



## Inverse Spectral Problems for Energy-Dependent Sturm-Liouville Equations with $\delta$ -Interaction

Manaf Dzh. Manafov<sup>a</sup>

<sup>a</sup>*Adiyaman University, Faculty of Science and Art, Department of Mathematics, 02040 Adiyaman, Turkey*

**Abstract.** In this study, inverse spectral problems for a energy-dependent Sturm-Liouville equations with  $\delta$ -interaction on a finite interval are considered. Some useful integral representations for the solutions of the considered equation have been derived and using these, properties of the spectral characteristics of the boundary value problem are investigated. The uniqueness theorems for the inverse problems of reconstruction of the boundary value problem from the Weyl function, from the spectral data, and from two spectra are proved.

### 1. Introduction

We consider inverse problems for the boundary value problem (BVP)  $L = L(q(x), h, H, \alpha, a)$  generated by the differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (0, a) \cup (a, \pi), \quad (1)$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0 \quad (2)$$

and conditions at the point  $x = a$ ,

$$I(y) := \begin{cases} y(a+0) = y(a-0) = y(a), \\ y'(a+0) - y'(a-0) = 2\alpha\lambda y(a), \end{cases} \quad (3)$$

where  $q(x)$  is a nonnegative real valued function in  $L_2(0, \pi)$ ;  $h > 0$ ,  $H > 0$  and  $\alpha$  are real numbers; and  $\lambda$  is spectral parameter.

Notice that, we can understand problem (1)+(3) as one of the treatments of the equation

$$y'' + (\lambda^2 - 2\lambda p(x) - q(x))y = 0, \quad x \in (0, \pi), \quad (4)$$

when  $p(x) = \alpha\delta(x - a)$ , where  $\delta(x)$  is the Dirac function (see [2]).

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*Email address:* [manafov@adiyaman.edu.tr](mailto:manafov@adiyaman.edu.tr) (Manaf Dzh. Manafov)

Sturm-Liouville spectral problems with potentials depending on the spectral parameter arise in various models of quantum and classical mechanics. For instance to this form can be reduced the corresponding evolution equations (such as the Klein-Gordon equation [15],[22]) that are used to model interactions between colliding relativistic spinless particles. Then  $\lambda^2$  is related to the energy of the system this explaining the term "energy-dependent" in the (4).

Spectral problems of differential operators are studied in two main branches, namely, direct and inverse spectral problems. Direct problems of spectral analysis consist in investigating the spectral properties of an operator. On the other hand, inverse problems aim at recovering operators from their spectral characteristics. One takes for the main spectral data, for instance, one, two, or more spectra, the spectral function, the spectrum, and the normalized constants, the Weyl function. Direct and inverse problems for the classical Sturm-Liouville operators have been extensively studied ( see [9], [16], [18], [23] and the references therein).

The presence of discontinuities generates important qualitative modifications in the investigation of the BVPs. Direct and inverse problems for discontinuous Sturm-Liouville (special case  $p(x) \equiv 0$ ) BVPs in various formulations have been studied in [3], [8], [13], [24].

Non-linear dependence of equation (4) on the spectral parameter  $\lambda$  should be regarded as a spectral problem for a quadratic operator pencil. The problem with  $p(x) \in W_2^1(0, 1)$  and  $q(x) \in L_2(0, 1)$  and with Robin boundary conditions was discussed in [10]. Such problems for separated and nonseparated boundary conditions were considered ( see [1], [4], [11], [14], [21], [25], [26] and the extensive references lists therein). In this aspect, the spectral problem for integral representation on the solutions of the equation (4) with Dirichlet boundary conditions recently has been investigated in [19], the inverse scattering problem for equation (4), with eigenparameter-dependent boundary condition on the line solved in [20].

In this paper, we give techniques to obtain the integral representations for solutions and also study the properties of solutions. The orthogonality of the eigenfunctions, realness and simplicity of the eigenvalues are investigated. Uniqueness theorems for the solution of the inverse problem with Weyl function, spectral data and two spectra are proved.

## 2. Integral Representations for Solutions of the Sturm-Liouville Equation with the $\delta$ -Interaction

In this section, an integral representation of the solution  $y(x, \lambda)$  of equation (1), satisfying the initial conditions

$$y(0, \lambda) = 1, \quad y'(0, \lambda) = i\lambda \quad (5)$$

and conditions (3) are constructed and also the properties of solutions are studied.

Using the standard successive approximation methods ( see [9] ), the following theorem is proved.

**Theorem 2.1.** *If  $q(x) \in L_2[-b, b]$  ( $0 < b < \pi$ ), then the solution  $y(x, \lambda)$  has the form*

$$y(x, \lambda) = y_0(x, \lambda) + \int_{-x}^x A(x, t) e^{i\lambda t} dt, \quad (6)$$

where

$$y_0(x, \lambda) = \begin{cases} e^{i\lambda x}, & x < a \\ (1 - i\alpha)e^{i\lambda x} + i\alpha e^{i\lambda(2a-x)}, & x > a \end{cases}$$

and the function  $A(x, t)$  satisfies the inequality

$$\int_{-x}^x |A(x, t)| dt \leq e^{c\sigma(x)} - 1 \quad (7)$$

with  $\sigma(x) = \int_0^x (x-t)|q(t)| dt$ ,  $c = 1 + 2|\alpha|$ .

*Proof.* It is clear that if we consider the equation (4) separately on the intervals  $(0, a)$  and  $(a, \pi)$ , we can write the solutions

$$e_0(x, \lambda) = e^{i\lambda x} + \int_{-x}^x K_0(x, t)e^{i\lambda t} dt, \quad 0 \leq x < a, \tag{8}$$

$$e_a(x, \lambda) = e^{i\lambda(x-a)} + \int_{-x+2a}^x K_a(x, t)e^{i\lambda(t-a)} dt, \quad x > a, \tag{9}$$

respectively, where  $K_a(x, t)$  satisfies the integral equation (see [18, p.10])

$$K_a(x, t) = \frac{1}{2} \int_a^{\frac{x+t}{2}} q(s)ds + \int_a^{\frac{x+t}{2}} du \int_0^{\frac{x-t}{2}} q(u+v)K_a(u+v, u-v)dv. \tag{10}$$

Because  $e_a(x, -\lambda)$  is also the solution of (4) on the interval  $a < x \leq \pi$ , the solution  $y(x, \lambda)$  has the following form:

$$y(x, \lambda) = \begin{cases} e_0(x, \lambda), & 0 \leq x < a, \\ c_1 e_a(x, \lambda) + c_2 e_a(x, -\lambda), & a < x \leq \pi \end{cases} \tag{11}$$

where the constants  $c_1, c_2$  are defined from the conditions (3). Hence, we have

$$y(x, \lambda) = \begin{cases} e_0(x, \lambda), & 0 \leq x < a, \\ e_0(a, \lambda) \frac{(1-2i\alpha)e_a(x, \lambda) + (1+2i\alpha)e_a(x, -\lambda)}{2} + \frac{1}{2i\lambda} e'_0(a, \lambda) [e_a(x, \lambda) - e_a(x, -\lambda)], & a < x \leq \pi. \end{cases} \tag{12}$$

Using (8) and (9), after some simple computations, we find the following expression for  $y(x, \lambda)$  ( $a < x \leq \pi$ ):

$$y(x, \lambda) = e(x, \lambda) + \int_{-x+2a}^x K_a(x, t)e(t, \lambda)dt, \tag{13}$$

where

$$\begin{aligned} e(x, \lambda) &= e_0(a, \lambda) [\cos \lambda(x-a) + 2\alpha \sin \lambda(x-a)] + e'_0(a, \lambda) \frac{\sin \lambda(x-a)}{\lambda} \\ &= (1-i\alpha)e^{i\lambda x} + i\alpha e^{i\lambda(2a-x)} + \int_{-x}^x A_1(x, t)e^{i\lambda t} dt, \end{aligned} \tag{14}$$

$$A_1(x, t) = \varsigma + \frac{1}{2} K_0(a, t+2a-x) + \frac{1}{2} K_0(a, t+x) + \frac{1}{2} \int_{t+2a-x}^{t+x} H(s)ds, \quad |t| < x,$$

$$\varsigma = \begin{cases} \frac{1}{2} \int_0^a q(t)dt + \frac{1}{4} \int_{-a}^a q\left(\frac{a+t}{2}\right)dt, & 2a-x < t < x \\ 0, & -x < t < 2a-x, \end{cases}$$

$$H(t) = \frac{1}{2} \int_{\frac{a-t}{2}}^a K_0(s, t-a+s)q(s)ds + \frac{1}{2} \int_{\frac{a+t}{2}}^a K_0(s, t+a-s)q(s)ds. \tag{15}$$

Here, we suppose that  $K_0(a, t) \equiv 0$ ,  $H(t) \equiv 0$ , for  $|t| > a$  and  $A_1(x, t) = 0$  for  $|t| > x$ . Now using the expression (14) in (13), we have for  $a < x \leq \pi$  ( $|t| < x$ )

$$y(x, \lambda) = (1-i\alpha)e^{i\lambda x} + i\alpha e^{i\lambda(2a-x)} + \int_{-x}^x A_2(x, t)e^{i\lambda t} dt, \tag{16}$$

where

$$A_2(x, t) = A_1(x, t) + K_a(x, t) + K_a(x, 2a - t) + \int_{-x+2a}^x K_a(x, s)A_1(s, t)ds. \tag{17}$$

From (8) and (15), we can write the formula (6) for the solution  $y(x, \lambda)$ , where

$$A(x, t) = \begin{cases} K_0(x, t), & \text{if } 0 \leq x \leq a, |t| < x, \\ A_2(x, t), & \text{if } a < x \leq \pi, |t| < x. \end{cases} \tag{18}$$

It is easy to obtain from (10) that

$$\int_{2a-x}^x |K_a(x, t)| dt \leq e^{c_a \sigma_a(x)} - 1, \tag{19}$$

where  $c_a > 0$  is a constant and  $\sigma_a(x) = \int_a^x (x-s) |q(s)| ds$ . Using (19), from (15) and (17), we have the estimation

$$\int_{-x}^x |A_2(x, t)| dt \leq e^{c \sigma(x)} - 1 \tag{20}$$

for  $c > 0$ . Hence, from (18) and (20), we arrive at (7).  $\square$

**Theorem 2.2.** *The kernel  $A(x, t)$  of the representation (6) are continuous at  $t \neq 2a - x, x \neq a$  belonging to the space  $L_1(-x, x)$  for every  $x \in [0, \pi]$  and the following relations hold:*

$$\left\{ \begin{array}{l} (i) \frac{d}{dx} A(x, x) = \begin{cases} \frac{1}{2}q(x), & 0 \leq x \leq a \\ \left(\frac{1}{2} - \frac{i\alpha}{2}\right)q(x), & a < x \leq \pi, \end{cases} \\ (ii) \frac{d}{dx} \left\{ A(x, t) \Big|_{t=2a-x-0}^{t=2a-x+0} \right\} = \frac{i\alpha}{2}q(x) \\ (iii) A(x, -x) = 0 \end{array} \right. \tag{21}$$

*Proof.* It follows from (17) and (18) that

$$A(x, t) = A_0(x, t) + \frac{1}{2} \int_0^x q(s) \int_{t-(x-s)}^{t+(x-s)} A(s, \xi) d\xi ds + \frac{i\alpha}{2} \int_0^a q(s) \left[ \int_{t-x+s}^{t+2a-x-s} A(s, \xi) d\xi + \int_{t-x+s}^{t+x+s-2a} A(s, \xi) d\xi \right] ds, \tag{22}$$

where

$$A_0(x, t) = \begin{cases} \frac{1}{2} \int_0^{\frac{x+t}{2}} q(s) ds + \frac{i\alpha}{2} \int_{\frac{t+2a-x}{2}}^{\frac{x+t}{2}} q(s) ds, & \text{for } -x < t < 2a - x, \\ \left(\frac{1}{2} - \frac{i\alpha}{2}\right) \int_0^{\frac{x+t}{2}} q(s) ds + \frac{i\alpha}{2} \int_{\frac{t+2a-x}{2}}^{\frac{x+2a-t}{2}} q(s) ds, & \text{for } 2a - x < t < x. \end{cases} \tag{23}$$

It is known that (see [18]) for  $x < a$  it holds

$$A(x, t) = A_0(x, t) + \frac{1}{2} \int_0^x q(s) \int_{t-x+s}^{t+x-s} K(s, \xi) d\xi ds, \tag{24}$$

where

$$A_0(x, t) = \frac{1}{2} \int_0^{\frac{x+t}{2}} q(s) ds. \tag{25}$$

Using the mathematical induction method, show that for each fixed  $x \in (0, a) \cup (a, \pi)$  the system of equations (22), (24) has the solution  $A(x, \cdot) \in L_1(-x, x)$ . Relations (i)-(iii) follows immediately from (22)-(25)  $\square$

Let  $s(x, \lambda), c(x, \lambda)$  be solutions of equation (4) with initial conditions

$$s(0, \lambda) = c'(0, \lambda) = 0, \quad s'(0, \lambda) = c(0, \lambda) = 1$$

and  $\varphi(x, \lambda), \psi(x, \lambda)$  be solutions of (4) under initial conditions at  $\pi$  :

$$\varphi(\pi, \lambda) = \psi'(\pi, \lambda) = 1, \quad \varphi'(\pi, \lambda) = \psi(\pi, \lambda) = 0.$$

Because  $y(x, \lambda)$  and  $y(x, -\lambda)$  are two linearly independent solutions of (4), then

$$s(x, \lambda) = \frac{y(x, \lambda) - y(x, -\lambda)}{2i\lambda} \text{ and } c(x, \lambda) = \frac{y(x, \lambda) + y(x, -\lambda)}{2}.$$

Using integral representation (6), we easily have

$$\begin{pmatrix} s(x, \lambda) \\ c(x, \lambda) \end{pmatrix} = \begin{pmatrix} s_0(x, \lambda) \\ c_0(x, \lambda) \end{pmatrix} + \int_0^x \begin{pmatrix} G_-(x, t) \frac{\sin \lambda t}{\lambda} \\ G_+(x, t) \cos \lambda t \end{pmatrix} dt, \tag{26}$$

where

$$\begin{pmatrix} s_0(x, \lambda) \\ c_0(x, \lambda) \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{\sin \lambda x}{\lambda} \\ \cos \lambda x \end{pmatrix}, & x < a \\ (1 - i\alpha) \begin{pmatrix} \frac{\sin \lambda x}{\lambda} \\ \cos \lambda x \end{pmatrix} + i\alpha \begin{pmatrix} \frac{\sin \lambda(2a-x)}{\lambda} \\ \cos \lambda(2a-x) \end{pmatrix}, & x > a, \end{cases} \tag{27}$$

$$G_{\pm}(x, t) = A(x, t) \pm A(x, -t).$$

Using (26) and (27), to obtain

$$\begin{pmatrix} \psi(x, \lambda) \\ \varphi(x, \lambda) \end{pmatrix} = \begin{pmatrix} \psi_0(x, \lambda) \\ \varphi_0(x, \lambda) \end{pmatrix} + \int_0^{\pi-x} \begin{pmatrix} \Psi(x, t) \frac{\sin \lambda t}{\lambda} \\ \Phi(x, t) \cos \lambda t \end{pmatrix} dt, \tag{28}$$

where

$$\begin{pmatrix} \psi_0(x, \lambda) \\ \varphi_0(x, \lambda) \end{pmatrix} = \begin{cases} (1 - i\alpha) \begin{pmatrix} -\frac{\sin \lambda(\pi-x)}{\lambda} \\ \cos \lambda(\pi-x) \end{pmatrix} + i\alpha \begin{pmatrix} \frac{\sin \lambda(2a-x-\pi)}{\lambda} \\ \cos \lambda(2a-x-\pi) \end{pmatrix}, & x < a \\ \begin{pmatrix} -\frac{\sin \lambda(\pi-x)}{\lambda} \\ \cos \lambda(\pi-x) \end{pmatrix}, & x > a, \end{cases} \tag{29}$$

and  $\Psi(x, t), \Phi(x, t) \in L_1(0, \pi - x)$  for each  $x \in [0, \pi]$ .

### 3. Properties of the Spectral Characteristics of the Energy-Dependent Sturm-Liouville Equation with $\delta$ -Interaction

In the section, properties of eigenvalues eigenfunctions and norming constants of problem  $L$  are investigated.

Let  $y(x)$  and  $z(x)$  be continuously differentiable functions on  $(0, a)$  and  $(a, \pi)$ . Denote  $\langle y, z \rangle := yz' - y'z$ . If  $y(x)$  and  $z(x)$  satisfy the matching conditions (3), then

$$\langle y, z \rangle_{x=a-0} = \langle y, z \rangle_{x=a+0}, \quad (30)$$

i.e. the function  $\langle y, z \rangle$  is continuous on  $(0, \pi)$ .

Let  $w(x, \lambda), \chi(x, \lambda)$  be solutions of (1) under the conditions

$$\begin{aligned} w(0, \lambda) &= \chi(\pi, \lambda) = 1, \\ w'(0, \lambda) &= h, \chi'(\pi, \lambda) = -H, \end{aligned}$$

and under the matching conditions (3).

Denote  $\Delta(\lambda) := \langle w(x, \lambda), \chi(x, \lambda) \rangle$ . By virtue of (30) and the Ostrogradskii-Liouville theorem (see [6]),  $\Delta(\lambda)$  does not depend on  $x$ . The function  $\Delta(\lambda)$  is called the characteristic function of  $L$ . Clearly,

$$\Delta(\lambda) = -V(w) = U(\chi). \quad (31)$$

Obviously, the function  $\Delta(\lambda)$  is entire in  $\lambda$  and it has at most a countable set of zeros  $\{\lambda_n\}$ .

**Lemma 3.1.** *The eigenvalues  $\{\lambda_n^2\}_{n \geq 0}$  of the BVP  $L$  coincide with zeros of the characteristic function. The functions  $w(x, \lambda_n)$  and  $\chi(x, \lambda_n)$  are eigenfunctions, and*

$$\chi(x, \lambda_n) = \beta_n w(x, \lambda_n), \quad \beta_n \neq 0. \quad (32)$$

Denote

$$\gamma_n = \int_0^\pi w^2(x, \lambda_n) dx - \frac{\alpha}{\lambda_n} w^2(a, \lambda_n). \quad (33)$$

The set  $\{\lambda_n, \gamma_n\}_{n \geq 0}$  is called the *spectral data* of  $L$ .

**Lemma 3.2.** *The equality*

$$\dot{\Delta}(\lambda_n) = 2\lambda_n \beta_n \gamma_n$$

holds. Here  $\dot{\Delta}(\lambda_n) = \frac{d}{d\lambda} \Delta(\lambda)$ .

**Lemma 3.3.** *Eigenfunctions  $y_1(x, \lambda_n)$  and  $y_2(x, \lambda_m)$  corresponding to different eigenvalues  $\lambda_n$  and  $\lambda_m$  of the problem  $L$  are orthogonal in the sense of the equality*

$$(\lambda_n + \lambda_m) \int_0^\pi y_1(x, \lambda_n) \overline{y_2(x, \lambda_m)} dx - 2\alpha y_1(a, \lambda_n) \overline{y_2(a, \lambda_m)} = 0.$$

We omit the proofs of Lemmas 3.1-3.3 since they are similar to those for the classical Sturm-Liouville operators (see [17]).

**Lemma 3.4.** *The eigenvalues of BVP  $L$  are real, nonzero and simple.*

*Proof.* Suppose that  $\lambda$  is an eigenvalue of BVP  $L$  and that  $y(x, \lambda)$  is a corresponding eigenfunction such that  $\int_0^\pi |y(x, \lambda)|^2 dx = 1$ . Multiplying both sides of (1) by  $\overline{y(x, \lambda)}$  and integrate the result with respect to  $x$  from 0 to  $\pi$  :

$$-\int_0^\pi y''(x, \lambda)\overline{y(x, \lambda)} dx + \int_0^\pi q(x)|y(x, \lambda)|^2 dx = \lambda^2 \int_0^\pi |y(x, \lambda)|^2 dx. \tag{34}$$

Using the formula of integration by parts and the conditions (2) and (3), we obtain

$$\int_0^\pi y''(x, \lambda)\overline{y(x, \lambda)} dx = -h|y(0, \lambda)|^2 - H|y(\pi, \lambda)|^2 - 2\alpha\lambda|y(a, \lambda)|^2 - \int_0^\pi |y'(x, \lambda)|^2 dx.$$

It follows from here and (34) that

$$\lambda^2 + B(\lambda)\lambda + C(\lambda) = 0, \tag{35}$$

where

$$B(\lambda) = -2\alpha|y(a, \lambda)|^2,$$

$$C(\lambda) = -h|y(0, \lambda)|^2 - H|y(\pi, \lambda)|^2 - \int_0^\pi q(x)|y(x, \lambda)|^2 dx - \int_0^\pi |y'(x, \lambda)|^2 dx.$$

Thus, the eigenvalue  $\lambda$  of the BVP  $L$  is a root of the quadratic equation (35). Therefore,  $B^2(\lambda) - 4C(\lambda) > 0$ . Consequently, the equation (35) has only real roots.

Let us show that  $\lambda_0$  is a simple eigenvalue. Assume that this is not true. Suppose that  $y_1(x)$  and  $y_2(x)$  are linearly independent eigenfunctions corresponding to the eigenvalue  $\lambda_0$ . Then for a given value of  $\lambda_0$ , each solution  $y_0(x)$  of (4) will be given as linear combination of solutions  $y_1(x)$  and  $y_2(x)$ . Moreover it will satisfy boundary conditions (2) and conditions (3) at the point  $x = a$ . However, it is impossible.  $\square$

Now, consider the solution  $w(x, \lambda)$ . Because  $w(x, \lambda) = c(x, \lambda) + hs(x, \lambda)$  by the virtue of (26), (27) from Theorem 2.1, we immediately have

$$w(x, \lambda) = \cos \lambda x + \left(h + \frac{1}{2} \int_0^x q(t)dt\right) \frac{\sin \lambda x}{\lambda} + o\left(\frac{1}{\lambda} \exp(|\tau|x)\right), \quad x < a, \tag{36}$$

$$\begin{aligned} w(x, \lambda) = & \sqrt{\alpha^2 + 1} \sin(\lambda x + \sigma) - \alpha \sin \lambda(2a - x) \\ & - \sqrt{\alpha^2 + 1} \frac{\cos(\lambda x + \sigma)}{\lambda} \left(h + \frac{1}{2} \int_0^x q(t)dt\right) \\ & + \alpha \frac{\cos \lambda(2a - x)}{\lambda} \left(h + \int_0^a q(t)dt - \frac{1}{2} \int_0^x q(t)dt\right) + o\left(\frac{1}{\lambda} \exp(|\tau|x)\right), \quad x > a, \end{aligned} \tag{37}$$

$$w'(x, \lambda) = -\lambda \sin \lambda x + \left(h + \frac{1}{2} \int_0^x q(t)dt\right) \cos \lambda x + o(\exp(|\tau|x)), \quad x < a, \tag{38}$$

$$\begin{aligned} w'(x, \lambda) = & \lambda \left(\sqrt{\alpha^2 + 1} \cos(\lambda x + \sigma) + \alpha \cos \lambda(2a - x)\right) \\ & + \sqrt{\alpha^2 + 1} \sin(\lambda x + \sigma) \left(h + \frac{1}{2} \int_0^x q(t)dt\right) \\ & + \alpha \sin \lambda(2a - x) \left(h + \int_0^a q(t)dt - \frac{1}{2} \int_0^x q(t)dt\right) + o(\exp(|\tau|x)), \quad x > a, \end{aligned} \tag{39}$$

where  $\tau = \text{Im}\lambda$  and  $\tan \sigma = \alpha^{-1}$ .

It follows from (31), (37) and (39) that

$$\Delta(\lambda) = -\lambda \left( \sqrt{\alpha^2 + 1} \cos(\lambda\pi + \sigma) + \alpha \cos \lambda(2a - \pi) - w_1 \sin(\lambda\pi + \sigma) - w_2 \sin \lambda(2a - \pi) + o(\exp(|\tau|\pi)) \right) \tag{40}$$

where

$$w_1 = \sqrt{\alpha^2 + 1} \left( H + h + \frac{1}{2} \int_0^\pi q(t) dt \right), w_2 = \alpha \left( -H + h + \int_0^a q(t) dt - \frac{1}{2} \int_0^\pi q(t) dt \right). \tag{41}$$

Let  $\Delta_0(\lambda) = -\lambda \left( \sqrt{\alpha^2 + 1} \cos(\lambda\pi + \sigma) + \alpha \cos \lambda(2a - \pi) \right)$  and  $\{\lambda_n^0\}$  are zeros of  $\Delta_0(\lambda)$ . Using (40), by the well-known methods (see, for example, [5]) one can obtain the following properties of the characteristic function  $\Delta(\lambda)$  of the BVP L :

- 1) For  $|\lambda| \rightarrow \infty, \Delta(\lambda) = O(|\lambda| \exp(|\tau|\pi))$ .
- 2) Denote  $G_\delta := \{\lambda : |\lambda - \lambda_n^0| \geq \delta\}$ . Then exist  $C_\delta > 0$  such that

$$|\Delta(\lambda)| \geq C_\delta |\lambda| \exp(|\tau|\pi) \text{ for all } \lambda \in G_\delta (\delta > 0). \tag{42}$$

- 3) For sufficiently large values of  $n$ , one has

$$|\Delta(\lambda) - \Delta_0(\lambda)| < \frac{C_\delta}{2} |\lambda| \exp(|\tau|\pi), \lambda \in \Gamma_n = \left\{ \lambda : |\lambda| = |\lambda_n^0| + \frac{1}{2} \inf_{n \neq m} |\lambda_n^0 - \lambda_m^0| \right\}. \tag{43}$$

**Lemma 3.5.** *If one denotes by  $\lambda_1, \lambda_2, \dots$  the positive eigenvalues arranged in increasing order and by  $\lambda_{-1}, \lambda_{-2}, \dots$  the negative eigenvalues arranged in decreasing order, then eigenvalues of the BVP L have the asymptotic behavior*

$$\lambda_n = \lambda_n^0 + \frac{\Theta_n}{\lambda_n^0} + \frac{k_n}{\lambda_n^0}, \quad |n| \rightarrow \infty \tag{44}$$

where  $k_n \in l_2$  and  $\Theta_n$  is a bounded sequence.

*Proof.* According to (42) and (43), if  $n$  is a sufficiently large and  $\lambda \in \Gamma_n$ , we have  $|\Delta_0(\lambda)| > |\Delta(\lambda) - \Delta_0(\lambda)|$ . Applying Rouché’s theorem [7, page 125], we conclude that for sufficiently large  $n$  inside the contour  $\Gamma_n$  the functions  $\Delta_0(\lambda)$  and  $\Delta(\lambda)$  have the same number of zeros counting their multiplicities. That is, there are exactly  $(n + 1)$  zeros  $\lambda_0, \lambda_1, \dots, \lambda_n$  in  $\Gamma_n$ . Analogously, by using Rouché’s theorem one can prove that for sufficiently large vales of  $n$ , the function  $\Delta(\lambda)$  has a unique zero inside circle  $|\lambda_n - \lambda_n^0| < \delta$ . Since  $\delta > 0$  is arbitrary, it follows that  $\lambda_n = \lambda_n^0 + \varepsilon_n$ , where  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . Further according to  $\Delta(\lambda_n) = 0$ , we have

$$\Delta_0(\lambda_n^0 + \varepsilon_n) + \int_0^\pi V(K_+(x, t))_{x=\pi} \cos(\lambda_n^0 + \varepsilon_n) t dt + \int_0^\pi V(K_-(x, t))_{x=\pi} \frac{\sin(\lambda_n^0 + \varepsilon_n) t}{\lambda_n^0 + \varepsilon_n} dt = 0. \tag{45}$$

On the other hand, since

$$\Delta_0(\lambda_n^0 + \varepsilon_n) = \dot{\Delta}(\lambda_n^0) \varepsilon_n + o(\varepsilon_n), \quad n \rightarrow \infty. \tag{46}$$

Further, substituting (46) into (45) after certain transformations, we have

$$\varepsilon_n = \frac{1}{2\dot{\Delta}(\lambda_n^0)\lambda_n^0} \left[ w_1 \sin(\lambda_n^0\pi + \sigma) + w_2 \sin \lambda_n^0(2a - \pi) \right] + \frac{k_n}{\lambda_n^0},$$

where

$$\Theta_n = \frac{1}{2\dot{\Delta}(\lambda_n^0)} \left[ w_1 \sin(\lambda_n^0\pi + \sigma) + w_2 \sin \lambda_n^0(2a - \pi) \right]$$

is a bounded sequence. Here  $w_1$  and  $w_2$  are defined by (41). The proof is completed.  $\square$

**Lemma 3.6.** Normalizing numbers  $\gamma_n$  of the problem  $L$  are positive and the formula

$$\gamma_n = \frac{a}{2} + \left( \alpha^2 + \frac{1}{2} + \alpha \sqrt{\alpha^2 + 1} \cos(2\lambda_n^0 a + \sigma) \right) (\pi - a) + \frac{\Theta_{n1}}{\lambda_n^0} + \frac{k_{n1}}{\lambda_n^0}, \tag{47}$$

holds, where

$$\begin{aligned} \Theta_{n1} = & \frac{1}{4} (1 + \alpha^2) \sin 2\lambda_n^0 a - \frac{\alpha}{2} \cos 2\lambda_n^0 a + \frac{\alpha^2 + 1}{4} (\sin 2(\lambda_n^0 \pi + \sigma) - \sin 2(\lambda_n^0 a + \sigma)) \\ & - \frac{\alpha^2}{4} \sin 2\lambda_n^0 (2a - \pi) + \alpha \sqrt{\alpha^2 + 1} \left[ \left( 2h + \int_0^a q(t) dt \right) \sin(2\lambda_n^0 a + \sigma) \right. \\ & \left. - \frac{1}{2} \sin(2\lambda_n^0 (\pi - a) + \sigma) \right], \quad k_{n1} \in l_2. \end{aligned}$$

*Proof.* The formula (47) can be easily obtained from the (33), by using (36), (37) and (44).  $\square$

#### 4. Inverse Problems

In this section, we study three inverse problems of recovering  $L$  from its spectral characteristics, namely,

- (i) from the Weyl function,
- (ii) from the so-called spectral data,
- (iii) from two spectra.

For each class of inverse problems we prove the corresponding uniqueness theorems and show connection between the different spectral characteristics.

##### 4.1. The Inverse Problem from the Weyl Function

Let  $\Phi(x, \lambda)$  be the solution of (4) under the conditions  $U(\Phi) = 1$  and  $V(\Phi) = 0$ . We set  $M(\lambda) := \Phi(0, \lambda)$ . The functions  $\Phi(x, \lambda)$  and  $M(\lambda)$  are called the Weyl solution and the Weyl function for the BVP  $L$ , respectively. The notion of the Weyl function introduced here is a generalization of the Weyl function for the classical Sturm-Liouville operators (see [9],[17]). Clearly,

$$\Phi(x, \lambda) = \frac{\chi(x, \lambda)}{\Delta(\lambda)} = s(x, \lambda) + M(\lambda)w(x, \lambda), \tag{48}$$

$$M(\lambda) = \frac{\chi(0, \lambda)}{\Delta(\lambda)}, \tag{49}$$

where  $\chi(x, \lambda)$  is a solution of (4) satisfying the conditions  $U(\chi) = 0, \chi(\pi, \lambda) = 0$  and  $s(x, \lambda)$  is defined from the equality

$$\chi(x, \lambda) = \Delta(\lambda)s(x, \lambda) + \chi(0, \lambda)w(x, \lambda). \tag{50}$$

Note that, by the virtue of equalities  $\langle w(x, \lambda), s(x, \lambda) \rangle \equiv 1$  and (48), one has

$$\langle \Phi(x, \lambda), w(x, \lambda) \rangle \equiv 1, \quad \langle w(x, \lambda), \chi(x, \lambda) \rangle \equiv \Delta(\lambda) \text{ for } x \neq a. \tag{51}$$

##### Inverse Problem 1

Given the Weyl function  $M(\lambda)$ , construct  $q(x), h, H, \alpha$  and  $a$ .

Let us prove the uniqueness theorem for the solution of the Inverse Problem 1. For this purpose we agree that together with  $L$  we consider a BVP  $\tilde{L}$  of the same form but with different coefficients  $q(\tilde{x}), \tilde{h}, \tilde{H}, \tilde{\alpha}$  and  $\tilde{a}$ . Everywhere below if a certain symbol  $e$  denotes an object related to  $L$ , then the corresponding symbol  $\tilde{e}$  with tilde denotes the analogous object related to  $\tilde{L}$ .

**Theorem 4.1.** *If  $M(\lambda) = \tilde{M}(\lambda)$ , then  $L = \tilde{L}$ . Thus, the specification of the Weyl function  $M(\lambda)$  uniquely determines  $L$ .*

*Proof.* Let us define the matrix  $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$  by the formula

$$P(x, \lambda) \begin{bmatrix} \tilde{w}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{w}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} w(x, \lambda) & \Phi(x, \lambda) \\ w'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}. \tag{52}$$

Using (51) and (52) we calculate for  $j = 1, 2$  :

$$\begin{aligned} P_{j1}(x, \lambda) &= w^{(j-1)}(x, \lambda)\tilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda)\tilde{w}'(x, \lambda), \\ P_{j2}(x, \lambda) &= \Phi^{(j-1)}(x, \lambda)\tilde{w}'(x, \lambda) - w^{(j-1)}(x, \lambda)\tilde{\Phi}(x, \lambda). \end{aligned} \tag{53}$$

Then we have

$$\begin{aligned} w(x, \lambda) &= P_{11}(x, \lambda)\tilde{w}(x, \lambda) + P_{12}(x, \lambda)\tilde{w}'(x, \lambda), \\ \Phi(x, \lambda) &= P_{11}(x, \lambda)\tilde{\Phi}(x, \lambda) + P_{12}(x, \lambda)\tilde{\Phi}'(x, \lambda). \end{aligned} \tag{54}$$

According to (48) and

$$|\Phi^{(v)}(x, \lambda)| \leq C_\delta |\lambda|^{v-1} \exp(-|\tau|x), \lambda \in G_\delta, v = 0, 1, \tag{55}$$

for each fixed  $x$ , the functions  $P_{jk}(x, \lambda)$  are meromorphic in  $\lambda$  with poles at points  $\lambda_n$  and  $\tilde{\lambda}_n$ . Denote  $G_\delta^\circ = G_\delta \cap \tilde{G}_\delta$ . By virtue of (53), (55) and

$$w^{(v)}(x, \lambda) = O(|\lambda|^v \exp(|\tau|x)), \lambda \in G_\delta^\circ,$$

we get

$$|P_{11}(x, \lambda) - 1| \leq C_\delta |\lambda|^{-1}, |P_{12}(x, \lambda)| \leq C_\delta |\lambda|^{-1}, \lambda \in G_\delta^\circ. \tag{56}$$

It follows from (48) and (53) that if  $M(\lambda) \equiv \tilde{M}(\lambda)$ , then for each fixed  $x$  the functions  $P_{1k}(x, \lambda)$  are entire in  $\lambda$ . Together with (56) this yields  $P_{12}(x, \lambda) \equiv 0, P_{11}(x, \lambda) \equiv A(x)$ . Now using (54), we obtain

$$w(x, \lambda) \equiv A(x)\tilde{w}(x, \lambda), \Phi(x, \lambda) \equiv A(x)\tilde{\Phi}(x, \lambda). \tag{57}$$

Therefore, for  $|\lambda| \rightarrow \infty, \arg \lambda \in [\varepsilon, \pi - \varepsilon] (\varepsilon > 0)$ , we have

$$w(x, \lambda) = \frac{b}{2} \exp(i(-\lambda x + \sigma)) \left(1 + O\left(\frac{1}{\lambda}\right)\right),$$

where  $b = 1$  for  $x < a$  and  $b = i\alpha - 1$  for  $x > a$ . Similarly, one can calculate

$$\Phi(x, \lambda) = (ib\lambda)^{-1} \exp(i(\lambda x - \sigma)) \left(1 + O\left(\frac{1}{\lambda}\right)\right).$$

Together with (51) and (55) this gives  $\alpha = \tilde{\alpha}, A(x) \equiv 1$ , that is  $w(x, \lambda) \equiv \tilde{w}(x, \lambda), \Phi(x, \lambda) = \tilde{\Phi}(x, \lambda)$  for all  $x$  and  $\lambda$ . Consequently,  $L \equiv \tilde{L}$ .  $\square$

4.2. The Inverse Problem from the Spectral Data

Let  $\{\lambda_n\}_{n=0,\pm 1,\pm 2,\dots}$  and  $\{\gamma_n\}_{n=0,\pm 1,\pm 2,\dots}$  be the eigenvalues and norming constants of  $L$ , respectively. We consider the following inverse problem.

**Inverse Problem 2**

Given the spectral data  $\{\lambda_n, \gamma_n\}_{n=0,\pm 1,\pm 2,\dots}$ , construct  $q(x), h, H, \alpha$  and  $a$ . Let us prove a uniqueness theorem for the solution of Inverse Problem 2.

**Theorem 4.2.** *If  $\lambda_n = \tilde{\lambda}_n, \gamma_n = \tilde{\gamma}_n, n = 0, \pm 1, \pm 2, \dots$  then  $L = \tilde{L}$ . Thus, the specification of the spectral data  $\{\lambda_n, \gamma_n\}_{n=0,\pm 1,\pm 2,\dots}$  uniquely determines the operator.*

*Proof.* It follows from (49) that the Weyl function  $M(\lambda)$  is meromorphic with simple poles at points  $\lambda_n^2$ . Using (49), (32) and equality  $\dot{\Delta}(\lambda_n) = 2\lambda_n\beta_n\gamma_n$ , we have

$$\operatorname{Res}_{\lambda=\lambda_n} M(\lambda) = \frac{\chi(0, \lambda)}{\dot{\Delta}(\lambda_n)} = \frac{\beta_n}{\dot{\Delta}(\lambda_n)} = \frac{1}{2\lambda_n\gamma_n}. \tag{58}$$

Since the Weyl function  $M(\lambda)$  is regular for  $\lambda \in \Gamma_n$ , applying the Rouché theorem [7, page 112], we conclude that

$$M(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{M(\mu)}{\mu - \lambda} d\mu, \quad \lambda \in \operatorname{int} F_n,$$

where the contour  $\Gamma_n$  is assumed to have the counterclockwise circuit.

Taking (42) and (49) into account, we arrive at  $|M(\lambda)| \leq C_\delta |\lambda|^{-1}, \lambda \in G_\delta$ . Hence, by the residue theorem, we have

$$M(\lambda) = \sum_{n=-\infty}^{\infty} \frac{1}{2\lambda_n\gamma_n(\lambda - \lambda_n)}. \tag{59}$$

Under the hypothesis of the theorem we get, in view of (59), that  $M(\lambda) = \tilde{M}(\lambda)$ , and consequently by Theorem 4.1,  $L = \tilde{L}$ .  $\square$

**Remark 4.3.** *By the virtue of (59), the specification of the Weyl function  $M(\lambda)$  is equivalent to the specification of the spectral data  $\{\lambda_n, \gamma_n\}_{n \geq 0}$ , that is, the Inverse Problem 1 is equivalent to the Inverse Problem 2.*

4.3. The Inverse Problem from Two Spectra

Let  $\{\lambda_n\}_{n=0,\pm 1,\pm 2,\dots}$  and  $\{\mu_n\}_{n=0,\pm 1,\pm 2,\dots}$  be the eigenvalues of the problem  $L$ . We consider the following inverse problem.

**Inverse Problem 3**

Given two spectra  $\{\lambda_n, \mu_n\}_{n=0,\pm 1,\pm 2,\dots}$ , construct  $q(x), h, H, \alpha$  and  $a$ . Let us prove a uniqueness theorem for the solution of Inverse Problem 3.

**Theorem 4.4.** *If  $\lambda_n = \tilde{\lambda}_n, \mu_n = \tilde{\mu}_n, n = 0, \pm 1, \pm 2, \dots$  then  $L = \tilde{L}$ . Thus, the specification of two spectra  $\{\lambda_n, \mu_n\}_{n=0,\pm 1,\pm 2,\dots}$  uniquely determines the operator.*

*Proof.* It is obvious that characteristic functions  $\Delta(\lambda)$  and  $\chi(0, \lambda)$  are uniquely determined by the sequences  $\{\lambda_n\}^2$  and  $\{\mu_n\}^2$  ( $n = 0, \pm 1, \pm 2, \dots$ ), respectively. If  $\lambda_n = \tilde{\lambda}_n, \mu_n = \tilde{\mu}_n, n = 0, \pm 1, \pm 2, \dots$ , then  $\Delta(\lambda) \equiv \Delta(\tilde{\lambda}), \chi(0, \lambda) = \tilde{\chi}(0, \lambda)$ . Together with (49) this yields  $M(\lambda) = \tilde{M}(\lambda)$ . By Theorem 4.1 we get  $L = \tilde{L}$ .  $\square$

**Remark 4.5.** *It follows from Theorems 4.1 and 4.4 that the specification of Weyl function  $M(\lambda)$  is equivalent to the specification of two spectra  $\{\lambda_n, \mu_n\}_{n=0,\pm 1,\pm 2,\dots}$ , that is, the Inverse Problem 1 is equivalent to the Inverse Problem 3.*

## References

- [1] A. M. Akhtyamov, V. A. Sadovnichy, Ya. I. Sultanaev, Inverse problem for an operator pencil with nonseparated boundary conditions, *Eurasian Math. J.* 1 (2010) 5–16.
- [2] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, H. Holden with an appendix by P. Exner, *Solvable models in quantum mechanics*, (second edition), AMS Chelsea Publ., 2005.
- [3] R. Kh. Amirov, On Sturm-Liouville operators with discontinuity conditions inside an interval, *J. of Math. Anal. Appl.* 317 (2006) 163–176.
- [4] R. Kh. Amirov, A. A. Nabiyev, Inverse Problems for the Quadratic Pencil of the Sturm-Liouville Equations with Impulse, *Abstract Appl. Anal.* 2013 (2013) 1–10 (361989).
- [5] R. Bellman, K. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.
- [6] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, USA, 1955.
- [7] J. B. Conway, *Functions of One Complex Variable*, (2nd edition), Springer, New York, NY, USA, 1995.
- [8] W. Eberhard, G. Freiling, A. Schneider, On the distribution of the eigenvalues of a class of indefinite eigenvalue problems, *Different. and Integral Equations* 3 (1990) 1167–1179.
- [9] G. Freiling, V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publ., Inc., Huntington, NY, 2001.
- [10] M. G. Gasymov, G. Sh. Guseinov, Determination of a diffusion operator from spectral data, *Dokl. Akad. Nauk Azerb. SSR* 37 (1980) 19–23.
- [11] G. Sh. Guseinov, Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval, in *Spectral Theory of Operators and its Applications*, Elm, Baku, Azerbaijan, (1986) 51–101.
- [12] I. M. Guseinov, I. M. Nabiyev, The inverse spectral problem for pencils of differential operators, *Mat. Sb.* 198 (2007) 47–66; Eng. version: *Sbornik: Mathematics* 198 (2007) 1579–1598.
- [13] O. H. Hald, Discontinuous inverse eigenvalue problems, *Comm. on Pure and Appl. Math.* 37(1986) 53–72.
- [14] R. Hryniv, N. Pronska, Inverse spectral problems for energy-dependent Sturm-Liouville equations, *Inverse Problems*, 28 (2012) 1–21 (085008).
- [15] P. Jonas, On the spectral theory of operators associated with perturbed Klein-Gordon and wave type equations, *J. Oper. Theory* 29 (1993) 207–224.
- [16] B. M. Levitan, *Inverse Sturm-Liouville Problems*, VSP, Zeist, 1987.
- [17] B. M. Levitan, I. S. Sargsyan, *Introduction to Spectral Theory*, AMS Transl. of Math. Monogr., Providence, 1975.
- [18] V. A. Marchenko, *Sturm-Liouville Operators and their Applications. Operator Theory: Advanced and Application*, Birkhauser, Basel, 1986.
- [19] L. I. Mammadova, Representation on the solution of Sturm-Liouville equation with discontinuity conditions interior to interval, *Proceedings of IMM of NAS of Azerb.* 33 (2010) 127–136.
- [20] M. Dzh. Manafov, A. Kaplan, Inverse scattering for energy-dependent Sturm-Liouville equations with point  $\delta$ -interaction and eigenparameter-dependent boundary condition, *Electronic J. of Differen. Equations* 237 (2013) 1–9.
- [21] I. M. Nabiev, The inverse spectral problem for the diffusion operator on an interval, *Matematicheskaya Fizika, Analiz, Geometriya*, (Mat. Fiz. Anal. Geom.) 11 (2004) 302–313.
- [22] B. Najman, Eigenvalues of the Klein-Gordon equation, *Proc. Edinburgh. Math. Soc.* 26 (1983) 181–190.
- [23] J. Pöschel, E. Trubowitz, *Inverse Spectral Theory*, Academic Press, New York, 1987.
- [24] D. G. Shepelsky, The inverse problem of reconstruction of the medium's conductivity in a class of discontinuous and increasing functions, *Advances in Soviet Mathematics* 19 (1994) 209–231.
- [25] C. F. Yang, Y. X. Guo, Determination of a differential pencil from interior spectral data, *J. Math. Anal. Appl.* 375 (2011) 284–293.
- [26] V. A. Yurko, An inverse problem for differential operator pencils, *Mat. Sb.* 191 (2000) 137–160; Eng. Version: *Sbornik: Mathematics* 191 (2000) 1561–1586.