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# Global Behavior of a Higher Order Rational Difference Equation

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**Abstract.**
In this paper, we derive the forbidden set and discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-k}}{B - C\prod_{i=0}^{k} x_{n-i}}, \qquad n = 0, 1, \ldots$$

where $A, B, C$ are positive real numbers and the initial conditions $x_{-k}, ..., x_{-1}, x_0$ are real numbers.

## 1. Introduction

No one can deny that, Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations. One can see [1]-[25] and the references therein.

In [4], M. Aloqeili discussed the stability properties and semicycle behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}, \quad n = 0, 1, \ldots \tag{1}$$

with real initial conditions and positive real number $a$.

In [23], D. Simsek et al. introduced the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, ...$$

with positive initial conditions.
In [11], E.M. Elsayed discussed the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, ...$$

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where the initial conditions are nonzero real numbers with $x_{-5}x_{-2} \neq 1$, $x_{-4}x_{-1} \neq 1$ and $x_{-3}x_0 \neq 1$.
He also in [9], determined the solutions to some difference equations. He obtained the solution to the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}x_{n-3}}, \quad n = 0, 1, ...$$

where the initial conditions are nonzero positive real numbers.
R. Karatas et al. [15] discussed the positive solutions and the attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}, \quad n = 0, 1, ...$$

where the initial conditions are nonnegative real numbers.
The authors in [14], discussed the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{a x_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}, \quad n = 0, 1, ...$$

where $a$, $x_{-(2k-2)}, ..., x_0$ are real numbers such that $x_{-(2k-2)}x_{-(2k-1)}...x_0 \neq a$ and $k$ is a nonnegative integer.
Elabbasy et al. [8] determined and discussed the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, ...$$

with nonnegative real numbers $\alpha$,$\beta$, $\gamma$, positive real initial conditions and positive integer $k$.
In [16], we investigated the behavior and periodic nature of the two difference equations

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots$$

In [2], we have also discussed the oscillation, periodicity, boundedness and the global behavior of all admissible solutions of the difference equation

$$x_{n+1} = \frac{A x_{n-2r-1}}{B - C \prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, \ldots$$

where $A, B, C$ are positive real numbers and $l, r, k$ are nonnegative integers, such that $l \leq k$.
Also in [1], we discussed the global stability of all solutions of the difference equation

$$x_{n+1} = \frac{A x_{n-2}}{B + C x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots$$

where $A, B, C$ are positive real numbers and the initial conditions $x_{-2}, x_{-1}, x_0$ are real numbers.

In this paper, we discuss the global behavior of all solutions of the difference equation

$$x_{n+1} = \frac{A x_{n-k}}{B - C \prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \ldots \tag{2}$$

where $A, B, C$ are positive real numbers and the initial conditions $x_{-k}, ..., x_{-1}, x_0$ are real numbers. The difference equation (2) is a more general case of the difference equation (1).

## 2. Linearized Stability and Solutions of Equation (2)

In this section we introduce an explicit formula for the solutions of the difference equation (2) and study its linearized stability.

It is convenient to reduce the parameters on which equation (2) depends on.

The change of variables $\sqrt[k+1]{\frac{C}{A}}x_n = y_n$ reduces equation (2) to the equation

$$y_{n+1} = \frac{y_{n-k}}{p - \prod_{i=0}^{k} y_{n-i}}, \qquad n = 0, 1, \dots \tag{3}$$

where $p = \frac{B}{A}$.
We will deal with equation (3) rather than equation (2).

To start navigating the global behavior of the difference equation (3), we classify the nontrivial solutions of equation (3) into two types of solutions:

- Solutions with initial points $(y_{-k}, y_{-k+1}, ..., y_0)$ such that $y_{-i} = 0$, for some but not all $i \in \{0, 1, ..., k\}$.
- Solutions with initial points $(y_{-k}, y_{-k+1}, ..., y_0)$ such that $y_{-i} \neq 0$, for all $i \in \{0, 1, ..., k\}$.

These two types of solutions exhibit a global behavior different from each other.

**Theorem 2.1.** *Let $y_{-k}, y_{-k+1}, ..., y_{-1}$ and $y_0$ be real numbers such that $y_{-i} = 0$ for some but not all $i \in \{0, 1, ..., k\}$. Then the solution $\{y_n\}_{n=-k}^{\infty}$ of equation (3) is*

$$y_n = \begin{cases} (\frac{1}{p})^{\frac{n-1}{k+1}+1} y_{-k} & , n = 1, k+2, 2k+3, ... \\ (\frac{1}{p})^{\frac{n-2}{k+1}+1} y_{-k+1} & , n = 2, k+3, 2k+4, ... \\ ..... \\ ..... \\ ..... \\ (\frac{1}{p})^{\frac{n-k}{k+1}+1} y_{-1} & , n = k, 2k+1, 3k+2, ... \\ (\frac{1}{p})^{\frac{n-k-1}{k+1}+1} y_{0} & , n = k+1, 2k+2, 3k+3, ... \end{cases} \tag{4}$$

*Proof.* *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (3) such that $y_{-i} = 0$ for some but not all $i \in \{0, 1, ..., k\}$. Using equation (3), we can write*

$$\prod_{l=0}^{k} y_{n+1-l} = \frac{\prod_{l=0}^{k} y_{n-l}}{p - \prod_{l=0}^{k} y_{n-l}}, \quad n = 0, 1, ....$$

*But as $\prod_{l=0}^{k} y_{-l} = 0$, we get $\prod_{l=0}^{k} y_{n-l} = 0$ for all $n \geq 1$.*
*It follows that*

$$y_{n+1} = \frac{y_{n-k}}{p - \prod_{l=0}^{k} y_{n-l}} = \frac{y_{n-k}}{p}$$

*for all $n \geq 0$, from which the result follows.* □

Now suppose that $y_{-i} \neq 0$, for all $i \in \{0, 1, ..., k\}$. From equation (3) and using the substitution $t_n = \frac{1}{y_n y_{n-1} \cdots y_{n-k}}$, we can obtain the linear nonhomogeneous difference equation

$$t_{n+1} = pt_n - 1, \quad t_0 = \frac{1}{y_0 y_{-1} \cdots y_{-k}}. \tag{5}$$

It is clear that the mapping $h(x) = px - 1$ is invertible and its inverse is $h^{-1}(x) = \frac{1}{p}x + \frac{1}{p}$.
We try to deduce the forbidden set of equation (3).

For,
suppose that we start from an initial point $(y_{-k}, y_{-k+1}, ..., y_0)$ such that $y_0y_{-1}...y_{-k} = p$.
The backward orbits, $v_n = \frac{1}{y_ny_{n-1}...y_{n-k}}$ satisfy the equation

$$v_n = h^{-1}(v_{n-1}) = \frac{1}{p}v_{n-1} + \frac{1}{p} \quad \text{with} \quad v_0 = \frac{1}{y_0y_{-1}...y_{-k}} = \frac{1}{p},$$

then we obtain $v_n = \frac{1}{y_ny_{n-1}...y_{n-k}} = h^{-n}(v_0) = \frac{1}{p}\sum_{l=0}^{n}(\frac{1}{p})^l$.
That is $y_ny_{n-1}...y_{n-k} = \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$.
On the other hand, we can observe that if we start from an initial point $(y_{-k}, ..., y_{-1}, y_0)$ such that $y_0y_{-1}...y_{-k} = \frac{p}{\sum_{l=0}^{n_0}(\frac{1}{p})^l}$ for some $n_0 \in \mathbb{N}$, then according to equation (5) we obtain

$$t_{n_0} = \frac{1}{y_{n_0}y_{n_0-1}...y_{n_0-k}} = \frac{1}{p}.$$

This implies that $p - y_{n_0}y_{n_0-1}...y_{n_0-k} = 0$.
Therefore, $y_{n_0+1}$ is undefined.
These observations lead us to conclude the following result.

**Proposition 2.2.** *The forbidden set F of equation (3) is*

$$F = \bigcup_{n=0}^{\infty}\{(u_0, u_1, ..., u_k) : \prod_{i=0}^{k} u_i = \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}\}.$$

**Theorem 2.3.** *Let $y_{-k}, y_{-k+1}, ..., y_{-1}$ and $y_0$ be real numbers such that $\alpha = y_0y_{-1}...y_{-k} \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$. Then the solution of equation (3) is*

$$y_n = \begin{cases} y_{-k}\prod_{j=0}^{\frac{n-1}{k+1}} \frac{p^{(k+1)j} - \alpha\sum_{l=0}^{(k+1)j-1} p^l}{p^{(k+1)j+1} - \alpha\sum_{l=0}^{(k+1)j} p^l} & , n = 1, k+2, 2k+3, ... \\ y_{-k+1}\prod_{j=0}^{\frac{n-2}{k+1}} \frac{p^{(k+1)j+1} - \alpha\sum_{l=0}^{(k+1)j} p^l}{p^{(k+1)j+2} - \alpha\sum_{l=0}^{(k+1)j+1} p^l} & , n = 2, k+3, 2k+4, ... \\ ..... \\ ..... \\ ..... \\ y_{-1}\prod_{j=0}^{\frac{n-k}{k+1}} \frac{p^{(k+1)j+k-1} - \alpha\sum_{l=0}^{(k+1)j+k-2} p^l}{p^{(k+1)j+k} - \alpha\sum_{l=0}^{(k+1)j+k-1} p^l} & , n = k, 2k+1, 3k+2, ... \\ y_0\prod_{j=0}^{\frac{n-k-1}{k+1}} \frac{p^{(k+1)j+k} - \alpha\sum_{l=0}^{(k+1)j+k-1} p^l}{p^{(k+1)j+k+1} - \alpha\sum_{l=0}^{(k+1)j+k} p^l} & , n = k+1, 2k+2, 3k+3, ... \end{cases} \tag{6}$$

*Proof.* *Let $y_{-k}, y_{-k+1}, ..., y_{-1}$ and $y_0$ be real numbers such that $\alpha = y_0y_{-1}...y_{-k} \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$.*
*The solution of the linear nonhomogeneous difference equation (5) is*

$$t_{n+1} = p^{n+1}t_0 - \sum_{r=0}^{n} p^r, \quad t_0 = \frac{1}{y_0y_{-1}...y_{-k}}.$$

*If we set $\alpha = y_0y_{-1}...y_{-k}$, then we can write*

$$\prod_{l=0}^{k} y_{n+1-l} = \frac{\alpha}{p^{n+1} - \alpha\sum_{r=0}^{n} p^r}.$$

*It follows that*

$$\frac{\prod_{l=0}^{k} y_{n+1-l}}{\prod_{l=0}^{k} y_{n-l}} = \frac{p^n - \alpha \sum_{r=0}^{n-1} p^r}{p^{n+1} - \alpha \sum_{r=0}^{n} p^r}.$$

*This implies that*

$$y_{n+1} = y_{n-k}\frac{p^n - \alpha \sum_{r=0}^{n-1} p^r}{p^{n+1} - \alpha \sum_{r=0}^{n} p^r},$$

*from which we can write the form (6).* □

**Corollary 2.4.** *Assume that $p = 1$ and $\alpha = y_0 y_{-1}...y_{-k} \neq \frac{1}{n+1}$ for any $n \in N$. Then the solution of equation (3) is*

$$y_n = \begin{cases} y_{-k} \prod_{j=0}^{\frac{n-1}{k+1}} \frac{1-(k+1)j\alpha}{1-((k+1)j+1)\alpha} & ,n = 1, k+2, 2k+3, ... \\ y_{-k+1} \prod_{j=0}^{\frac{n-2}{k+1}} \frac{1-((k+1)j+1)\alpha}{1-((k+1)j+2)\alpha} & ,n = 2, k+3, 2k+4, ... \\ ..... \\ ..... \\ ..... \\ y_{-1} \prod_{j=0}^{\frac{n-k}{k+1}} \frac{1-((k+1)j+k-1)\alpha}{1-((k+1)j+k)\alpha} & ,n = k, 2k+1, 3k+2, ... \\ y_0 \prod_{j=0}^{\frac{n-k-1}{k+1}} \frac{1-((k+1)j+k)\alpha}{1-((k+1)j+k+1)\alpha} & ,n = k+1, 2k+2, 3k+3, ... \end{cases} \quad (7)$$

*Proof.* *It is sufficient to note that, $\sum_{r=0}^{n} p^r = \sum_{r=0}^{n} (\frac{1}{p})^r = n + 1$ when $p = 1$.*
*Using this fact, the solution form (6) reduced to the form (7) and the result follows.* □

**Corollary 2.5.** *Assume that $p < 1$ and let $\{y_n\}_{n=-k}^{\infty}$ be a nontrivial solution of equation (3). If $\alpha = y_0 y_{-1}...y_{-k} = 0$, then the solution $\{y_n\}_{n=-k}^{\infty}$ is unbounded.*

**Example (1)** Figure 1. shows that if $\{y_n\}_{n=-2}^{\infty}$ is a solution of the equation

$$y_{n+1} = \frac{y_{n-2}}{0.5 - y_n y_{n-1} y_{n-2}}, \qquad n = 0, 1, \ldots$$

with initial conditions $y_{-2} = 2$, $y_{-1} = 0$, $y_0 = 1$ ($\alpha = 0$) where $k = 2$ and $p = 0.5$, then the solution $\{y_n\}_{n=-2}^{\infty}$is unbounded.

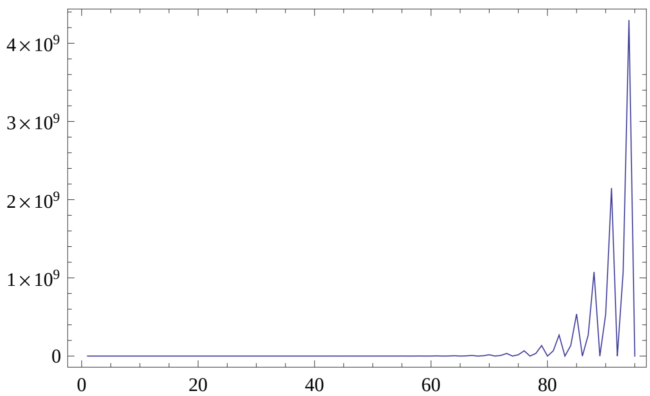


Figure 1: The difference equation $y_{n+1} = \frac{y_{n-2}}{0.5 - y_n y_{n-1} y_{n-2}}$

We end this section with the discussion of the local stability of the equilibrium points of equation (3).
It is clear that the equilibrium point $\bar{y} = 0$ is always an equilibrium point of equation (3) and the nonzero equilibrium points depend on whether $k$ is even or odd.

When $k$ is odd, we have the nonzero equilibrium points $\bar{y} = \pm \sqrt[k+1]{p-1}$ if $p > 1$.
When $k$ is even, we have the nonzero equilibrium point $\bar{y} = \sqrt[k+1]{p-1}$, $p \neq 1$.

**Lemma 2.6.** *Assume that $P(x)$ is the polynomial*

$$x^k + x^{k-1} + ... + x + 1.$$

*Then the zeros of $P(x)$ are of modulus one.*

The following theorem describes the local behavior of the equilibrium points.

**Theorem 2.7.** *The following statements are true.*

1. *The equilibrium point $\bar{y} = 0$ is locally asymptotically stable if $p > 1$ and unstable if $p < 1$.*
2. *If $k$ is even, then $\bar{y} = \sqrt[k+1]{p-1}$ is unstable if $p > 1$ and nonhyperbolic if $p < 1$.*
3. *If $k$ is odd, then the equilibrium points $\bar{y} = \pm \sqrt[k+1]{p-1}$ are unstable equilibrium points.*

*Proof.* *The linearized equation associated with equation (3) about an equilibrium point $\bar{y}$ is*

$$z_{n+1} - \frac{\bar{y}^{k+1}}{(p-\bar{y}^{k+1})^2} \sum_{i=0}^{k-1} z_{n-i} - \frac{p}{(p-\bar{y}^{k+1})^2} z_{n-k} = 0 \qquad , n = 0, 1, 2, \ldots \tag{8}$$

*Its characteristic equation associated with this equation is*

$$\lambda^{k+1} - \frac{\bar{y}^{k+1}}{(p-\bar{y}^{k+1})^2} \sum_{i=0}^{k-1} \lambda^{k-i} - \frac{p}{(p-\bar{y}^{k+1})^2} = 0. \tag{9}$$

*Therefore, (1) follows directly.*
*Equation (8) about a nonzero equilibrium point $\bar{y}$ is*

$$z_{n+1} - (p-1) \sum_{i=0}^{k-1} z_{n-i} - p z_{n-k} = 0 \qquad , n = 0, 1, 2, \ldots \tag{10}$$

*Also equation (9) becomes*

$$\lambda^{k+1} - (p-1) \sum_{i=0}^{k-1} \lambda^{k-i} - p = 0. \tag{11}$$

*Let*

$$f(\lambda) = \lambda^{k+1} - (p-1) \sum_{i=0}^{k-1} \lambda^{k-i} - p.$$

*We can see that*

$$f(\lambda) = (\lambda - p) \sum_{l=0}^{k} \lambda^l = (\lambda - p) P(\lambda).$$

*Then the roots of equation (11) are the zeros of $f(\lambda)$. Using lemma* (2.6)*, we see that, the roots of equation (11) are $p$ and $k$ other roots with modulus* 1.
*Therefore, (2) and (3) follow directly.* □

## 3. Global Behavior of Equation (3)

The solution of equation (3) can be written as

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \frac{p^{(k+1)j+i-1} - \alpha \sum_{l=0}^{(k+1)j+i-2} p^l}{p^{(k+1)j+i} - \alpha \sum_{l=0}^{(k+1)j+i-1} p^l}, \quad i = 1,2,...,k+1 \quad \text{and} \quad m = 0,1,... \tag{12}$$

But as

$$\frac{p^{(k+1)j+i-1} - \alpha \sum_{l=0}^{(k+1)j+i-2} p^l}{p^{(k+1)j+i} - \alpha \sum_{l=0}^{(k+1)j+i-1} p^l} = \frac{p^{(k+1)j+i-1}\mu - \alpha}{p^{(k+1)j+i}\mu - \alpha}, \quad \text{where} \quad \mu = 1 - p + \alpha.$$

We can write

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \beta_i(j), \quad i = 1,2,...,k+1 \quad \text{and} \quad m = 0,1,...$$

where

$$\beta_i(j) = \frac{p^{(k+1)j+i-1}\mu - \alpha}{p^{(k+1)j+i}\mu - \alpha}, \quad i = 1,2,...,k+1.$$

**Theorem 3.1.** *Assume that $\{y_n\}_{n=-k}^{\infty}$ is a solution of equation (3) such that $\alpha \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$. If $\alpha = p - 1$, then $\{y_n\}_{n=-k}^{\infty}$ is a periodic solution with period $k + 1$.*

*Proof.* It is sufficient to see that if $\alpha = p - 1$, then $\mu = 0$. Therefore,

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \frac{p^{(k+1)j+i-1}\mu - \alpha}{p^{(k+1)j+i}\mu - \alpha} = y_{-(k+1)+i}, \quad i = 1,2,...,k+1.$$

□

**Proposition 3.2.** *Assume that $p < 1$ and let $\alpha \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$. Then there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.*

*Proof.* *We have three situations:*

1. *If $\alpha < p - 1 < 0$, then $0 < \mu - \alpha < -\alpha$. Hence for each $j \in \mathbb{N}$, $p^{(k+1)j+i-1}\mu - \alpha > \mu - \alpha > 0$, $i = 1,2,...,k+1$. Then*

$$\beta_i(j) = \frac{p^{(k+1)j+i-1}\mu - \alpha}{p^{(k+1)j+i}\mu - \alpha} > 0 \quad \text{for all} \quad j \geq 0.$$

2. *If $p - 1 < \alpha < 0$, then $0 < -\alpha < \mu - \alpha$.*
   *But*

$$\lim_{j\to\infty} \beta_i(j) = \lim_{j\to\infty} \frac{p^{(k+1)j+i-1}\mu - \alpha}{p^{(k+1)j+i}\mu - \alpha} = 1.$$

   *Then there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.*
3. *When $p - 1 < 0 < \alpha$, the situation is similar to that in (2).*
   *In all cases there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$.*

□

**Theorem 3.3.** *Assume that $\{y_n\}_{n=-k}^{\infty}$ is a solution of equation (3) such that $\alpha \neq p - 1$ and $\alpha \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$. Then the following statements are true.*

1. *If $p > 1$, then $\{y_n\}_{n=-k}^{\infty}$ converges to $\bar{y} = 0$.*
2. *If $p < 1$ and $\alpha \neq 0$, then $\{y_n\}_{n=-k}^{\infty}$ is bounded.*

*Proof.* *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (3) such that $\alpha \neq \frac{p}{\sum_{i=0}^{n}(\frac{1}{p})^i}$ for any $n \in N$.*
*The condition $\alpha \neq p-1$ ensures that the solution $\{y_n\}_{n=-k}^{\infty}$ is not a $(k+1)$-periodic solution.*

1. *Suppose that $p > 1$. It is clear that, as the equilibrium point $\frac{1}{p-1}$ of equation (5) is repelling, every non-constant solution of equation (5) approaches $\infty$ or $-\infty$ according to the value of $t_0 = \frac{1}{\alpha}$.*
   *We shall consider the following situations:*
   (a) *If $\alpha = \frac{1}{t_0} < 0$, then according to equation (5), we have*
   $\prod_{i=0}^{k} y_{n-i} = \frac{1}{t_n} < 0$, *for each $n \in \mathbb{N}$. Therefore,*
   $$| y_{n+1} |= \frac{| y_{n-k} |}{| p - \prod_{i=0}^{k} y_{n-i} |} < \frac{| y_{n-k} |}{p}, \quad n = 0, 1, \ldots$$
   (b) *If $0 < \alpha = \frac{1}{t_0} < p-1$, then according to equation (5), $\prod_{i=0}^{k} y_{n-i} = \frac{1}{t_n} \to 0$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that $0 < \prod_{i=0}^{k} y_{n-i} < p-1$ for each $n > n_0$. Therefore,*
   $$| y_{n+1} |= \frac{| y_{n-k} |}{| p - \prod_{i=0}^{k} y_{n-i} |} <| y_{n-k} |, \quad n \geq n_0.$$
   (c) *If $p-1 < \alpha = \frac{1}{t_0} < p$, then according to equation (5), there exists $n_0 \in \mathbb{N}$ such that $\prod_{i=0}^{k} y_{n-i} = \frac{1}{t_n} < 0$ for each $n \geq n_0$. Therefore,*
   $$| y_{n+1} |= \frac{| y_{n-k} |}{| p - \prod_{i=0}^{k} y_{n-i} |} < \frac{| y_{n-k} |}{p}, \quad n \geq n_0.$$
   (d) *If $\alpha = \frac{1}{t_0} > p > 0$, then according to equation (5), $\prod_{i=0}^{k} y_{n-i} = \frac{1}{t_n} < 0$ for each $n > 0$. Therefore,*
   $$| y_{n+1} |= \frac{| y_{n-k} |}{| p - \prod_{i=0}^{k} y_{n-i} |} < \frac{| y_{n-k} |}{p}, \quad n = 0, 1, \ldots$$

   *In all cases, $y_n \to 0$ as $n \to \infty$.*
2. *Suppose that $p < 1$. Using proposition (3.2), there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) > 0$ for all $j \geq j_0$. Hence for each $i \in \{1, 2, ..., k+1\}$, we have for large $m$*
   $$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \beta_i(j) = y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \beta_i(j) \prod_{j=j_0}^{m} \beta_i(j)$$
   $$= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp(\ln \prod_{j=j_0}^{m} \beta_i(j))$$
   $$= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp\,(\sum_{j=j_0}^{m} \ln \beta_i(j)).$$
   *It is sufficient to test the convergence of the series $\sum_{j=j_0}^{\infty} | \ln \beta_i(j) |$.*
   *But*
   $$\lim_{j\to\infty} \frac{\ln \beta_i(j+1)}{\ln \beta_i(j)} = \frac{0}{0}.$$

*Then*

$$\lim_{j\to\infty}\frac{\ln\beta_i(j+1)}{\ln\beta_i(j)} = \lim_{j\to\infty}\frac{\frac{d}{dj}(\ln\beta_i(j+1))}{\frac{d}{dj}(\ln\beta_i(j))}$$
$$= \lim_{j\to\infty}\frac{\frac{(p-1)(\ln p)\mu(k+1)p^{(k+1)(j+1)+i-1}}{(p^{(k+1)(j+1)+i-1}\mu-\alpha)(p^{(k+1)(j+1)+i}\mu-\alpha)}}{\frac{(p-1)(\ln p)\mu(k+1)p^{(k+1)j+i-1}}{(p^{(k+1)j+i-1}\mu-\alpha)(p^{(k+1)j+i}\mu-\alpha)}}$$
$$= p^{k+1} < 1.$$

*It follows from D' Alemberts' test that the series $\sum_{j=j_0}^{\infty} \mid \ln\beta_i(j) \mid$ is convergent. This ensures that the solution $\{y_n\}_{n=-k}^{\infty}$ is bounded.*

□

**Example (2)** Figure 2. shows that if $\{y_n\}_{n=-2}^{\infty}$ is the solution of the equation

$$y_{n+1} = \frac{y_{n-2}}{2 - y_n y_{n-1} y_{n-2}}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-2} = 2$, $y_{-1} = 1$, $y_0 = 2$ ($\alpha \neq p-1$ and $\alpha \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$) where $k = 2$ and $p = 2$, then the solution $\{y_n\}_{n=-2}^{\infty}$ converges to zero.

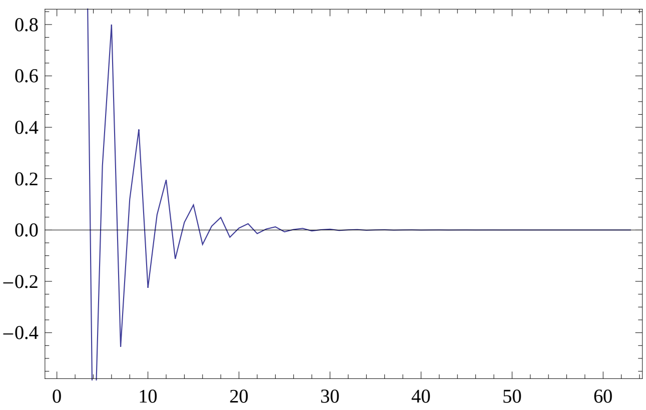


Figure 2: The difference equation $y_{n+1} = \frac{y_{n-2}}{2-y_n y_{n-1} y_{n-2}}$

We can observe in case $p < 1$ that, the behavior of the solution $\{y_n\}_{n=-k}^{\infty}$ is totally different according to whether $\alpha = 0$ or $\alpha \neq 0$. This is obvious in corollary (2.5) and theorem (3.3).

**Theorem 3.4.** *Assume that $p < 1$ and let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (3) such that $\alpha \neq \frac{p}{\sum_{l=0}^{n}(\frac{1}{p})^l}$ for any $n \in N$. Then $\{y_n\}_{n=-k}^{\infty}$ converges to a $(k+1)$-periodic solution $\{\rho_0, \rho_1, ..., \rho_k\}$ of equation (3) with $\rho_0\rho_1...\rho_k = p - 1$.*

*Proof.* *By theorem* (3.3), *there exist $k+1$ real numbers $\rho_i \in \mathbb{R}$ such that*

$$\lim_{j\to\infty} y_{(k+1)m+i} = \rho_i, \quad i \in \{0, 1, ..., k\}.$$

*If we set $n = (k+1)m + i - 1$, $i = 0, 1, ..., k$ in equation (3), we get*

$$y_{(k+1)m+i} = \frac{y_{(k+1)(m-1)+i}}{p - \prod_{l=0}^{k} y_{(k+1)(m-1)+i-l+k}}, \quad i = 0, 1, ..., k \quad and \quad m = 0, 1, ...$$

*By taking the limit as $m \to \infty$, we obtain*

$$\rho_i = \frac{\rho_i}{p - \prod_{l=0}^{k} \rho_{i-l+k}}, \quad i = 0, 1, ..., k.$$

*But from equation* (5) *we have* $\prod_{l=0}^{k} y_{n-l} = y_n y_{n-1} ... y_{n-k} = \frac{1}{t_n} \to p - 1$ *as* $n \to \infty$.
*This implies that* $\prod_{i=0}^{k} y_{(k+1)m+i} \to \rho_0 \rho_1 ... \rho_k = p - 1$ *as* $m \to \infty$.
*Therefore,* $\{y_n\}_{n=-k}^{\infty}$ *converges to the* $(k+1)$*-periodic solution*

$$\{..., \rho_0, \rho_1, ..., \rho_{k-1}, \frac{p-1}{\rho_0 \rho_1 ... \rho_{k-1}}, \rho_0, \rho_1, ..., \rho_{k-1}, \frac{p-1}{\rho_0 \rho_1 ... \rho_{k-1}}, ...\}$$

□

**Example (3)** Figure 3. shows that if $\{y_n\}_{n=-3}^{\infty}$ is the solution of the equation

$$y_{n+1} = \frac{y_{n-3}}{0.8 - y_n y_{n-1} y_{n-2} y_{n-3}}, \quad n = 0, 1, \ldots$$

with initial conditions $y_{-3} = 2$, $y_{-2} = 2.6$, $y_{-1} = 0.2$, $y_0 = 2.2$ ($\alpha \neq 0$ and $\alpha \neq \frac{p}{\sum_{l=0}^{n} (\frac{1}{p})^l}$ for any $n \in N$) where $k = 3$ and $p = 0.8$, then the solution $\{y_n\}_{n=-2}^{\infty}$ is bounded.
Moreover, the solution $\{y_n\}_{n=-2}^{\infty}$ converges to 4-periodic solution.

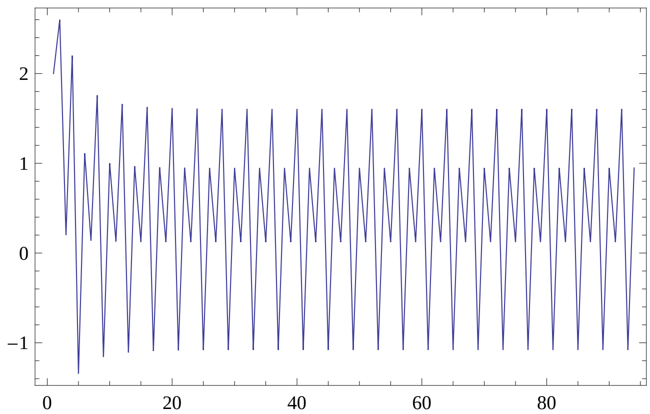


Figure 3: The difference equation $y_{n+1} = \frac{y_{n-3}}{0.8 - y_n y_{n-1} y_{n-2} y_{n-3}}$

## 4. Case $p = 1$

We end this work with the discussion of the case $p = 1$.
If we set $p = 1$ in equation (12), we get

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \zeta_i(j), \quad i = 1, 2, ..., k+1 \quad \text{and} \quad m = 0, 1, ... \tag{13}$$

where

$$\zeta_i(j) = \frac{1 - \alpha((k+1)j + i - 1)}{1 - \alpha((k+1)j + i)}, \quad i = 1, 2, ..., k+1.$$

**Proposition 4.1.** *Assume that $p = 1$ and let $\alpha \neq \frac{1}{n+1}$ for any $n \in N$. Then there exists $j_0 \in \mathbb{N}$ such that $\zeta_i(j) > 0$ for all $j \geq j_0$.*

*Proof.* *When $\alpha < 0$, the result is obvious where $\zeta_i(j) > 0$ for each $j \in \mathbb{N}$.*
*When $\alpha > 0$, It is sufficient to see that,*

$$\lim_{j\to\infty} \zeta_i(j) = \lim_{j\to\infty} \frac{1-\alpha((k+1)j+i-1)}{1-\alpha((k+1)j+i)} = 1.$$

*This implies that, there exists $j_0 \in \mathbb{N}$ such that $\zeta_i(j) > 0$ for all $j \geq j_0$.* □

**Theorem 4.2.** *Assume that $p = 1$. Then any solution $\{y_n\}_{n=-k}^{\infty}$ of equation (3) with $\alpha \neq 0$ and $\alpha \neq \frac{1}{n+1}$ for any $n \in N$ converges to zero.*

*Proof.* *Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of equation (3) such that $\alpha \neq \frac{1}{n+1}$ for any $n \in N$.*
*The condition $\alpha \neq 0$ ensures that the solution $\{y_n\}_{n=-k}^{\infty}$ is not a $(k+1)$-periodic solution.*
*Using proposition (4.1), there exists $j_0 \in \mathbb{N}$ such that $\zeta_i(j) > 0$ for all $j \geq j_0$. Hence for each $i \in \{1, 2, ..., k+1\}$, we have for large $m$*

$$\begin{aligned} y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{m} \zeta_i(j) &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \prod_{j=j_0}^{m} \zeta_i(j) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(\ln \prod_{j=j_0}^{m} \zeta_i(j)) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(\sum_{j=j_0}^{m} \ln \zeta_i(j)) \\ &= y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(-\sum_{j=j_0}^{m} \ln \frac{1}{\zeta_i(j)}). \end{aligned}$$

*We shall show that $\sum_{j=j_0}^{\infty} \ln \frac{1}{\zeta_i(j)} = \sum_{j=j_0}^{\infty} \ln \frac{1-\alpha((k+1)j+i)}{1-\alpha((k+1)j+i-1)} = \infty$, by considering the series $\sum_{j=j_0}^{\infty} \frac{\alpha}{-1+\alpha((k+1)j+i)}$. But as*

$$\lim_{j\to\infty} \frac{\ln(1-\alpha((k+1)j+i))/(1-\alpha((k+1)j+i-1))}{\alpha/(-1+\alpha((k+1)j+i))} = 1,$$

*using the limit comparison test, we get $\sum_{j=j_0}^{\infty} \ln \frac{1}{\zeta_i(j)} = \infty$.*
*Therefore,*

$$y_{(k+1)m+i} = y_{-(k+1)+i} \prod_{j=0}^{j_0-1} \zeta_i(j) \exp(-\sum_{j=j_0}^{m} \ln \frac{1}{\zeta_i(j)})$$

*converges to zero as $m \to \infty$.* □

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