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On Asymptotic Stability of Solutions to Third Order Nonlinear Delay Differential Equation

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Abstract. In this article, we study the asymptotic stability of solutions for the non-autonomous third order delay differential equation by constructing Lyapunov functionals.

1. introduction

Asymptotic properties of solutions of delay differential equations of the third order have been subject of intensive studying in the literature. This problem has received considerable attention in recent years, see for instance: Andres [3, 4], Burton [6–8], Krasovskii [10] and Yoshizawa [23] which contain the general results on the subject matter. Other authors include Ademola et al. [1, 2], Oudjedi et al. [12], Tunç [16 – 22], Zhang and Yu [24] and Zhu [25] on functional or delay differential equations.

Sadek in [13, 14] and recently Omeike [11] established some sufficient conditions for the asymptotic stability of the solution $x = 0$ to the following third order non-linear delay differential equation:

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t - r)) = 0. \tag{1}$$

Tunç in [18] and recently Yuzhen and Cuixia [5] studied the stability of solutions for the non-autonomous third order differential equation with a deviating argument, r:

$$x'''(t) + a(t)x''(t) + b(t)g_1(x'(t - r)) + g_2(x'(t)) + h(x(t - r)) = 0. \tag{2}$$

In the present paper, we study the asymptotic behavior of solutions of a certain third order delay differential equation of the form

$$[P(x(t))x'(t)]'' + a(t)[Q(x(t))x'(t)]' + b(t)[R(x(t))x'(t)] + c(t)f(x(t - r)) = 0, \tag{3}$$

where $a(t), b(t), c(t), P(x), Q(x), R(x)$ and $f(x)$ are continuous and depend (at most) only on the arguments displayed explicitly, r is a positive constant, fixed delay ($f(0) = 0$).

Equation (1) is a particular case to our preceding non-autonomous differential equation with the deviating argument r if $P(x) = Q(x) = R(x) = 1$. On the other hand, we can find the same result for the equation

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(3) without delay by putting $r = 0$, which is generalization of Hara [9] and Swick [15] results.

If $P(x) = 1$, then (3) takes the form

$$x''' + a(t)Q(x)x'' + g(t, x, x') + c(t)f(x(t - r)) = 0,$$

where

$$g(t, x, x') = a(t)Q'(x)x'^2 + b(t)R(x)x',$$

which is similar to (2) in the case $g_1(x'(t - r)) = 0$.

The motivation for the present work comes from the paper of Tunç [18], Omeike [11] and Sadek [13, 14] and the papers mentioned above. Our purpose is to find two similar results for (3). Namely, we will show sufficient conditions to improve that all solutions of (3) are uniformly bounded and converge to zero as $t \to \infty$. We shall use Lyapunov's second (or direct) method as our tool to achieve the desired results. The results obtained in this investigation improve the existing results on the third-order non-linear differential equations in the literature.

2. Preliminaries

In order to prove our results, we give the following definitions and lemmas. Consider the equation

$$x' = f(t, x_t), \ \ x_t(\theta) = x(t + \theta), \quad -r \le \theta \le 0, \ \ t \ge 0, \tag{4}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, $C_H := \{\phi \in C([-r, 0], \ \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t, \phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 2.1. *([8]). An element $\psi \in C$ is in the $\omega -$ limit set of ϕ, say $\Omega(\phi)$, if $x(t, 0, \phi)$ is defined on $[0, +\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \le \theta \le 0$.*

Definition 2.2. *([8]). A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.*

Lemma 2.3. *([6]). If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (4) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \ \ as \ \ t \to \infty.$$

Lemma 2.4. *([6]). Let $V(t, \phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t, 0) = 0$, and such that:*
(i) *$W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|)$ where $W_1(r)$, $W_2(r)$ are wedges.*
(ii) *$V'_{(4)}(t, \phi) \le 0$, for $\phi \in C_H$.*
Then the zero solution of (4) is uniformly stable.
If $Z = \{\phi \in C_H : V'_{(4)}(t, \phi) = 0\}$, then the zero solution of (4) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3. Statement of Results

We shall state here some assumptions which will be used on the functions that appeared in equation (3). Assume that there are positive constants $a_0, b_0, c_0, d, A, B, C, p_0, p_1, q_0, q_1, r_0, r_1, \delta_0$, and δ_1 such that:

A_0) $P(x), Q(x), R(x)$ and $f(x)$ are continuously differentiable functions on $\mathbb{R}$,

$a(t), b(t)$ and $c(t)$ are continuously differentiable functions on $[0, +\infty[$.

A_1) $0 < a_0 \leq a(t) \leq A;\ 0 < b_0 \leq b(t) \leq B; 0 < c_0 \leq c(t) \leq C.$

A_2) $0 < p_0 \leq P(x) \leq p_1;\ 0 < q_0 \leq Q(x) \leq q_1;\ \ 0 < r_0 \leq R(x) \leq r_1.$

A_3) $f(0) = 0, \dfrac{f(x)}{x} \geq \delta_0 > 0\ (x \neq 0)$, and $|f'(x)| \leq \delta_1$ for all x.

A_4) $\displaystyle\int_{-\infty}^{+\infty} (|P'(u)| + |Q'(u)| + |R'(u)|)du < \infty.$

For the sake of convenience, we introduce the following functions:

$$\theta_1(t) = \frac{P'(x(t))}{P^2(x(t))}x'(t),$$

$$\theta_2(t) = \frac{Q'(x(t))P(x(t)) - Q(x(t))P'(x(t))}{P^2(x(t))}x'(t),$$

and

$$\theta_3(t) = \frac{R'(x(t))P(x(t)) - R(x(t))P'(x(t))}{P^2(x(t))}x'(t).$$

The following theorems present uniform asymptotic criterion for (3):

Theorem 3.1. *Further to the basic assumptions (A_0)-(A_4) being satisfied, suppose that the following conditions hold:*

i) $c(t) \leq b(t),\ -L \leq b'(t) \leq c'(t) \leq 0$ *for* $t \in [0, \infty)$.

ii) $\dfrac{p_1\delta_1}{r_0} < d < a_0q_0.$

iii) $\frac{1}{2}da'(t)Q(x) - b_0(dr_0 - p_1\delta_1) \leq -\varepsilon < 0.$

Then the zero solution of (3) is uniformly asymptotically stable provided that

$$r < \min\left\{\frac{2(a_0q_0 - d)}{p_1C\delta_1}, \frac{2p_0^3\varepsilon}{C\delta_1p_1^2(d + dp_0^2 + p_0)}\right\}.$$

Theorem 3.2. *In addition to the basic assumptions (A_0)-(A_4), suppose that the following conditions hold:*

j) $\dfrac{p_1C}{b_0r_0}\delta_1 < d < a_0q_0.$

jj) $da'(t)Q(x) + b'(t)P(x)R(x) - P^2(x)\dfrac{\delta_1}{d}c'(t) < db_0r_0 - p_1C\delta_1.$

jjj) $\displaystyle\int_0^\infty |c'(s)|\,ds \leq N_1 < \infty$ *and* $c'(t) \to 0$ *as* $t \to \infty$.

Then the zero solution of (3) is uniformly asymptotically stable, provided that

$$r < \min\left\{\frac{2(a_0q_0 - d)}{p_1C\delta_1}, \frac{p_0^3(db_0r_0 - p_1C\delta_1)}{p_1^2C\delta_1(d + dp_0^2 + p_0)}\right\}.$$

Proof. **[Proof of Theorem 3.1]** Equation (3) can be transformed to the following system:

$$\begin{aligned} x' &= \frac{1}{P(x)}y \\ y' &= z \\ z' &= -a(t)\theta_2(t)y - \frac{a(t)Q(x)}{P(x)}z - \frac{b(t)R(x)}{P(x)}y - c(t)f(x) + c(t)\int_{t-r}^{t}\frac{y(s)}{P(x(s))}f'(x(s))ds. \end{aligned} \tag{5}$$

Define the Lyapunov functional $U = U(t, x_t, y_t, z_t)$ as follows:

$$U(t, x_t, y_t, z_t) = (\exp(-\frac{\gamma(t)}{\mu}))V(t, x_t, y_t, z_t) = (\exp(-\frac{\gamma(t)}{\mu}))V, \tag{6}$$

where

$$\gamma(t) = \int_0^t (|\theta_1(s)| + |\theta_2(s)| + |\theta_3(s)|)ds, \tag{7}$$

and

$$V = dc(t)F(x) + c(t)f(x)y + \frac{b(t)R(x)}{2P(x)}y^2 + \frac{1}{2}z^2 + \frac{d}{P(x)}yz + \frac{da(t)Q(x)}{2P^2(x)}y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \tag{8}$$

such that $F(x) = \int_0^x f(u)du$. μ and λ are positive constants which will be determined later. Define

$$V_1 = dc(t)F(x) + c(t)f(x)y + \frac{b(t)R(x)}{2P(x)}y^2,$$

in view of the hypotheses of Theorem 3.1, and after some rearrangements we have

$$\begin{aligned} V_1 &= dc(t)F(x) + \frac{b(t)R(x)}{2P(x)}\left\{y + \frac{c(t)f(x)P(x)}{b(t)R(x)}\right\}^2 - \frac{c^2(t)P(x)f^2(x)}{2b(t)R(x)} \\ &\geq dc(t)\int_0^x \left[1 - \frac{c(t)P(x)f'(u)}{db(t)R(x)}\right]f(u)du \\ &\geq dc(t)\int_0^x \left[1 - \frac{p_1\delta_1}{dr_0}\right]f(u)du \\ &\geq \delta_2 F(x), \end{aligned}$$

where

$$\delta_2 = dc_0\left(1 - \frac{p_1\delta_1}{dr_0}\right) > dc_0\left(1 - \frac{d}{d}\right) = 0.$$

Thus from (A_3) we obtain

$$V_1 \geq \frac{\delta_2\delta_0}{2}x^2.$$

We note that

$$V_2 = \frac{1}{2}z^2 + \frac{d}{P(x)}yz + \frac{da(t)Q(x)}{2P^2(x)}y^2,$$

is obviously positive definite, this follows from the conditions $a(t) \geq a_0$, $Q(x) \geq q_0$ and (ii). Hence,

$$V_2 \geq \frac{1}{2}\left(z + \frac{d}{P(x)}y\right)^2 + \frac{d(a_0q_0 - d)}{2P^2(x)}y^2 > 0.$$

We can therefore, find a constant $\delta = \delta(\delta_0, \delta_2, a_0, q_0, p_1, d) > 0$, such that

$$V \geq \delta(x^2 + y^2 + z^2), \tag{9}$$

since the integral is non-negative.

After a change of variables in the integral of (7) and by (A_2) and (A_4), we get

$$\begin{aligned} \gamma(t) &\leq (1 + r_1 + q_1)\int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|P'(u)|}{P^2(u)}du + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|R'(u)| + |Q'(u)|}{P(u)}du \\ &\leq \frac{(1 + r_1 + q_1)}{p_0^2}\int_{-\infty}^{+\infty} |P'(u)|\, du + \frac{1}{p_0}\int_{-\infty}^{+\infty} (|R'(u)| + |Q'(u)|)du \\ &\leq N < \infty, \end{aligned}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$. Now, we can deduce that there exists a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\Phi(0)|) \leq U(t, \Phi).$$

By (A_1), (A_2) and (A_3), it is not difficult to show that

$$V \leq \delta_3(x^2 + y^2 + z^2) + \delta_4 \int_{-r}^{0}\int_{t+s}^{t} (x^2(\xi) + y^2(\xi) + z^2(\xi))d\xi ds, \tag{10}$$

where $\delta_3 = \frac{1}{2}\max\{C\delta_1(1 + d), C\delta_1 + \frac{Br_1}{p_0} + \frac{d}{p_0} + \frac{dAq_1}{p_0^2}, 1 + \frac{d}{p_0}\}$, and $\delta_4 = \max\{1, \lambda\}$.
Then there exist a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $U(t, \phi) \leq W_2(\|\phi\|)$.

Now, we show that the derivative of $V(t, x_t, y_t, z_t)$ with respect to t along the solution path of system (5) is negative definite

$$\begin{aligned} \frac{d}{dt}V &= dc'(t)F(x) + c'(t)yf(x) + \frac{b'(t)R(x)}{2P(x)}y^2 - \lambda\int_{t-r}^{t} y^2(\xi)d\xi \\ &- d\theta_1(t)(yz + \frac{a(t)Q(x)}{2P(x)}y^2) - a(t)\theta_2(t)(yz + \frac{d}{2P(x)}y^2) + \frac{b(t)}{2}\theta_3(t)y^2 \\ &+ c(t)(z + \frac{d}{P(x)}y)\int_{t-r}^{t} \frac{y(s)}{P(x(s))}f'(x(s))ds + \frac{1}{P(x)}(d - a(t)Q(x))z^2 \\ &+ \left[\frac{da'(t)Q(x) + 2c(t)P(x)f'(x) - 2db(t)R(x)}{2P^2(x)}\right]y^2 + \lambda ry^2. \end{aligned}$$

From hypotheses of Theorem 3.1, we obtain

$$\frac{d}{dt}V \leq W_1 + W_2 + W_3 - \lambda\int_{t-r}^{t} y^2(\xi)d\xi - \frac{1}{p_1}(a_0q_0 - d)z^2 - (\frac{\varepsilon}{p_1^2} - \lambda r)y^2, \tag{11}$$

where

$$W_1 = dc'(t)F(x) + c'(t)yf(x) + \frac{b'(t)R(x)}{2P(x)}y^2,$$

$$W_2 = d\,|\theta_1(t)|\,(\left|zy\right| + \frac{a(t)Q(x)}{2P(x)}y^2) + \frac{B}{2}|\theta_3(t)|y^2 + a(t)|\theta_2(t)|(|yz| + \frac{d}{2P(x)}y^2),$$

and

$$W_3 = c(t)(z + \frac{dy}{P(x)})\int_{t-r}^{t}\frac{y(s)}{P(x(s))}f'(x(s))ds.$$

First, we show that W_1 is a negative-definite function, we have two cases for all x, y and $t \geq 0$. If $c'(t) = 0$, then

$$W_1 = \frac{b'(t)R(x)}{2P(x)}y^2 \leq 0.$$

If $c'(t) < 0$, the quantity in the brackets below can be written as,

$$\begin{aligned} W_1 &= dc'(t)\left[F(x) + \frac{1}{d}yf(x) + \frac{b'(t)R(x)}{2dP(x)c'(t)}y^2\right] \\ &= dc'(t)\left[F(x) + \frac{b'(t)R(x)}{2dP(x)c'(t)}\left\{y + \frac{c'(t)P(x)f(x)}{b'(t)R(x)}\right\}^2 - \frac{c'(t)P(x)f^2(x)}{2db'(t)R(x)}\right], \end{aligned}$$

from the assumption (i), we get

$$\begin{aligned} W_1 &\leq dc'(t)\int_0^x (1 - \frac{P(x)f'(u)}{dR(x)})f(u)du \\ &\leq dc'(t)\int_0^x (1 - \frac{p_1\delta_1}{dr_0})f(u)du \\ &\leq c'(t)\frac{\delta_2}{c_0}F(x) \leq 0. \end{aligned}$$

Thus, on combining the two cases, we get $W_1 \leq 0$ for all $t \geq 0$, x and y. Similarly by $2ab \leq a^2 + b^2$, we have the following :

$$\begin{aligned} W_2 &\leq \left[\frac{d}{2}|\theta_1(t)|\,(1 + \frac{Aq_1}{p_0}) + \frac{A}{2}|\theta_2(t)|(1 + \frac{d}{p_0})\right](y^2 + z^2) + \frac{B}{2}|\theta_3(t)|y^2 \\ &\leq k_1\,[|\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|]\,(y^2 + z^2), \end{aligned} \tag{12}$$

and

$$W_3 \leq \frac{C\delta_1 r}{2}z^2 + \frac{C\delta_1 dr}{2p_0}y^2 + \frac{C\delta_1}{2p_0^2}(1 + \frac{d}{p_0})\int_{t-r}^{t}y^2(\xi)d\xi, \tag{13}$$

where $k_1 = \max(\frac{d}{2}(1 + \frac{Aq_1}{p_0}), \frac{A}{2}(1 + \frac{d}{p_0}), \frac{B}{2})$.
Estimates for W_1, W_2 and W_3 into (11), yields

$$\begin{aligned} \frac{d}{dt}V &\leq -\left[\frac{\varepsilon}{p_1^2} - (\lambda + \frac{dC\delta_1}{2p_0})r\right]y^2 - \left[\frac{a_0q_0 - d}{p_1} - \frac{C\delta_1 r}{2}\right]z^2 + \left[\frac{C\delta_1}{2p_0^2}(1 + \frac{d}{p_0}) - \lambda\right]\int_{t-r}^{t}y^2(\xi)d\xi \\ &\quad + k_1\,[|\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|]\,(y^2 + z^2). \end{aligned}$$

Choosing $\lambda = \frac{C\delta_1}{2p_0^2}(1 + \frac{d}{p_0})$, we obtain

$$\frac{d}{dt}V \leq -\left[\frac{\varepsilon}{p_1^2} - \frac{C\delta_1}{2p_0}(d + \frac{1}{p_0} + \frac{d}{p_0^2})r\right]y^2 - \left[\frac{a_0q_0 - d}{p_1} - \frac{C\delta_1 r}{2}\right]z^2 + k_1\,[|\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|]\,(y^2 + z^2).$$

From (9), (6) and taking $\mu = \dfrac{\delta}{k_1}$ we see at once that

$$\begin{aligned}\frac{d}{dt}U &= (\exp(-\frac{k_1\gamma(t)}{\delta}))(\frac{d}{dt}V - \frac{k_1(|\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|)}{\delta}V)\\ &\leq K[-(\frac{\varepsilon}{p_1^2} - \frac{C\delta_1 r}{2p_0}(d + \frac{1}{p_0} + \frac{d}{p_0^2}))y^2 - (\frac{a_0q_0 - d}{p_1} - \frac{C\delta_1 r}{2})z^2],\end{aligned}$$

where $K = \exp(-\dfrac{k_1 N}{\delta})$. Therefore, if

$$r < \min\left\{\frac{2(a_0q_0 - d)}{p_1 C\delta_1}, \frac{2p_0^3\varepsilon}{C\delta_1 p_1^2(d + dp_0^2 + p_0)}\right\}.$$

Then

$$\frac{d}{dt}U(t, x_t, y_t, z_t) \leq -\beta(y^2 + z^2), \text{ for some } \beta > 0. \tag{14}$$

On combining the inequalities in (9), (10) and (14), the hypotheses of Lemma 2.4 are satisfied. Namely, the only solution of system (3) for which $\dfrac{d}{dt}U(t, x_t, y_t, z_t) = 0$ is the solution $x = y = z = 0$. Thus, under the above discussion, we conclude that the trivial solution of equation (3) is uniformly asymptotically stable. □

Proof. **[Proof of Theorem 3.2]** The proof depend on some fundamental properties of a continuously differentiable Lyapunov functional, we define $W = W(t, x_t, y_t, z_t)$ as

$$W(t, x_t, y_t, z_t) = (\exp(-\beta(t)))(V(t, x_t, y_t, z_t), \tag{15}$$

where

$$\beta(t) = \int_0^t \left[\frac{1}{\mu}(|\theta_1(s)| + |\theta_2(s)| + |\theta_3(s)|) + \frac{1}{c_0}|c'(s)|\right] ds,$$

and $V = V(t, x, y, z)$ is already defined in Theorem 3.1. To show that V is a positive definite function with the conditions in Theorem 3.2, we rewrite V thus:

$$V = dc(t)U_0 + U_1 + V_2 + \lambda\int_{-r}^0\int_{t+s}^t y^2(\xi)d\xi ds,$$

where

$$\begin{aligned}U_0 &= F(x) + \frac{1}{d}yf(x) + \frac{\delta_1}{2d^2}y^2,\\ U_1 &= \frac{1}{2}\left[-\frac{c(t)\delta_1}{d} + \frac{b(t)R(x)}{P(x)}\right]y^2.\end{aligned}$$

From the assumptions of Theorem 3.2, we obtain:

$$\begin{aligned}dc(t)U_0 &= dc(t)\left[F(x) + \frac{\delta_1}{2d^2}(y + \frac{d}{\delta_1}f(x))^2 - \frac{1}{2\delta_1}f^2(x)\right]\\ &\geq dc(t)\left[\int_0^x(1 - \frac{f'(u)}{\delta_1})f(u)du\right]\\ &\geq dc(t)\left[\int_0^x(1 - \frac{dr_0b_0}{p_1C}\frac{1}{\delta_1})f(u)du\right] \geq 0.\end{aligned}$$

Condition (j) implies that $U_1 \geq 0$. Hence, since V_2 is positive definite, there exist sufficiently small positive constant, such that

$$V \geq \delta(x^2 + y^2 + z^2).$$

Therefore we can find a continuous function $W_1(|\varphi(0)|)$ with

$$W_1(|\varphi(0)|) \geq 0 \quad \text{and} \quad W_1(|\varphi(0)|) \leq W(t, \varphi),$$

by the fact that $\beta(t) \leq \frac{N}{\mu} + \frac{N_1}{c_0}$. The existence of a continuous function $W_2(\|\varphi\|)$ which satisfies the inequality $W(t, \varphi) \leq W_2(\|\varphi\|)$, is easily verified (see (10)).
Let (x, y, z) be a solution of (5). Differentiating the Lyapunov functional $V(t, x_t, y_t, z_t)$ along this solution, we find

$$\begin{aligned}
\frac{d}{dt}V &= dc'(t)F(x) + c'(t)yf(x) + \frac{c'(t)\delta_1}{2d}y^2 - \lambda \int_{t-r}^{t} y^2(\xi)d\xi + \lambda r y^2 \\
&- d\theta_1(t)(yz + \frac{a(t)Q(x)}{2P(x)}y^2) - a(t)\theta_2(t)(yz + \frac{d}{2P(x)}y^2) + \frac{b(t)}{2}\theta_3(t)y^2 \\
&+ c(t)(z + \frac{d}{P(x)}y)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{P(x(s))}ds + \frac{1}{P(x)}(d - a(t)Q(x))z^2 \\
&+ (\frac{c(t)f'(x)}{P(x)} - \frac{db(t)R(x)}{P^2(x)})y^2 + (\frac{da'(t)Q(x)}{2P^2(x)} + \frac{b'(t)R(x)}{2P(x)} - \frac{c'(t)\delta_1}{2d})y^2.
\end{aligned}$$

Making use of the definitions of W_2 and W_3, it is clear that

$$\begin{aligned}
\frac{d}{dt}V &\leq dc'(t)F(x) + c'(t)yf(x) + \frac{c'(t)\delta_1}{2d}y^2 - \lambda \int_{t-r}^{t} y^2(\xi)d\xi + \lambda r y^2 \\
&+ W_2 + W_3 + \frac{1}{P(x)}(d - a(t)Q(x))z^2 + (\frac{c(t)f'(x)}{P(x)} - \frac{db(t)R(x)}{P^2(x)})y^2 \\
&+ \left(\frac{da'(t)Q(x)}{2P^2(x)} + \frac{b'(t)R(x)}{2P(x)} - \frac{c'(t)\delta_1}{2d}\right)y^2.
\end{aligned}$$

By the assumptions (A_1)-(A_3) and the inequalities (12), (13), we find

$$\begin{aligned}
\frac{d}{dt}V &\leq dc'(t)\left[F(x) + \frac{1}{d}yf(x) + \frac{\delta_1}{2d^2}y^2\right] + \left[\frac{C\delta_1}{2p_0^2}(1 + \frac{d}{p_0}) - \lambda\right]\int_{t-r}^{t} y^2(\xi)d\xi \\
&- \left[\frac{db_0r_0 - p_1C\delta_1}{P^2(x)} - (\lambda + \frac{C\delta_1 d}{2p_0})r\right]y^2 - \left[\frac{1}{p_1}(a_0q_0 - d) - \frac{C\delta_1 r}{2}\right]z^2 \\
&+ \left[da'(t)Q(x) + b'(t)R(x)P(x) - P^2(x)\frac{c'(t)\delta_1}{d}\right]\frac{y^2}{2P^2(x)} \\
&+ k_1\left[|\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|\right](y^2 + z^2),
\end{aligned}$$

choosing $\lambda = \frac{C\delta_1}{2p_0^2}(1 + \frac{d}{p_0})$, we have

$$\begin{aligned}
\frac{d}{dt}V &\leq dc'(t)\left[F(x) + \frac{1}{d}yf(x) + \frac{\delta_1}{2d^2}y^2\right] - \left[\frac{db_0r_0 - p_1C\delta_1}{2p_1^2} - \frac{C\delta_1}{2p_0}(d + \frac{1}{p_0} + \frac{d}{p_0^2})r\right]y^2 \\
&- \left[\frac{a_0q_0 - d}{p_1} - \frac{C\delta_1 r}{2}\right]z^2 + k_1\left[|\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|\right](y^2 + z^2).
\end{aligned} \tag{16}$$

From (15), it is easily verified that

$$\frac{d}{dt}W(t, x_t, y_t, z_t) = (\exp(-\beta(t)))(\frac{d}{dt}V - (\frac{1}{\mu}\gamma'(t) + \frac{1}{c_0}|c'(t)|)V),$$

on applying the inequalities (9), (16) and since

$$dc'(t)[F(x) + \frac{1}{d}yf(x) + \frac{\delta_1}{2d^2}y^2] = dc'(t)U_0,$$

we get

$$\begin{aligned}\frac{d}{dt}V - (\frac{1}{\mu}\gamma'(t) + \frac{1}{c_0}|c'(t)|)V \leq & -\left[\frac{db_0r_0 - p_1C\delta_1}{2p_1^2} - \frac{C\delta_1}{2p_0}(d + \frac{1}{p_0} + \frac{d}{p_0^2})r\right]y^2 \\ & -\left[\frac{a_0q_0 - d}{p_1} - \frac{C\delta_1 r}{2}\right]z^2 + d\,|c'(t)|\,U_0 \\ & -\frac{\delta}{\mu}\gamma'(t)(x^2 + y^2 + z^2) - \frac{1}{c_0}|c'(t)|\,V \\ & +k_1\gamma'(t)(y^2 + z^2).\end{aligned}$$

Putting $\mu = \dfrac{\delta}{k_1}$ and by the fact that $V \geq dc_0U_0$, we obtain

$$\frac{d}{dt}W \leq - M\left[\frac{db_0r_0 - p_1C\delta_1}{2p_1^2} - \frac{C\delta_1}{2p_0}(d + \frac{1}{p_0} + \frac{d}{p_0^2})r\right]y^2 - M\left[\frac{a_0q_0 - d}{p_1} - \frac{C\delta_1 r}{2}\right]z^2,$$

where $M = \exp -\left(\dfrac{k_1N}{\delta} + \dfrac{N_1}{c_0}\right)$. Therefore, if

$$r < \min\left\{\frac{2(a_0q_0 - d)}{p_1C\delta_1}, \frac{p_0^3(db_0r_0 - p_1C\delta_1)}{p_1^2C\delta_1(d + dp_0^2 + p_0)}\right\}.$$

Then

$$\frac{d}{dt}W(t, x_t, y_t, z_t) \leq -\alpha(y^2 + z^2), \text{ for some } \alpha > 0.$$

Thus, under the above discussion, we conclude that the trivial solution of equation (3) is uniformly asymptotically stable. □

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