequalities

$$G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(\overline{x})\right) \geqq \alpha_{g_j}(x,\overline{x})\, G'_{g_j}\left(g_j(\overline{x})\right)\left\langle \zeta_j, \eta(x,\overline{x})\right\rangle, \ j \in J(\overline{x}) \tag{20}$$

hold for all $x \in D$ and each $\zeta_j \in \partial g_j(\overline{x})$, $j \in J(\overline{x})$. Therefore, they are also satisfied for $x = \widetilde{x} \in D$. Multiplying each inequality (20) by the corresponding Lagrange multiplier $\overline{\mu}_j$, we get

$$\overline{\mu}_j \left[ G_{g_j}\left(g_j(\widetilde{x})\right) - G_{g_j}\left(g_j(\overline{x})\right)\right] \geqq \alpha_{g_j}(\widetilde{x},\overline{x})\,\overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\left\langle \zeta_j, \eta(\widetilde{x},\overline{x})\right\rangle, \ j \in J(\overline{x}).$$

Hence, using the $G$-Fritz John type necessary optimality condition (3.8) together with the $G$-$V$-constraint qualification, we have

$$\overline{\mu}_j \left[ G_{g_j}\left(g_j(\widetilde{x})\right) - G_{g_j}\left(g_j(\overline{x})\right)\right] < 0, \ j \in J(\overline{x}).$$

Combining above inequalities, we get

$$\overline{\mu}_j \alpha_{g_j}(\widetilde{x},\overline{x})\, G'_{g_j}\left(g_j(\overline{x})\right)\left\langle \zeta_j, \eta(\widetilde{x},\overline{x})\right\rangle < 0, \ j \in J(\overline{x}).$$

Since $\alpha_{g_j}(\widetilde{x},\overline{x}) > 0$, $j \in J(\overline{x})$, the following inequalities

$$\overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\left\langle \zeta_j, \eta(\widetilde{x},\overline{x})\right\rangle < 0, \ j \in J(\overline{x}) \tag{21}$$

hold. Taking into account $\overline{\mu}_j = 0$, $j \notin J(\overline{x})$, and then adding both sides of inequalities (21), we obtain that the following inequality

$$\sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\left\langle \zeta_j, \eta(\widetilde{x},\overline{x})\right\rangle < 0$$

holds for each $\zeta_j \in \partial g_j(\overline{x})$, $j \in J$, contradicting (19). This means that $\overline{\lambda} \geq 0$ and the proof of this theorem is completed. □

Now, we prove the sufficient optimality conditions for the considered nonsmooth vector optimization problem under hypotheses that the functions constituting it are nondifferentiable $G$-$V$-invex at a feasible point satisfying the $G$-Karush-Kuhn-Tucker type necessary optimality conditions (16)-(18).

**Theorem 3.7.** *Let $\overline{x} \in D$, $G_{f_i}$, $i \in I$, be a strictly increasing differentiable real-valued function defined on $I_{f_i}(D)$ and $G_{g_j}$, $j \in J$, be a strictly increasing differentiable real-valued function defined on $I_{g_j}(D)$ such that the G-Karush-Kuhn-Tucker necessary optimality conditions (16)-(18) be satisfied at $\overline{x}$ with $G_{f_i}$, $i \in I$, and $G_{g_j}$, $j \in J$. Further, assume that $f$ is a $G_f$-V-invex function at $\overline{x}$ on D and g is a $G_g$-V-invex function at $\overline{x}$ on D with respect to the same vector-valued function $\eta : D \times D \to R^n$. Then $\overline{x}$ is a weak Pareto solution in problem (MOP).*

*Proof.* Assume that $\overline{x}$ is such a feasible solution in problem (MOP) at which the $G$-Karush-Kuhn-Tucker necessary optimality conditions (16)-(18) are satisfied with Lagrange multipliers $\overline{\lambda} \in R^k$, $\overline{\mu} \in R^m$ and with respect to $G_f = \left(G_{f_1}, ..., G_{f_k}\right)$ and $G_g = \left(G_{g_1}, ..., G_{g_m}\right)$, where $G_{f_i}$, $i \in I$, is a strictly increasing differentiable real-valued function defined on $I_{f_i}(D)$ and $G_{g_j}$, $j \in J$, is a strictly increasing differentiable real-valued function defined on $I_{g_j}(D)$. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution in the considered nondifferentiable multiobjective programming problem (MOP). Then, by Definition 3.1, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{22}$$

By assumption, $f$ is a $G_f$-$V$-invex function at $\overline{x}$ on $D$ with respect to $\eta$. Thus, $G_{f_i}$, $i \in I$, is a strictly increasing differentiable real-valued function defined on $I_{f_i}(D)$. Hence, (22) implies

$$G_{f_i}\left(f_i(\widetilde{x})\right) < G_{f_i}\left(f_i(\overline{x})\right), \ i \in I. \tag{23}$$

By Definition 2.9, the following inequalities

$$G_{f_i}(f_i(x)) - G_{f_i}\left(f_i(\overline{x})\right) \geqq \alpha_{f_i}(x,\overline{x})\, G'_{f_i}\left(f_i(\overline{x})\right)\left\langle \xi_i, \eta(x,\overline{x})\right\rangle, \ i \in I$$

hold for all $x \in D$ and each $\xi_i \in \partial f_i(\overline{x})$, $i \in I$. Therefore, they are also satisfied for $x = \widetilde{x} \in D$. Thus,

$$G_{f_i}(f_i(\widetilde{x})) - G_{f_i}(f_i(\overline{x})) \geqq \alpha_{f_i}(\widetilde{x}, \overline{x}) G'_{f_i}(f_i(\overline{x})) \langle \xi_i, \eta(\widetilde{x}, \overline{x}) \rangle, \; i \in I. \tag{24}$$

Combining (23) and (24), we have

$$\alpha_{f_i}(\widetilde{x}, \overline{x}) G'_{f_i}(f_i(\overline{x})) \langle \xi_i, \eta(\widetilde{x}, \overline{x}) \rangle < 0, \; i \in I.$$

Since $\alpha_{f_i}(\widetilde{x}, \overline{x}) > 0$, $i \in I$, the above inequalities yield

$$G'_{f_i}(f_i(\overline{x})) \langle \xi_i, \eta(\widetilde{x}, \overline{x}) \rangle < 0, \; i \in I. \tag{25}$$

Multiplying (25) by the corresponding Lagrange multiplier $\overline{\lambda}_i$ and then adding both sides of the obtained inequalities, we get that the following inequality

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x})) \langle \xi_i, \eta(\widetilde{x}, \overline{x}) \rangle < 0, \; i \in I \tag{26}$$

holds for each $\xi_i \in \partial f_i(\overline{x})$, $i \in I$. By assumption, $g$ is a $G_g$-$V$-invex function at $\overline{x}$ on $D$ with respect to $\eta$. By Definition 2.9, it follows that

$$G_{g_j}(g_j(\widetilde{x})) - G_{g_j}\left(g_j(\overline{x})\right) \geqq \alpha_{g_j}(\widetilde{x}, \overline{x}) G'_{g_j}\left(g_j(\overline{x})\right) \left\langle \zeta_j, \eta(\widetilde{x}, \overline{x}) \right\rangle, \; j \in J. \tag{27}$$

Multiplying (27) by the corresponding Lagrange multiplier $\overline{\mu}_j$ and then using the $G$-Karush-Kuhn-Tucker necessary optimality condition (17), we obtain that the inequalities

$$\alpha_{g_j}(\widetilde{x}, \overline{x}) \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right) \left\langle \zeta_j, \eta(\widetilde{x}, \overline{x}) \right\rangle \leqq 0, \; j \in J$$

hold for each $\zeta_j \in \partial g_j(\overline{x})$, $j \in J$. Since $\alpha_{g_j}(\widetilde{x}, \overline{x}) > 0$, $j \in J$, the inequalities above yield

$$\overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right) \left\langle \zeta_j, \eta(\widetilde{x}, \overline{x}) \right\rangle \leqq 0, \; j \in J.$$

Then, adding both sides of the inequalities above, we get that the inequality

$$\sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right) \left\langle \zeta_j, \eta(\widetilde{x}, \overline{x}) \right\rangle \leqq 0 \tag{28}$$

holds for each $\zeta_j \in \partial g_j(\overline{x})$, $j \in J$. Adding both sides of inequalities (26) and (28), we obtain the inequality

$$\left\langle \sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}(f_i(\overline{x})) \xi_i + \sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right) \zeta_j, \eta(\widetilde{x}, \overline{x}) \right\rangle < 0,$$

which contradicts the $G$-Karush-Kuhn-Tucker necessary optimality condition (16). Hence, $\overline{x}$ is a weak Pareto solution for (MOP) and this means that the proof of this theorem is completed. □

In order to prove that a feasible point $\overline{x}$ satisfying the $G$-Karush-Kuhn-Tucker necessary optimality conditions (16)-(18) is Pareto optimal for (MOP), some slightly stronger hypotheses imposed on the functions constituting it are needed.

**Theorem 3.8.** *Let $\overline{x} \in D$, $\eta : D \times D \to R^n$ be a vector-valued function, $G_{f_i}$, $i \in I$, be a strictly increasing differentiable real-valued function defined on $I_{f_i}(D)$ and $G_{g_j}$, $j \in J$, be a strictly increasing differentiable real-valued function defined on $I_{g_j}(D)$ such that the $G$-Karush-Kuhn-Tucker necessary optimality conditions (16)-(18) be satisfied at $\overline{x}$ with functions $G_{f_i}$, $i \in I$, and $G_{g_j}$, $j \in J$. Further, assume that $f$ is a strictly $G_f$-$V$-invex function at $\overline{x}$ on $D$ with respect to $\eta$ and $g$ is a $G_g$-$V$-invex at $\overline{x}$ on $D$ with respect to $\eta$. Then $\overline{x}$ is a Pareto solution in problem (MOP).*

Now, we give an example of a nonconvex nondifferentiable multiobjective programming problem and we prove Pareto optimality of a feasible point by using the optimality conditions established in the paper. Based on this example, we show that the sufficient optimality conditions under $G$-$V$-invexity are helpful in proving Pareto optimality a feasible point satisfying the $G$-Karush-Kuhn-Tucker necessary optimality conditions. We also show that neither the sufficient optimality conditions under invexity nor under $G$-invexity are not applicable for the nondifferentiable multiobjective programming problem considered in this example.

**Example 3.9.** *Consider the following nonconvex nondifferentiable multiobjective programming problem*

$$\begin{array}{c} f(x) = \left(\ln\left(\frac{|x|}{x^2+x+1}+1\right),\ \arctan\left(e^{-x}|x|\right)\right) \to \min \\ g(x) = 1 - e^{x} \leqq 0. \end{array} \quad (MOP1)$$

*Note that $D = \{x \in R : x \geqq 0\}$. Let $G_f(t) = e^t$ and $G_g(t) = \tan(t)$. It is not difficult to note that a feasible solution $\overline{x} = 0$ satisfies the G-Karush-Kuhn-Tucker necessary optimality conditions (16)-(18) with $G_f$ and $G_g$ defined above. In order to prove that $\overline{x} = 0$ is a Pareto solution in the considered nondifferentiable multiobjective programming problem (MOP1), we use the sufficient optimality conditions established in the paper. Hence, we have to prove that the functions constituting problem (MOP1) are G-V-invex at $\overline{x} = 0$ on D with respect to the functions $G_f$ and $G_g$ defined above and with respect to the same function $\eta : D \times D \to R$. Let $\eta$ be defined as follows $\eta(x,\overline{x}) = |x| - |\overline{x}|$ and, moreover, $\alpha_{f_1}(x,\overline{x}) = \frac{1}{x^2+x+1}$, $\alpha_{f_2}(x,\overline{x}) = e^{-x}$ and $\alpha_g(x,\overline{x}) = 1$. Then, by Definition 2.9, it can be shown that the objective function f is strictly $G_f$-V-invex at $\overline{x} = 0$ on D with respect to the functions $\eta$, $G_f$ and $\alpha_f = \left(\alpha_{f_1}, \alpha_{f_2}\right)$ and the constraint function g is $G_g$-V-invex at $\overline{x} = 0$ on D with respect to the functions $\eta$, $G_g$ and $\alpha_g$. Thus, all hypotheses of Theorem 3.8 are fulfilled and, therefore, we conclude that $\overline{x} = 0$ is a Pareto solution in the considered nondifferentiable multiobjective programming problem (MOP1). Note, moreover, that we are not in a position to use the sufficient optimality conditions under invexity (see, for example, [19], [22]). It follows from the fact that there doesn't exist any function $\eta$ defined by $\eta : D \times D \to R$ with respect to which the functions constituting problem (MOP1) are invex at $\overline{x} = 0$ on D. Further, also the sufficient optimality conditions under nondifferentiable G-invexity (see Remark 2.11) are not applicable in the considered case since the functions constituting problem (MOP1) are not G-invex with respect to $\eta$, $G_f$ and $G_g$ defined above.*

## 4. Nondifferentiable *G*-Mond-Weir Duality

In this section, for the considered nondifferentiable multiobjective programming problem (MOP), we define a vector $G$-dual problem in the sense of Mond-Weir as follows

$$\begin{array}{c} V\text{-maximize } f(x) = (f_1(y), ..., f_k(y)) \\ \text{s.t. } 0 \in \sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y))\, \partial f_i(y) + \sum_{j=1}^{m} \mu_j G'_{g_j}\left(g_j(y)\right) \partial g_j(y), \\ \mu_j\left[G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(y)\right)\right] \leqq 0, \quad j \in J, \forall\, x \in D, \\ \lambda \geq 0,\ \xi \geqq 0, \end{array} \quad (G\text{-DVP})$$

where $f_i : X \to R$, $i \in I = \{1, ..., k\}$ and $g_j : X \to R$, $j \in J = \{1, ...m\}$, are locally Lipschitz functions, $G_f$ and $G_g$ are fixed and they verified that $G_f = \left(G_{f_1}, ..., G_{f_k}\right) : R \to R^k$ is a differentiable vector-valued function such that each its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain, and $G_g = \left(G_{g_1}, ..., G_{g_m}\right) : R \to R^m$ is a differentiable vector-valued function such that each its component $G_{g_j} : I_{g_j}(X) \to R$ is a strictly increasing function on its domain.

Let

$$\begin{array}{c} \Omega = \Big\{(y,\lambda,\mu) : 0 \in \sum_{i=1}^{k} \lambda_i G'_{f_i}(f_i(y))\, \partial f_i(y) + \sum_{j=1}^{m} \mu_j G'_{g_j}\left(g_j(y)\right) \partial g_j(y), \\ \mu_j\left[G_{g_j}\left(g_j(x)\right) - G_{g_j} g_j(y)\right)\right] \leqq 0, \quad j \in J, \forall\, x \in D, \lambda \geq 0, \mu \geqq 0\Big\} \end{array}$$

be the set of all feasible solutions in ($G$-DVP). Let us denote by $pr_X\Omega$, the projection of $\Omega$ on $X$ and by $Y$ the set $D \cup pr_X\Omega$.

Note that this vector optimization problem as dual one is formulated in terms of maximization instead of minimization. It implies that we have to reconsider the definition of a (weakly) efficient solution for optimization problems of this type. But it is enough to say that $\overline{y} \in pr_X\Omega$ is a (weakly) efficient solution of a maximum type in (DVP) if there exists no $y \in pr_X\Omega$ such that $f(\overline{y}) - f(y)(<) \leq 0$ is verified. That is, just the reverse definition for a minimization process.

**Theorem 4.1 (Weak duality).** *Let $x$ and $(y, \lambda, \mu)$ be any feasible solutions in problems (MOP) and (G-DVP), respectively. Assume that $f$ is a $G_f$-$V$-invex function at $y$ on $Y$ with respect to $\eta$ and $g$ is a $G_g$-$V$-invex function at $y$ on $Y$ with respect to $\eta$. Then, $f(x) \not< f(y)$.*

*Proof.* Let $x$ and $(y, \lambda, \mu)$ be any feasible solutions in problems (MOP) and ($G$-DVP), respectively. Suppose, contrary to the result, that

$$f(x) < f(y).$$

By assumption, $f$ is a $G_f$-$V$-invex function at $y$ on $Y$ with respect to $\eta$ and $g$ is a $G_g$-$V$-invex function at $y$ on $Y$ with respect to $\eta$. Since $G_f = \left(G_{f_1}, ..., G_{f_k}\right) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to R$ is a strictly increasing function on its domain, the inequality above implies

$$G_{f_i}(f(x)) < G_{f_i}(f(y)), \ i = 1, ..., k. \tag{29}$$

Since $(y, \lambda, \mu) \in \Omega$, multiplying each above inequality by the corresponding Lagrange multiplier $\lambda_i$, $i = 1, ..., k$, we obtain

$$\lambda_i G_{f_i}(f(x)) \leqq \lambda_i G_{f_i}(f(y)), \ i = 1, ..., k, \tag{30}$$

$$\lambda_{i_0} G_{f_{i_0}}(f(x)) < \lambda_{i_0} G_{f_{i_0}}(f(y)) \text{ for at least one } i_0 \in \{1, ..., k\}. \tag{31}$$

Thus, (30) and (31) yield

$$\sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(x)) < \sum_{i=1}^{k} \lambda_i G_{f_i}(f_i(y)). \tag{32}$$

By Definition 2.9, it follows that the following inequalities

$$G_{f_i}(f_i(z)) - G_{f_i}(f_i(y)) \geqq \alpha_{f_i}(z, y) G'_{f_i}(f_i(y)) \langle \xi_i, \eta(z, y) \rangle, \ i \in I, \tag{33}$$

$$G_{g_j}\left(g_j(z)\right) - G_{g_j}\left(g_j(y)\right) \geqq \alpha_{g_j}(z, y) G'_{g_j}\left(g_j(y)\right) \left\langle \zeta_j, \eta(z, y) \right\rangle, \ j \in J \tag{34}$$

hold for each $\xi_i \in \partial f_i(y)$, $i \in I$, each $\zeta_j \in \partial g_j(y)$, $j \in J$, and for all $z \in Y$. Therefore, they are also satisfied for $z = x \in D$. Multiplying each inequality (33) and (34) by the associated Lagrange multiplier $\lambda_i$, $i \in I$, and $\mu_j$, $j \in J$, respectively, we obtain, respectively,

$$\lambda_i G_{f_i}(f_i(x)) - \lambda_i G_{f_i}(f_i(y)) \geqq \alpha_{f_i}(x, y) \lambda_i G'_{f_i}(f_i(y)) \langle \xi_i, \eta(x, y) \rangle, \ i \in I, \tag{35}$$

$$\mu_j G_{g_j}\left(g_j(x)\right) - \mu_j G_{g_j}\left(g_j(y)\right) \geqq \alpha_{g_j}(x, y) \mu_j G'_{g_j}\left(g_j(y)\right) \left\langle \zeta_j, \eta(x, y) \right\rangle, \ j \in J. \tag{36}$$

Combining (30), (31) and (35), we have

$$\alpha_{f_i}(x, y) \lambda_i G'_{f_i}(f_i(y)) \langle \xi_i, \eta(x, y) \rangle \leqq 0, \ i \in I,$$

$$\alpha_{f_{i_0}}(x, y) \lambda_{i_0} G'_{f_{i_0}}(f_{i_0}(y)) \langle \xi_{i_0}, \eta(x, y) \rangle < 0 \text{ for at least one } i_0 \in \{1, ..., k\}.$$

Since $\alpha_{f_i}(x, y) > 0$, $i \in I$, the above inequalities yield, respectively,

$$\lambda_i G'_{f_i}(f_i(y)) \langle \xi_i, \eta(x, y) \rangle \leqq 0, \ i \in I, \tag{37}$$

$$\lambda_{i_0} G'_{f_{i_0}}\left(f_{i_0}(y)\right)\left\langle \xi_{i_0}, \eta(x,y)\right\rangle < 0 \text{ for at least one } i_0 \in \{1,...,k\}. \tag{38}$$

Thus, (37) and (38) imply that the inequality

$$\sum_{i=1}^{k} \lambda_i G'_{f_i}\left(f_i(y)\right)\left\langle \xi_i, \eta(x,y)\right\rangle < 0 \tag{39}$$

holds for each $\xi_i \in \partial f_i(y)$, $i \in I$. Combining (36) and the second constraint of problem ($G$-DVP), we have that the inequalities

$$\alpha_{g_j}(x,y)\, G'_{g_j}\left(g_j(y)\right)\left\langle \zeta_j, \eta(x,y)\right\rangle \leqq 0, \ \ j \in J$$

hold for each $\zeta_j \in \partial g_j(y)$, $j \in J$. Since $\alpha_{g_j}(x,y) > 0$, $j \in J$, the inequalities above yield

$$G'_{g_j}\left(g_j(y)\right)\left\langle \zeta_j, \eta(x,y)\right\rangle \leqq 0, \ \ j \in J. \tag{40}$$

Hence, (40) implies that the following inequality

$$\sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(y)\right)\left\langle \zeta_j, \eta(x,y)\right\rangle \leqq 0 \tag{41}$$

holds for each $\zeta_j \in \partial g_j(y)$, $j \in J$. By (39) and (41), it follows that the following inequality

$$\left\langle \sum_{i=1}^{k} \lambda_i G'_{f_i}\left(f_i(y)\right)\xi_i + \sum_{j=1}^{m} \mu_j G'_{g_j}\left(g_j(y)\right)\zeta_j, \eta(x,y)\right\rangle < 0$$

holds, which is a contradiction to the first constraint of ($G$-DVP). Hence, the proof of this theorem is completed. □

In order to prove a stronger result, some stronger hypotheses of $G$-$V$-invexity should be assumed.

**Theorem 4.2.** *(Weak duality). Let $x$ and $(y,\lambda,\mu)$ be any feasible solutions in problems (MOP) and (G-DVP), respectively. Assume that $f$ is a strictly $G_f$-invex function at $y$ on $Y$ with respect to $\eta$ and $g$ is a $G_g$-invex function at $y$ on $Y$ with respect to $\eta$. Then, $f(x) \nleq f(y)$.*

**Theorem 4.3 (Strong duality).** *(Strong duality). Let $\overline{x} \in D$ be a weakly efficient (efficient) solution for problem (MOP) and the G-V-constraint qualification (G-V-CQ) be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^k$ and $\overline{\mu} \in R^m$ such that $(\overline{x},\overline{\lambda},\overline{\mu})$ is feasible in (G-DVP). Further, if all hypotheses of weak duality Theorem 4.1 (Theorem 4.2) are fulfilled, then $\overline{x}$ is a weakly efficient (efficient) solution of a maximum type for problem (G-DVP).*

*Proof.* By assumption, all hypotheses of Theorem 3.6 are fulfilled at $\overline{x}$. Hence, by Theorem 3.6, there exist $\overline{\lambda} \in R^k$ and $\overline{\mu} \in R^m$ such that the following conditions are satisfied

$$\sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i(\overline{x})\right)\partial f_i(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{x})\right)\partial g_j(\overline{x}) = 0,$$

$$\overline{\mu}_j\left[G_{g_j}\left(g_j(x)\right) - G_{g_j}\left(g_j(\overline{x})\right)\right] \leqq 0, \quad j \in J, \forall x \in D,$$

$$\overline{\lambda} \geq 0, \ \overline{\mu} \geqq 0.$$

Thus, $(\overline{x},\overline{\lambda},\overline{\mu})$ is a feasible solution in the vector $G$-Mond-Weir dual problem ($G$-DVP). From the weak duality theorem (Theorem 4.1 or Theorem 4.3, respectively), $(f(\overline{x}) < f(y))$ $f(\overline{x}) \leq f(y)$ is not verified, where $y \in pr_X\Omega$. This means that $(\overline{x},\overline{\lambda},\overline{\mu})$ is a weakly efficient (efficient) solution of a maximum type in dual problem ($G$-DVP). □

The following result follows directly from weak duality.

**Theorem 4.4 (Converse duality).** *Let $(\overline{x}, \overline{\lambda}, \overline{\mu})$ be a feasible solution in G-dual problem in the sense of Mond-Weir (G-DVP) such that $\overline{x} \in D$. If $f$ is a strictly $G_f$-$V$-invex ($G_f$-$V$-invex) function at $\overline{x}$ on $Y$ with respect to $\eta$ and $g$ is a $G_g$-$V$-invex function at $\overline{x}$ on $Y$ with respect to $\eta$, then $\overline{x}$ is a (weakly efficient) efficient solution in problem (MOP).*

*Proof.* Proof follows directly from weak duality (Theorem 4.1 or Theorem 4.2, respectively). □

**Theorem 4.5.** *(Strict converse duality). Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be feasible solutions in (MOP) and (G-DVP), respectively, such that*

$$f(\overline{x}) = f(\overline{y}). \tag{42}$$

*Further, assume that $f$ is a strictly $G_f$-$V$-invex function at $\overline{x}$ on $Y$ with respect to $\eta$ and $g$ is a $G_g$-$V$-invex function at $\overline{x}$ on $Y$ with respect to $\eta$. Then $\overline{x} = \overline{y}$ and, therefore, $\overline{x}$ is a Pareto solution in (MOP) whereas $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is efficient of a maximum type in (G-DVP).*

*Proof.* By assumption, $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ are feasible solutions in (MOP) and ($G$-DVP), respectively. Let us suppose $\overline{x} \neq \overline{y}$ as, if not, the result would be proved. By assumption, $f$ is a strictly $G_f$-$V$-invex function at $\overline{x}$ on $Y$ with respect to $\eta$ and $g$ is a $G_g$-$V$-invex function at $\overline{x}$ on $Y$ with respect to $\eta$. Then, by Definition 2.9, the inequalities

$$G_{f_i}\left(f_i(\overline{x})\right) - G_{f_i}\left(f_i(\overline{y})\right) > \alpha_{f_i}\left(\overline{x}, \overline{y}\right) G'_{f_i}\left(f_i(\overline{y})\right) \langle \xi_i, \eta(\overline{x}, \overline{y}) \rangle, \ i \in I, \tag{43}$$

$$G_{g_j}\left(g_j(\overline{x})\right) - G_{g_j}\left(g_j(\overline{y})\right) \geqq \alpha_{g_j}\left(\overline{x}, \overline{y}\right) G'_{g_j}\left(g_j(\overline{y})\right) \left\langle \zeta_j, \eta(\overline{x}, \overline{y}) \right\rangle, \ j \in J \tag{44}$$

hold. Multiplying (43) and (44) by the corresponding Lagrange multiplier and using (42) together with the second constraint of ($G$-DVP), we get

$$\overline{\lambda}_i \alpha_{f_i}\left(\overline{x}, \overline{y}\right) G'_{f_i}\left(f_i(\overline{y})\right) \langle \xi_i, \eta(\overline{x}, \overline{y}) \rangle \leqq 0, \ i \in I, \tag{45}$$

$$\overline{\lambda}_i \alpha_{f_i}\left(\overline{x}, \overline{y}\right) G'_{f_i}\left(f_i(\overline{y})\right) \langle \xi_i, \eta(\overline{x}, \overline{y}) \rangle < 0, \text{ for at least one } i \in I, \tag{46}$$

$$\overline{\mu}_i \alpha_{g_j}\left(\overline{x}, \overline{y}\right) G'_{g_j}\left(g_j(\overline{y})\right) \left\langle \zeta_j, \eta(\overline{x}, \overline{y}) \right\rangle \leqq 0, \ j \in J. \tag{47}$$

Since $\alpha_{f_i}(\overline{x}, \overline{y}) > 0$, $i \in I$, and $\alpha_{g_j}(\overline{x}, \overline{y}) > 0$, $j \in J$, the above inequalities imply, respectively,

$$\overline{\lambda}_i G'_{f_i}\left(f_i(\overline{y})\right) \langle \xi_i, \eta(\overline{x}, \overline{y}) \rangle \leqq 0, \ i \in I, \tag{48}$$

$$\overline{\lambda}_i G'_{f_i}\left(f_i(\overline{y})\right) \langle \xi_i, \eta(\overline{x}, \overline{y}) \rangle < 0, \text{ for at least one } i \in I, \tag{49}$$

$$\overline{\mu}_i G'_{g_j}\left(g_j(\overline{y})\right) \left\langle \zeta_j, \eta(\overline{x}, \overline{y}) \right\rangle \leqq 0, \ j \in J. \tag{50}$$

Adding both sides of inequalities (48)-(50) respectively, we get that the following inequalities

$$\sum_{i=1}^{k} \lambda_i G'_{f_i}\left(f_i(\overline{y})\right) \langle \xi_i, \eta(\overline{x}, \overline{y}) \rangle < 0, \tag{51}$$

$$\sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{y})\right) \left\langle \zeta_j, \eta(\overline{x}, \overline{y}) \right\rangle \leqq 0. \tag{52}$$

hold for each $\xi_i \in \partial f_i(\overline{y})$, $i \in I$ and $\zeta_j \in \partial g_j(\overline{y})$, $j \in J$, respectively. Hence, by (51) and (52), it follows that that the following inequality

$$\left\langle \sum_{i=1}^{k} \overline{\lambda}_i G'_{f_i}\left(f_i\left(\overline{y}\right)\right) \xi_i + \sum_{j=1}^{m} \overline{\mu}_j G'_{g_j}\left(g_j\left(\overline{y}\right)\right) \zeta_j, \eta(\overline{x}, \overline{y}) \right\rangle < 0$$

holds for each $\xi_i \in \partial f_i(\overline{y})$, $i \in I$ and $\zeta_j \in \partial g_j(\overline{y})$, $j \in J$, contradicting the feasibility of $(\overline{y}, \overline{\lambda}, \overline{\mu})$ in ($G$-DVP). This means that $\overline{x} = \overline{y}$. Efficiency of $\overline{x}$ in (MOP) and efficiency of $(\overline{y}, \overline{\lambda}, \overline{\mu})$ in (DVP) follow directly from weak duality (Theorem 4.2). Thus, the proof of this theorem is completed. □

## 5. Conclusion

In the paper, new Fritz John type and new Karush-Kuhn-Tucker type necessary optimality conditions have been established for a class of nondifferentiable multiobjective programming problems involving functions with locally Lipschitz components. Further, a new concept of nondifferentiable generalized invexity notion was introduced. The so-called $G$-$V$-invexity is a generalization both the nondifferentiable $V$-invexity notion defined by Jeyakumar and Mond [16] and the concept of differentiable vectorial $G$-invexity defined by Antczak [6]. Using the introduced concept of nondifferentiable $G$-$V$-invexity, the sufficiency of the $G$-Karush-Kuhn-Tucker type necessary optimality conditions has been established for a class of nonsmooth vector optimization problems in which the functions involved are $G$-$V$-invex with respect to the same function $\eta$ and with respect to, not necessarily, the same function $G$. This result has been illustrated by an example of a nondifferentiable multiobjective programming problem with $G$-$V$-invex functions (with respect to the same $\eta$). It has been noted that, for such a class of nondifferentiable vector optimization problems, this result can be proved neither under nondifferentiable invexity hypotheses nor the concept of nondifferentiable $G$-invexity. Furthermore, for the considered nonsmooth multiobjective programming problem, its nondifferentiable vector $G$-dual problem in the sense of Mond-Weir has been defined and several duality results have been established between these nonsmooth vector optimization problems also under nondifferentiable $G$-$V$-invexity hypotheses. In this regard, we prove optimality conditions and duality results for a new class of nonconvex nonsmooth multiobjective programming problems for which some of the generalized convexity notions previously defined in optimization theory may avoid.

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