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Generalized Inverses of a Linear Combination of Moore-Penrose Hermitian Matrices

Milan Mišić[a], Marina Tošić[a], Zoran J. Popović[a]

[a] *High Technical School of Professional Studies, Zvečan, Serbia.*

Abstract. In this paper we give a representation of the Moore-Penrose inverse and the group inverse of a linear combination of Moore-Penrose Hermitian matrices, i.e., square matrices satisfying $A^\dagger = A$. Also, we consider the invertibility of some linear combination of commuting Moore-Penrose Hermitian matrices.

1. Introduction

Let $\mathbb{C}^{n\times m}$ denote the set of all $n \times m$ complex matrices. The symbols A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $r(A)$ will denote the conjugate transpose, the range (column space), the null space and the rank of a matrix A, respectively. By $\mathbb{C}_r^{n\times n}$ we will denote the set of all matrices from $\mathbb{C}^{n\times n}$ with a rank r. The symbol $\oplus$ denotes a direct sum. We say that k and l are congruent modulo m, and we use the notation $k \equiv_m l$, if $m|(k-l)$. The Moore-Penrose inverse of A, is the unique matrix $A^\dagger$ satisfying the equations

$$(1)\ AA^\dagger A = A, \quad (2)\ A^\dagger AA^\dagger = A^\dagger, \quad (3)\ AA^\dagger = (AA^\dagger)^*, \quad (4)\ A^\dagger A = (A^\dagger A)^*.$$

For a square matrix A there exists a unique reflexive generalized inverse of A which commutes with A if and only if A is of index 1, that is, $r(A) = r(A^2)$ ([2], Theorem 1). This generalized inverse is called the group inverse of A and is denoted by $A^\sharp$.

By I_n we will denote the identity matrix of order n. We use the notations C_n^P, C_n^{OP} and C_n^{EP} for the subsets of $\mathbb{C}^{n\times n}$ consisting of projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices) and EP (range-Hermitian) matrices, respectively, i.e.,

$$\begin{aligned} C_n^P &= \{A \in \mathbb{C}^{n\times n} : A^2 = A\},\\ C_n^{OP} &= \{A \in \mathbb{C}^{n\times n} : A^2 = A = A^*\},\\ C_n^{EP} &= \{A \in \mathbb{C}^{n\times n} : \mathcal{R}(A) = \mathcal{R}(A^*)\} = \{A \in \mathbb{C}^{n\times n} : AA^\dagger = A^\dagger A\}.\end{aligned}$$

P_S denotes the orthogonal projector onto subspace S. Also, recall that a matrix $A \in \mathbb{C}^{n\times n}$ is generalized projector if $A^2 = A^*$ and hypergeneralized k-projector for $A^k = A^\dagger$, where $k \in \mathbb{N}$ and $k > 1$. Specially, if $k = 2$, we get the class of hypergeneralized projectors ($A^2 = A^\dagger$).

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Email addresses: m.misic@vts-zvecan.edu.rs (Milan Mišić), marinatosic@ymail.com (Marina Tošić), z.popovic@vts-zvecan.edu.rs (Zoran J. Popović)

A characterization of nonnegative matrices such that $A = A^{\dagger}$ is derived by Berman [3]. In [4], the author introduced the following concept: Consider a $\mathbb{C}^{*}$-algebra A. A regular element $a \in A$ will be called Moore-Penrose Hermitian, if $a^{\dagger} = a$. In this paper our interest is the subset of the class of square matrices A with the property $A^{\dagger} = A$, called as Moore-Penrose Hermitian matrices. Its basic properties are that $A^3 = A$ and A^2 is a orthogonal projector onto $\mathcal{R}(A)$.

The inspiration for this paper were [9], [10] in which authors considered the nonsingularity, i.e., the Moore-Penrose inverse of a linear combination of commuting generalized and hypergeneralized projectors, respectively, and [11] which includes the results related to the Moore-Penrose inverse of commuting hypergeneralized k-projectors.

The first and main objective of the present work is to give a form of the Moore-Penrose inverse, i.e., the group inverse of a linear combination $c_1A^m + c_2B^k$ under various conditions, where $A, B \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices, $m, k \in \mathbb{N}$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^2 - c_2^2 \neq 0$. Also, we study the nonsingularity of $c_1A^m + c_2B^k + c_3C^l$ and, in particular, $c_1I_n + c_2A^m + B^k$, where $m, k, l \in \mathbb{N}$ and A, B and C are commuting Moore-Penrose Hermitian matrices and we give necessary and sufficient conditions for the simultaneous invertibility of $A - B$ and $A + B$, in the case when A and B are commuting Moore-Penrose Hermitian matrices.

2. Results

Using the fact that the Moore-Penrose Hermitian matrix $A \in \mathbb{C}_r^{n\times n}$ is EP-matrix, by Theorem 4.3.1 [5] we can conclude that A can be represented by

$$A = U\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}U^{*}, \tag{1}$$

where $U \in C^{n\times n}$ is unitary and $K \in \mathbb{C}^{r\times r}$ is such that $K^2 = I_r$.

The following fact will be used very often:
If $X, Y \in \mathbb{C}^{n\times n}$ and $c_1, c_2 \in \mathbb{C}$, then

$$X^2 = Y^2 = I_n, XY = YX \Rightarrow (c_1X + c_2Y)(c_1X - c_2Y) = (c_1^2 - c_2^2)I_n \tag{2}$$

We first present the form of the Moore-Penrose inverse, i.e., the group inverse of $c_1A^m + c_2B^k$, where $m, k \in \mathbb{N}$ and A, B are commuting Moore-Penrose Hermitian matrices.

Theorem 2.1. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be commuting Moore-Penrose Hermitian matrices, $m, k \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1^2 - c_2^2 \neq 0$. Then*

$$(c_1A^m + c_2B^k)^{\dagger} = (c_1A^m + c_2AA^{\dagger}B^k)^{\dagger} + c_2^{-1}(I_n - AA^{\dagger})B^k. \tag{3}$$

Furthermore, $c_1A^m + c_2B^k$ is nonsingular if and only if $(I_n - AA^{\dagger})B + AA^{\dagger}$ is nonsingular and in this case $(c_1A^m + c_2B^k)^{-1}$ is given by (3).

Proof. Let a Moore-Penrose Hermitian matrix $A \in \mathbb{C}^{n\times n}$ be of the form (1) and $r(A) = r$. We get that the condition $AB = BA$ is equivalent to the fact that B has the form

$$B = U\begin{bmatrix} D & 0 \\ 0 & G \end{bmatrix}U^{*}, \tag{4}$$

where $D \in C^{r\times r}$and $G \in C^{(n-r)\times(n-r)}$ are Moore-Penrose Hermitian matrices and $KD = DK$. Now,

$$c_1A^m + c_2B^k = U\begin{bmatrix} c_1K^m + c_2D^k & 0 \\ 0 & c_2G^k \end{bmatrix}U^{*},$$

where $U \in C^{n\times n}$ is unitary, $K, D \in \mathbb{C}^{r\times r}$ are such that

$$K^m = \begin{cases} I_r, & m \equiv_2 0 \\ K, & m \equiv_2 1. \end{cases} \tag{5}$$

$D^\dagger = D$, $KD = DK$ and $G \in C^{(n-r)\times(n-r)}$ is a Moore-Penrose Hermitian matrix such that

$$G^k = \begin{cases} P_{\mathcal{R}(G)}, & k \equiv_2 0 \\ G, & k \equiv_2 1. \end{cases} \tag{6}$$

Since $(D^k)^2$ is an orthogonal projector, $K^{2m} = I_r$ and $(c_1K^m)^2 - (c_2D^k)^2 = c_1^2 I_r - c_2^2 P_{\mathcal{R}(D)}$, we get that $(c_1K^m)^2 - (c_2D^k)^2$ is nonsingular for all constants $c_1, c_2 \in \mathbb{C}$ such that $c_1 \neq 0$ and $c_1^2 - c_2^2 \neq 0$. From the invertibility of $(c_1K^m)^2 - (c_2D^k)^2$, it follows that $c_1K^m + c_2D^k$ is nonsingular.

Let

$$W = U \begin{bmatrix} (c_1K^m + c_2D^k)^{-1} & 0 \\ 0 & c_2^{-1}(G^k)^\dagger \end{bmatrix} U^*,$$

i.e., the right hand side of (3), where

$$(G^k)^\dagger = \begin{cases} P_{\mathcal{R}(G)}, & k \equiv_2 0 \\ G, & k \equiv_2 1. \end{cases} \tag{7}$$

Obviously, W is the Moore-Penrose inverse of $c_1A^m + c_2B^k$.

Also, $c_1A^m + c_2B^k$ is nonsingular if and only if G is nonsingular, i.e., $c_1A^m + c_2B^k$ is nonsingular if and only if $(I_n - AA^\dagger)B + AA^\dagger$ is nonsingular and in this case $(c_1A^m + c_2B^k)^{-1}$ is given by (3). □

With the additional requirements of Theorem 2.1 it is possible to give a more precise form of Moore-Penrose inverse, i.e., the group inverse.

Corollary 2.2. *Let $m, k \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If $A, B \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices such that $AB = 0$, then*

$$(c_1A^m + c_2B^k)^\dagger = c_1^{-1}A^m + c_2^{-1}B^k. \tag{8}$$

In the next theorem, we present the form of Moore-Penrose inverse, i.e., the group inverse of $c_1A^m + c_2B^k$, where A and B are commuting Moore-Penrose Hermitian matrices such that $AB = A^2 = BA$.

Theorem 2.3. *Let $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, $c_1^2 - c_2^2 \neq 0$ and $m, k \in \mathbb{N}$. If $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices such that $AB = A^2 = BA$, then*

$$(c_1A^m + c_2B^k)^\dagger = \frac{1}{c_1^2 - c_2^2}(c_1A^m - c_2A^k) + c_2^{-1}(I - AA^\dagger)B^k. \tag{9}$$

Proof. Suppose that A has the form (1) and B has the form given by (4). From $AB = A^2 = BA$ we get that

$$B = U \begin{bmatrix} K & 0 \\ 0 & G \end{bmatrix} U^*,$$

where $G \in \mathbb{C}^{(n-r)\times(n-r)}$ is a Moore-Penrose Hermitian matrix. Now $c_1A^m + c_2B^k$ has the form

$$c_1A^m + c_2B^k = U \begin{bmatrix} c_1K^m + c_2K^k & 0 \\ 0 & c_2G^k \end{bmatrix} U^*,$$

where

$$c_1K^m + c_2K^k = \begin{cases} (c_1+c_2)I_r, & m \equiv_2 0,\ k \equiv_2 0 \\ c_1I_r + c_2K, & m \equiv_2 0,\ k \equiv_2 1 \\ c_1K + c_2I_r, & m \equiv_2 1,\ k \equiv_2 0 \\ (c_1+c_2)K, & m \equiv_2 1,\ k \equiv_2 1 \end{cases} \tag{10}$$

and G^k is given by (6). By (2) it follows that $c_1K^m + c_2K^k$ is nonsingular for every $m, k \in \mathbb{N}$ and

$$(c_1K^m + c_2K^k)^{-1} = \begin{cases} (c_1+c_2)^{-1}I_r, & m \equiv_2 0,\ k \equiv_2 0 \\ \frac{1}{c_1^2-c_2^2}(c_1I_r - c_2K), & m \equiv_2 0,\ k \equiv_2 1 \\ \frac{1}{c_1^2-c_2^2}(c_1K - c_2I_r), & m \equiv_2 1,\ k \equiv_2 0 \\ (c_1+c_2)^{-1}K, & m \equiv_2 1,\ k \equiv_2 1 \end{cases}$$

Obviously $(c_1A^m + c_2B^k)^\dagger = U\begin{bmatrix} (c_1K^m + c_2K^k)^{-1} & 0 \\ 0 & c_2^{-1}(G^k)^\dagger \end{bmatrix}U^*$, i.e., $(c_1A^m + c_2B^k)^\dagger$ is defined by (9). □

Corollary 2.4. *Let $A \in \mathbb{C}_r^{n\times n}$ be a Moore-Penrose Hermitian matrix, $c_1, c_2 \in \mathbb{C}$, $c_1^2 - c_2^2 \neq 0$ and $m, k \in \mathbb{N}$. Then*

$$(c_1A^m + c_2A^k)^\dagger = \frac{1}{c_1^2 - c_2^2}(c_1A^m - c_2A^k).$$

In the following we study the invertibility of linear combinations of Moore-Penrose Hermitian matrices.

First, we state an auxiliary result.

Lemma 2.5. [7] *Let $A, B \in \mathbb{C}^{n\times n}$. Then*

$$\mathcal{R}(A^*) + \mathcal{R}(B^*) = \mathbb{C}^{n\times 1} \Leftrightarrow \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\},$$

$$\mathcal{R}(A^*) \cap \mathcal{R}(B^*) = \{0\} \Leftrightarrow \mathcal{N}(A) + \mathcal{N}(B) = \mathbb{C}^{n\times 1}.$$

The following theorem presents some necessary and sufficient conditions for the simultaneous invertibility of $A - B$ and $A + B$, in the case when A and B are commuting Moore-Penrose Hermitian matrices.

Theorem 2.6. *Let A, $B \in \mathbb{C}^{n\times n}$ be Moore-Penrose Hermitian matrices and $AB = BA$. The following conditions are equivalent:*

(i) $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathbb{C}^{n\times 1}$,

(ii) $\mathcal{N}(A) \oplus \mathcal{N}(B) = \mathbb{C}^{n\times 1}$,

(iii) $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ *and* $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$,

(iv) $A - B$, $A + B$ *are nonsingular.*

Proof. The part $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows by Lemma 2.1 and the fact that $\mathcal{R}(A^*) = \mathcal{R}(A)$ and $\mathcal{R}(B^*) = \mathcal{R}(B)$.

$(iii) \Rightarrow (iv)$ We prove that $A - B$ is bijective.

Let $(A - B)x = 0$. Then $Ax = Bx \in \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$, so $x \in \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$. Thus $A - B$ is injective, so it is bijective.

The proof for the invertibility of $A + B$ is similar, so we omit it.

$(iv) \Rightarrow (i)$ Since $AB = BA$, we have

$$A^2 - B^2 = (A - B)(A + B).$$

From (iv) it follows that $A^2 - B^2$ is nonsingular. Then from Theorem 1.2 [7] we get that $\mathcal{R}(A^2) \oplus \mathcal{R}(B^2) = \mathbb{C}^{n\times 1}$ which is equivalent to (i). □

In subsequent consideration, the first part of Theorem 2.1 in [8] plays a crucial role.

Theorem 2.7. [8] *Let $A, B \in C_n^{EP}$ and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If $AB = 0$, then the following conditions are equivalent:*

(i) $\mathcal{R}(A) \oplus \mathcal{R}(B) = \mathbb{C}^{n\times 1}$,

(ii) $\mathcal{N}(A) \oplus \mathcal{N}(B) = \mathbb{C}^{n\times 1}$,

(iii) $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ *and* $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$,

(iv) $c_1A + c_2B$ *is nonsingular.*

It is obvious that any Moore-Penrose Hermitian matrix is a EP-matrix and if A is a Moore-Penrose Hermitian matrix, then A^k, $k \in N$ is also a Moore-Penrose Hermitian matrix. Thus, applies the following corollary:

Corollary 2.8. *Let $A, B \in \mathbb{C}^{n\times n}$ be commuting Moore-Penrose Hermitian matrices and let $k, l \in N$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If $AB = 0$, then the following conditions are equivalent:*

(i) $c_1A^k + c_2B^l$ *is nonsingular,*

(ii) $A + B$ *is nonsingular.*

Also, we need the following lemma:

Lemma 2.9. *Let $P_1 \in \mathbb{C}_r^{n\times n}$ and $P_2 \in \mathbb{C}^{n\times n}$ be orthogonal projectors, $c_1, c_2, c_3 \in \mathbb{C}$, $c_1 \neq 0$, $c_1 - c_2 \neq 0$ and $c_1 - c_3 \neq 0$. If $P_1P_2 = 0 = P_2P_1$, then $c_1I_n - c_2P_1 - c_3P_2$ is nonsingular.*

Proof. Since $P_1 \in C_n^{OP}$ and $r(P_1) = r$, then we get that P_1 has the form

$$P_1 = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

where $U \in C^{n\times n}$ is unitary (by Lemma 1 [1]). The condition $P_1P_2 = 0 = P_2P_1$ is equivalent to the fact that P_2 has the form

$$P_2 = U\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}U^*,$$

where $G \in C^{(n-r)\times(n-r)}$ is an orthogonal projector. Now,

$$c_1I_n - c_2P_1 - c_3P_2 = U\begin{bmatrix} (c_1 - c_2)I_r & 0 \\ 0 & c_1I_{n-r} - c_3G \end{bmatrix}U^*.$$

Since $c_1I_{n-r} - c_3G$ is the sum of the identity matrix and an orthogonal projector, then $c_1I_{n-r} - c_3G$ is nonsingular for every constants $c_1, c_3 \in \mathbb{C}$ such that $c_1 \neq 0$ and $c_1 - c_3 \neq 0$. Hence, $c_1I_n - c_2P_1 - c_3P_2$ is nonsingular for every constants $c_1, c_2, c_3 \in \mathbb{C}$ such that $c_1 \neq 0$, $c_1 - c_2 \neq 0$ and $c_1 - c_3 \neq 0$. □

The following theorem presents necessary and sufficient conditions for the invertibility of $c_1A^m + c_2B^k + c_3C^l$.

Theorem 2.10. *Let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$, $c_1^2 - c_3^2 \neq 0$ and $m, k, l \in \mathbb{N}$. If $A, B, C \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices such that $BC = 0$, then $c_1A^m + c_2B^k + c_3C^l$ is nonsingular if and only if $(I_n - AA^\dagger)(B + C) + AA^\dagger$ is nonsingular.*

Proof. Let $A, B, C \in \mathbb{C}^{n\times n}$ be commuting Moore-Penrose Hermitian matrices. Let $A \in \mathbb{C}^{n\times n}$ be of the form (1) and $r(A) = r$. The condition $AB = BA$ implies that B has the form (4).

The condition $AC = CA$ implies that C has the form

$$C = U \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} U^*,$$

where $M \in C^{r\times r}$ and $N \in C^{(n-r)\times(n-r)}$ are Moore-Penrose Hermitian matrices and $KM = MK$. From $BC = 0 = CB$ it follows that $DM = 0 = MD$ and $GN = 0 = NG$. Now,

$$c_1A^m + c_2B^k + c_3C^l = U \begin{bmatrix} c_1K^m + c_2D^k + c_3M^l & 0 \\ 0 & c_2G^k + c_3N^l \end{bmatrix} U^*,$$

where K^m is given by (5), D^k, M^l G^k and N^l are given by (6).

Notice that $(c_1K^m)^2 - (c_2D^k + c_3M^l)^2 = c_1^2K^2 - c_2^2D^2 - c_3^2M^2 = c_1^2I_r - c_2^2D^2 - c_3^2M^2$. Since D^2 and M^2 are orthogonal projectors, then $c_1^2I_r - c_2^2D^2 - c_3^2M^2$, i.e. $(c_1K^m)^2 - (c_2D^k + c_3M^l)^2$ is nonsingular for every constants $c_1, c_2, c_3 \in \mathbb{C}$ such that $c_1 \neq 0$, $c_1^2 - c_2^2 \neq 0$ and $c_1^2 - c_3^2 \neq 0$ (by Lemma 2.9). From the invertibility of $(c_1K^m)^2 - (c_2D^k + c_3M^l)^2$, it follows that $c_1K^m + c_2D^k + c_3M^l$ is nonsingular.

Also,

$$(I_n - AA^\dagger)(B + C) + AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & G + N \end{bmatrix} U^*.$$

Remark that the invertibility of $c_2G^k + c_3N^l$ is equivalent to the invertibility of $G + N$ for every constants $c_2, c_3 \in \mathbb{C} \setminus \{0\}$ (by Corollary 2.8). Hence, $c_1A^m + c_2B^k + c_3C^l$ is nonsingular if and only if $(I_n - AA^\dagger)(B + C) + AA^\dagger$ is nonsingular. □

As corollaries we get:

Corollary 2.11. *Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$, $c_1^2 - c_3^2 \neq 0$ and $m, k, l \in \mathbb{N}$. If $A, B, C \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices such that $BC = 0$, then the invertibility of $c_1A^m + c_2B^k + c_3C^l$ is independent of the choice of the constants c_1, c_2, c_3, m, k, l.*

Corollary 2.12. *Let $A, B, C \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices such that $BC = 0$, $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$, $c_1^2 - c_3^2 \neq 0$ and $m, k, l \in \mathbb{N}$. If A is nonsingular, then $c_1A^m + c_2B^k + c_3C^l$ is nonsingular.*

Corollary 2.13. *Let $A, B \in \mathbb{C}^{n\times n}$ be commuting Moore-Penrose Hermitian matrices and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$ and $m, k \in N$. Then $c_1A^m + c_2B^k$ is nonsingular if and only if $(I_n - AA^\dagger)B + AA^\dagger$ is nonsingular.*

Notice that Corollary 2.13 is the part of Theorem 2.1.

Corollary 2.14. *Let A, $B \in \mathbb{C}^{n\times n}$ be commuting $c_1^2 - c_2^2 \neq 0$, $c_1^2 - c_3^2 \neq 0$ and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$ and $m, k \in N$. If A is nonsingular, then $c_1A^m + c_2B^k$ is nonsingular.*

By Theorem 2.10 we conclude that $c_1I_n + c_2A^m + c_3B^k$ is nonsingular, in the case when A, B are commuting Moore-Penrose Hermitian matrices such that $AB = 0$ and $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$ such that $c_1^2 - c_2^2 \neq 0$, $c_1^2 - c_3^2 \neq 0$. In the following theorem, we give the form $(c_1I_n + c_2A^m + c_3B^k)^{-1}$.

Theorem 2.15. *Let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$, $c_1^2 - c_3^2 \neq 0$ and $m, k \in \mathbb{N}$. If A, $B \in \mathbb{C}^{n\times n}$ are commuting Moore-Penrose Hermitian matrices such that $AB = 0$, then $c_1I_n + c_2A^m + c_3B^k$ is nonsingular and*

$$(c_1I_n + c_2A^m + c_3B^k)^{-1} = \frac{1}{c_1^2 - c_2^2}\Big[c_1A^{2m} - c_2A^m\Big] + (I - AA^\dagger)\Big[c_1I_n + c_3B^k\Big]^{-1}. \tag{11}$$

Proof. Let $A, B \in \mathbb{C}^{n\times n}$ be commuting generalized projectors such that $AB = 0$. If A is given by (1) and $r(A) = r$, then B has the form

$$B = U\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}U^*, \tag{12}$$

where $G \in C^{(n-r)\times(n-r)}$ is a Moore-Penrose Hermitian matrix. Then

$$c_1 I_n + c_2 A^m + c_3 B^k = U\begin{bmatrix} c_1 I_r + c_2 K^m & 0 \\ 0 & c_1 I_{n-r} + c_3 G^k \end{bmatrix}U^*,$$

where K^m and G^k are given by (5) and (6), respectively. Obviously, $c_1 I_n + c_2 A^m + c_3 B^k$ is nonsingular if and only if $c_1 I_r + c_2 K^m$ and $c_1 I_{n-r} + c_3 G^k$ are nonsingular. By (2) it follows that $c_1 I_r + c_2 K^m$ is nonsingular for every $m \in N$ and

$$(c_1 I_r + c_2 K^m)^{-1} = \begin{cases} (c_1 + c_2)^{-1} I_r, & m \equiv_2 0 \\ \frac{1}{c_1^2 - c_2^2}(c_1 I_r - c_2 K), & m \equiv_2 1 \end{cases}. \tag{13}$$

By Theorem 2.1 we conclude that $c_1 I_{n-r} + c_3 G^k$ is nonsingular. Now,

$$(c_1 I_n + c_2 A^m)^{-1} = U\begin{bmatrix} (c_1 I_r + c_2 K^m)^{-1} & 0 \\ 0 & (c_1 I_{n-r} + c_3 G^k)^{-1} \end{bmatrix}U^*, \tag{14}$$

where $(c_1 I_r + c_2 K^m)^{-1}$ is given by (13). Obviously, the form (14) is equivalent to the form (11). □

As a corollary, we get the form $(c_1 I_n + c_2 A^m)^{-1}$ in the case when A is a Moore-Penrose Hermitian matrix and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^2 - c_2^2 \neq 0$.

Corollary 2.16. *Let $A \in \mathbb{C}_r^{n\times n}$ be a commuting Moore-Penrose Hermitian matrix, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^2 - c_2^2 \neq 0$ and $m \in \mathbb{N}$. Then $c_1 I_n + c_2 A^m$ is nonsingular and*

$$(c_1 I_n + c_2 A^m)^{-1} = \frac{1}{c_1^2 - c_2^2}\Big[c_1 A^{2m} - c_2 A^m\Big] + c_1^{-1}(I - AA^\dagger).$$

Remark: If we consider a finite commuting family $A_i \in \mathbb{C}^{n\times n}$, $i = \overline{1, m}$, where all of the members are commuting Moore-Penrose Hermitian matrices, then $\prod_{i=1}^m A_i^{k_i}$ is also a Moore-Penrose Hermitian matrix. Then $c_1 I_n + c_2 \prod_{i=1}^m A_i^{k_i}$ is nonsingular, where $m, k_1, \ldots, k_m \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1^2 - c_2^2 \neq 0$.

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