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Common Fixed Point Theorems for Asymptotically Regular Mappings on Ordered Orbitally Complete Metric Spaces with an Application to Systems of Integral Equations

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Abstract. In this paper, we prove existence and uniqueness results for common fixed points of two or three relatively asymptotically regular mappings satisfying the orbital continuity of one of the involved maps on ordered orbitally complete metric spaces under generalized Φ-contractive condition. Also, we introduce and use orbitally dominating maps and orbitally weakly increasing maps. We furnish suitable examples to demonstrate the usability of the hypotheses of our results. As an application, we prove the existence of solutions for certain system of integral equations.

1. Introduction and Preliminaries

Let f be a self-map on a metric space $(\mathcal{X}, d)$. The following concepts were introduced by Browder and Petryshyn.

Definition 1.1. *[1]*

1. *The map f is said to be asymptotically regular at a point $x \in \mathcal{X}$ if* $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = 0$.
2. *The map f is said to be orbitally continuous at a point z in $\mathcal{X}$ if for any sequence $\{x_n\} \subset O(x; f)$ (for some $x \in \mathcal{X}$), $x_n \to z$ as $n \to \infty$ implies $fx_n \to fz$ as $n \to \infty$.*

Recall that the set $O(x; f) = \{f^n x : n = 0, 1, 2, \dots\}$ is called the orbit of the self-map f at the point $x \in \mathcal{X}$. Clearly, every continuous self-map of a metric space is orbitally continuous, but not conversely.

Definition 1.2. *[2] The space $(\mathcal{X}, d)$ is said to be f-orbitally complete if every Cauchy sequence contained in $O(x; f)$ (for some x in $\mathcal{X}$) converges in $\mathcal{X}$.*

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Obviously, every complete metric space is f-orbitally complete for any f, but an f-orbitally complete metric space need not be complete.

Sastry et al. [3] extended the above concepts to two and three mappings and employed them to prove common fixed point results for commuting mappings.

Definition 1.3. *Let f, g, h be three self-mappings defined on a metric space $(\mathcal{X}, d)$.*

1. *If for a point $x_0 \in \mathcal{X}$, there exits a sequence $\{x_n\}$ in $\mathcal{X}$ such that $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2, \dots$, then the set $O(x_0; f, g, h) = \{hx_n : n = 1, 2, \dots\}$ is called the orbit of (f, g, h) at x_0.*
2. *The space $(\mathcal{X}, d)$ is said to be (f, g, h)-orbitally complete at x_0 if every Cauchy sequence in $O(x_0; f, g, h)$ converges in $\mathcal{X}$.*
3. *The map h is said to be (f, g, h)-orbitally continuous at x_0 if it is continuous on $O(x_0; f, g, h)$.*
4. *The pair (f, g) is said to be asymptotically regular (in short a.r.) with respect to h at x_0 if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $hx_{2n+1} = fx_{2n}$, $hx_{2n+2} = gx_{2n+1}$, $n = 0, 1, 2, \dots$, and $d(hx_n, hx_{n+1}) \to 0$ as $n \to \infty$.*
5. *If h is the identity mapping on $\mathcal{X}$, we omit "h" in the respective definitions.*

Starting with the results of Ran and Reurings [4] and Nieto and Rodríguez-López [5], fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. Several authors obtained a lot of fixed point theorems and applied them in various situations (see, e.g., [6–10] and the references cited therein). In particular, the following notation and definitions will be used in the sequel.

If $(\mathcal{X}, \preceq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be well ordered if every two elements of $\mathcal{K}$ are comparable. If $g : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preceq y$ implies $gx \preceq gy$, then the mapping g is said to be non-decreasing.

Definition 1.4. *Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $f, g, h : \mathcal{X} \to \mathcal{X}$.*

1. *[11] The mapping g is called dominating if $x \preceq gx$ for each $x \in \mathcal{X}$.*
2. *[12, 13] The pair (f, g) is called weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in \mathcal{X}$.*
3. *[9] Let h be such that $f\mathcal{X} \subseteq h\mathcal{X}$ and $g\mathcal{X} \subseteq h\mathcal{X}$, and denote $h^{-1}(x) := \{u \in \mathcal{X} : hu = x\}$, for $x \in \mathcal{X}$. We say that (f, g) is weakly increasing with respect to h if for all $x \in \mathcal{X}$, we have:*
$$gx \preceq fy, \ \forall\, y \in h^{-1}(gx) \tag{1}$$
and
$$fx \preceq gy, \ \forall\, y \in h^{-1}(fx). \tag{2}$$
4. *The mapping f is called (f, g, h)-orbitally dominating at $x_0 \in \mathcal{X}$ if $x \preceq fx$ holds for $x \in O(x_0; f, g, h)$.*
5. *The pair (f, g) is orbitally (at x_0) weakly increasing with respect to h if* (1) *and* (2) *hold for all $x \in O(x_0; f, g, h)$.*

Remark 1.5. *(1) None of two weakly increasing mappings need be non-decreasing. There exist some examples to illustrate this fact in [14].*

(2) If h is the identity mapping ($hx = x$ for all $x \in \mathcal{X}$), then (f, g) is weakly increasing with respect to h if and only if f and g are weakly increasing mappings.

Example 1.6. *[11]* Let $\mathcal{X} = [0, 1]$ be endowed with the usual order $\le$. Let $g : \mathcal{X} \to \mathcal{X}$ be defined by $gx = \sqrt[n]{x}$. Since $x \le \sqrt[n]{x} = gx$ for all $x \in \mathcal{X}$, g is a dominating map.

Example 1.7. Let $\mathcal{X} = [0, +\infty)$ be endowed with the usual order $\le$. Define the mappings $f, g, h : \mathcal{X} \to \mathcal{X}$ by

$$gx = \begin{cases} x, & \text{if } 0 \le x < 1, \\ 0, & \text{if } 1 \le x, \end{cases} \qquad fx = \begin{cases} \sqrt{x}, & \text{if } 0 \le x < 1, \\ 0, & \text{if } 1 \le x, \end{cases} \qquad hx = \begin{cases} x^2, & \text{if } 0 \le x < 1, \\ 0, & \text{if } 1 \le x. \end{cases}$$

Then (f, g) is weakly increasing with respect to h.

Example 1.8. Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preceq y \iff x \geq y.$$

Consider the following self-mappings on $\mathcal{X}$:

$$hx = 6x, \qquad fx = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2}, \\ x, & x > \frac{1}{2}, \end{cases} \qquad gx = \begin{cases} \frac{1}{3}x, & 0 \leq x \leq \frac{1}{3}, \\ x, & x > \frac{1}{3}. \end{cases}$$

Take $x_0 = \frac{1}{2}$. It is easy to show that

$$O(x_0; f, g, h) \subset \left\{ \frac{1}{2^k \cdot 3^l} : k, l \in \mathbb{N} \right\}.$$

Then (f, g) is orbitally (at $x_0 = \frac{1}{2}$) weakly increasing with respect to h.

Throughout this paper, $(\mathcal{X}, d, \preceq)$ will be called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

The space $(\mathcal{X}, d, \preceq)$ will be called regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in $\mathcal{X}$ with respect to $\preceq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preceq z$.

In this paper, we prove existence and uniqueness results for common fixed points of two or three relatively asymptotically regular mappings satisfying the orbital continuity of one of the involved maps on ordered orbitally complete metric spaces. The so-called generalized Φ-contractive condition is utilized, which was introduced by Pathak and Tiwari in [15]. Also, we use orbitally dominating maps and orbitally weakly increasing maps.

We furnish suitable examples to demonstrate the usability of the hypotheses of our results. Finally, we apply these results to prove the existence of solutions of a system of integral equations.

2. Common Fixed Points for Relatively Orbitally Weakly Increasing Mappings

In the rest of the paper, following [15], we denote by Φ the collection of all functions $\varphi : [0, \infty) \to [0, \infty)$ which are upper semicontinuous from the right, non-decreasing and satisfy $\limsup_{s \to t^+} \varphi(s) < t$, $\varphi(t) < t$, for all $t > 0$.

The first result of this section is the following

Theorem 2.1. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let f, g and h be self-maps on $\mathcal{X}$ satisfying*

$$\begin{aligned} &[d^p(fx, gy) + a d^p(hx, hy)] d^p(fx, gy) \\ &\quad \leq a \max\{d^p(fx, hx) d^p(gy, hy), d^q(fx, hy) d^{q'}(gy, hx)\} \\ &\qquad + \max\Big\{\varphi_1(d^{2p}(hx, hy)), \varphi_2(d^r(fx, hx) d^{r'}(gy, hy)), \varphi_3(d^s(fx, hy) d^{s'}(gy, hx)), \\ &\qquad\quad \varphi_4\Big(\tfrac{1}{2}\Big[d^l(fx, hy) d^{l'}(fx, hx) + d^l(gy, hx) d^{l'}(gy, hy)\Big]\Big)\Big\}, \end{aligned} \tag{3}$$

for all $x, y \in \overline{O(x_0; f, g, h)}$ (for some x_0) such that hx and hy are comparable, and some $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \geq 0$ such that $2p = q + q' = r + r' = s + s' = l + l' \leq 1$.

We assume the following hypotheses:

(i) (f, g) is a.r. with respect to h at some $x_0 \in \mathcal{X}$;

(ii) $\mathcal{X}$ is (f, g, h)-orbitally complete at x_0;

(iii) (f, g) *is orbitally weakly increasing with respect to* h *at* x_0*;*

(iv) g *and* f *are* (f, g, h)*-orbitally dominating maps at* x_0*;*

(v) h *is monotone and orbitally continuous at* x_0*.*

Assume either

(a) f *and* h *are compatible; or*
(b) g *and* h *are compatible.*

Then f, g *and* h *have a common fixed point. Moreover, the set of common fixed points of* f, g *and* h *in* $\overline{O(x_0; f, g, h)}$ *is well ordered if and only if it is a singleton.*

Recall that the mappings f and h are said to be compatible if $\lim_{n\to\infty} d(fhx_n, hfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n$.

Proof. Since (f, g) is a.r. with respect to h at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$hx_{2n+1} = fx_{2n}, \qquad hx_{2n+2} = gx_{2n+1}, \qquad \forall\, n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, \tag{4}$$

and

$$\lim_{n\to\infty} d(hx_n, hx_{n+1}) = 0 \tag{5}$$

holds. We claim that

$$hx_n \preceq hx_{n+1}, \ \forall\, n \in \mathbb{N}_0. \tag{6}$$

To this aim, we will use the increasing property with respect to h satisfied by the pair (f, g). From (4), we have

$$hx_1 = fx_0 \preceq gy, \ \forall\, y \in h^{-1}(fx_0).$$

Since $hx_1 = fx_0$, then $x_1 \in h^{-1}(fx_0)$, and we get

$$hx_1 = fx_0 \preceq gx_1 = hx_2.$$

Again,

$$hx_2 = gx_1 \preceq fy, \ \forall\, y \in h^{-1}(gx_1).$$

Since $x_2 \in h^{-1}(gx_1)$, we get

$$hx_2 = gx_1 \preceq fx_2 = hx_3.$$

Hence, by induction, (6) holds. Therefore, we can apply (3) for $x = x_p$ and $y = x_q$ for all p and q.

Now, we assert that $\{hx_n\}$ is a Cauchy sequence in the metric space $O(x_0; f, g, h)$. We proceed by negation and suppose that $\{hx_{2n}\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \ d(hx_{2m(k)}, hx_{2n(k)}) \geq \varepsilon, \ d(hx_{2m(k)}, hx_{2n(k)-2}) < \varepsilon. \tag{7}$$

From (7) and using the triangular inequality, we get

$$\begin{aligned}\varepsilon &\leq d(hx_{2m(k)}, hx_{2n(k)})\\ &\leq d(hx_{2m(k)}, hx_{2n(k)-2}) + d(hx_{2n(k)-2}, hx_{2n(k)-1}) + d(hx_{2n(k)-1}, hx_{2n(k)})\\ &< \varepsilon + d(hx_{2n(k)-2}, hx_{2n(k)-1}) + d(hx_{2n(k)-1}, hx_{2n(k)}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (5), we obtain

$$\lim_{k\to\infty} d(hx_{2m(k)}, hx_{2n(k)}) = \varepsilon. \tag{8}$$

By making use of the triangle inequalities, for $\rho \in [0,1]$, we have

$$\begin{aligned} d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)+1}) \le{}& d^{\rho}(hx_{2m(k)+2}, hx_{2m(k)+1}) + d^{\rho}(hx_{2m(k)+1}, hx_{2m(k)}) \\ &+ d^{\rho}(hx_{2m(k)}, hx_{2n(k)}) + d^{\rho}(hx_{2n(k)}, hx_{2n(k)+1}), \end{aligned}$$

i.e., denoting $d_i := d(hx_i, hx_{i-1})$,

$$d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)+1}) - d^{\rho}(hx_{2m(k)}, hx_{2n(k)}) \le d^{\rho}_{2m(k)+2} + d^{\rho}_{2m(k)+1} + d^{\rho}_{2n(k)+1},$$

and

$$\begin{aligned} d^{\rho}(hx_{2m(k)}, hx_{2n(k)}) \le{}& d^{\rho}(hx_{2m(k)}, hx_{2m(k)+1}) + d^{\rho}(hx_{2m(k)+1}, hx_{2m(k)+2}) \\ &+ d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)+1}) + d^{\rho}(hx_{2n(k)+1}, hx_{2n(k)}), \end{aligned}$$

i.e.,

$$d^{\rho}(hx_{2m(k)}, hx_{2n(k)}) - d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)+1}) \le d^{\rho}_{2m(k)+1} + d^{\rho}_{2m(k)+2} + d^{\rho}_{2n(k)+1}.$$

Thus, we obtain

$$|d^{\rho}(hx_{2m(k)}, hx_{2n(k)}) - d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)+1})| \le d^{\rho}_{2m(k)+1} + d^{\rho}_{2m(k)+2} + d^{\rho}_{2n(k)+1}. \tag{9}$$

Similarly we have

$$|d^{\rho}(hx_{2m(k)+1}, hx_{2n(k)}) - d^{\rho}(hx_{2n(k)}, hx_{2m(k)})| \le d^{\rho}_{2m(k)+1}, \tag{10}$$

$$|d^{\rho}(hx_{2m(k)+1}, hx_{2n(k)+1}) - d^{\rho}(hx_{2n(k)}, hx_{2m(k)})| \le d^{\rho}_{2n(k)+1} + d^{\rho}_{2m(k)+1}, \tag{11}$$

and

$$|d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)}) - d^{\rho}(hx_{2n(k)}, hx_{2m(k)})| \le d^{\rho}_{2m(k)+1} + d^{\rho}_{2m(k)+2}. \tag{12}$$

Using relations (5), (9)–(12), we have

$$\begin{aligned} \lim_{k\to\infty} d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)+1}) &= \lim_{k\to\infty} d^{\rho}(hx_{2m(k)+1}, hx_{2n(k)}) = \lim_{k\to\infty} d^{\rho}(hx_{2m(k)+1}, hx_{2n(k)+1}) \\ &= \lim_{k\to\infty} d^{\rho}(hx_{2m(k)+2}, hx_{2n(k)}) = \epsilon. \end{aligned} \tag{13}$$

Now, using (3) with $x = x_{2m(k)+2}$ and $y = x_{2n(k)+1}$, we obtain

$$\begin{aligned} &[d^{p}(fx_{2m(k)+2}, gx_{2n(k)+1}) + ad^{p}(hx_{2m(k)+2}, hx_{2n(k)+1})]d^{p}(fx_{2m(k)+2}, gx_{2n(k)+1}) \\ &\quad\le a\max\{d^{p}(fx_{2m(k)+2}, hx_{2m(k)+2})d^{p}(gx_{2n(k)+1}, hx_{2n(k)+1}), d^{q}(fx_{2m(k)+2}, hx_{2n(k)+1})d^{q'}(gx_{2n(k)+1}, hx_{2m(k)+2})\} \\ &\qquad+ \max\{\varphi_1(d^{2p}(hx_{2m(k)+2}, hx_{2n(k)+1})), \varphi_2(d^{r}(fx_{2m(k)+2}, hx_{2m(k)+2})d^{r'}(hx_{2n(k)+1}, hx_{2n(k)+1})), \\ &\qquad\varphi_3(d^{s}(fx_{2m(k)+2}, hx_{2n(k)+1})d^{s'}(gx_{2n(k)+1}, hx_{2m(k)+2})), \\ &\qquad\varphi_4(\frac{1}{2}[d^{l}(fx_{2m(k)+2}, hx_{2n(k)+1})d^{l'}(fx_{2m(k)+2}, hx_{2m(k)+2}) + d^{l}(gx_{2n(k)+1}, hx_{2m(k)+2})d^{l'}(gx_{2n(k)+1}, hx_{2n(k)+1})])\}, \end{aligned}$$

i.e.,

$$\begin{aligned}&[d^p(hx_{2m(k)+2},hx_{2n(k)+1})+ad^p(hx_{2m(k)+1},hx_{2n(k)})]d^p(hx_{2m(k)+2},hx_{2n(k)+1})\\&\quad\le a\max\{d^p(hx_{2m(k)+2},hx_{2m(k)+1})d^p(hx_{2n(k)+1},hx_{2n(k)}),d^q(hx_{2m(k)+2},hx_{2n(k)+1})d^{q'}(hx_{2n(k)+1},hx_{2m(k)+1})\}\\&\qquad+\max\Big\{\varphi_1(d^{2p}(hx_{2m(k)+1},hx_{2n(k)})),\varphi_2(d^r(hx_{2m(k)+2},hx_{2m(k)+1})d^{r'}(hx_{2n(k)+1},hx_{2n(k)})),\\&\qquad\varphi_3(d^s(hx_{2m(k)+2},hx_{2n(k)})d^{s'}(hx_{2n(k)+1},hx_{2m(k)+1})),\\&\qquad\varphi_4\Big(\frac{1}{2}\Big[d^l(hx_{2m(k)+2},hx_{2n(k)})d^{l'}(hx_{2m(k)+2},hx_{2m(k)+1})+d^l(hx_{2n(k)+1},hx_{2m(k)+1})d^{l'}(hx_{2n(k)+1},hx_{2n(k)})\Big]\Big)\Big\}.\end{aligned}$$

Passing to the limit as $k\to\infty$ and using (5), (8) and (13) and the fact that $\varphi_i\in\Phi$ ($i=1,2,3,4$), we have

$$\epsilon^{2p}+a\epsilon^{2p}\le a\epsilon^{q+q'}+\max\{\varphi_1(\epsilon^{2p}),\varphi_2(0),\varphi_3(\epsilon^{s+s'}),\varphi_4(0)\},$$

i.e.,

$$\epsilon^{2p}\le\max\{\varphi_1(\epsilon^{2p}),\varphi_2(0),\varphi_3(\epsilon^{s+s'}),\varphi_4(0)\},$$

i.e.,

$$\epsilon^{2p}\le\varphi(\epsilon^{2p})<\epsilon^{2p},$$

a contradiction. Hence, $\{hx_{2n}\}$ is a Cauchy sequence in $\mathcal{X}$. This proves that $\{hx_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0;f,g,h)$. Since $\mathcal{X}$ is (f,g,h)-orbitally complete at x_0, there exists some $z\in\mathcal{X}$ such that

$$hx_n\to z \text{ as } n\to\infty. \tag{14}$$

We will prove that z is a common fixed point of the three mappings f,g and h.

We have

$$fx_{2n}=hx_{2n+1}\to z \text{ as } n\to\infty \tag{15}$$

and

$$gx_{2n+1}=hx_{2n+2}\to z \text{ as } n\to\infty. \tag{16}$$

Suppose that (a) holds, i.e., f and h are compatible. Then, using condition (v),

$$\lim_{n\to\infty}fhx_{2n+2}=\lim_{n\to\infty}hfx_{2n+2}=hz. \tag{17}$$

From (14) and the orbitally continuity of h, we have also

$$h(hx_n)\to hz \text{ as } n\to\infty. \tag{18}$$

Now, using (iv), $x_{2n+1}\preceq gx_{2n+1}=hx_{2n+2}$ and since h is monotone, hx_{2n+1} and hhx_{2n+2} are comparable. Thus, we can apply (3) to obtain

$$\begin{aligned}&[d^p(fhx_{2n+2},gx_{2n+1})+ad^p(hhx_{2n+2},hx_{2n+1})]d^p(fhx_{2n+2},gx_{2n+1})\\&\le a\max\{d^p(fhx_{2n+2},hhx_{2n+2})d^p(gx_{2n+1},hx_{2n+1}),d^q(fhx_{2n+2},hx_{2n+1})d^{q'}(gx_{2n+1},hhx_{2n+2})\}\\&\quad+\max\Big\{\varphi_1(d^{2p}(hhx_{2n+2},hx_{2n+1})),\varphi_2(d^r(fhx_{2n+2},hhx_{2n+2})d^{r'}(gx_{2n+1},hx_{2n+1})),\\&\quad\varphi_3(d^s(fhx_{2n+2},hx_{2n+1})d^{s'}(gx_{2n+1},hhx_{2n+2})),\\&\quad\varphi_4\Big(\tfrac{1}{2}\Big[d^l(fhx_{2n+2},hx_{2n+1})d^{l'}(fhx_{2n+2},hhx_{2n+2})+d^l(gx_{2n+1},hhx_{2n+2})d^{l'}(gx_{2n+1},hx_{2n+1})\Big]\Big)\Big\}.\end{aligned} \tag{19}$$

Passing to the limit as $n \to \infty$ in (19), using (14)–(18), we obtain

$$d^{2p}(hz,z) \le ad^{q+q'}(z,hz) + \max\{\varphi_1(d^{2p}(hz,z)), \varphi_2(0), \varphi_3(d^{s+s'}(z,hz)), \varphi_4(0)\}$$

i.e.,

$$d^{2p}(hz,z) \le \frac{a}{(1+a)} d^{2p}(z,hz) + \frac{1}{(1+a)} \varphi(d^{2p}(hz,z))$$

that is

$$d^{2p}(hz,z) < d^{2p}(hz,z)$$

unless

$$hz = z. \tag{20}$$

Now, $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \to z$ as $n \to \infty$, so by the assumption we have $x_{2n+1} \preceq z$ and hx_{2n+1} and hz are comparable. Hence (3) gives

$$\begin{aligned}
&[d^p(fz, gx_{2n+1}) + ad^p(hz, hx_{2n+1})]d^p(fz, gx_{2n+1}) \\
&\quad \le a \max\{d^p(fz,hz)d^p(gx_{2n+1}, hx_{2n+1}), d^q(fz, hx_{2n+1})d^{q'}(gx_{2n+1}, hz)\} \\
&\qquad + \max\Big\{\varphi_1(d^{2p}(hz, hx_{2n+1})), \varphi_2(d^r(fz,hz)d^{r'}(gx_{2n+1}, hx_{2n+1})), \varphi_3(d^s(fz, hx_{2n+1})d^{s'}(gx_{2n+1}, hz)), \\
&\qquad\quad \varphi_4\Big(\tfrac{1}{2}\Big[d^l(fz, hx_{2n+1})d^{l'}(fz,hz) + d^l(gx_{2n+1}, hz)d^{l'}(gx_{2n+1}, hx_{2n+1})\Big]\Big)\Big\},
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality and using (20), it follows that

$$d^{2p}(fz,z) \le \max\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(\tfrac{1}{2}d^{l+l'}(fz,z))\} < d^{2p}(fz,z))$$

which holds unless

$$fz = z. \tag{21}$$

Similarly, $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to z$ as $n \to \infty$, implies that $x_{2n} \preceq z$, hence hx_{2n} and hz are comparable. From (3) we get

$$\begin{aligned}
&[d^p(fx_{2n}, gz) + ad^p(hx_{2n}, hz)]d^p(fx_{2n}, gz) \\
&\quad \le a \max\{d^p(fx_{2n}, hx_{2n})d^p(gz,hz), d^q(fx_{2n}, hz)d^{q'}(gz, hx_{2n})\} \\
&\qquad + \max\Big\{\varphi_1(d^{2p}(hx_{2n}, hz)), \varphi_2(d^r(fx_{2n}, hx_{2n})d^{r'}(gz,hz)), \varphi_3(d^s(fx_{2n}, hz)d^{s'}(gz, hx_{2n})), \\
&\qquad\quad \varphi_4\Big(\tfrac{1}{2}\Big[d^l(fx_{2n}, hz)d^{l'}(fx_{2n}, hx_{2n}) + d^l(gz, hx_{2n})d^{l'}(gz,hz)\Big]\Big)\Big\}.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$d^{2p}(z,gz) \le \max\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(\tfrac{1}{2}d^{l+l'}(gz,z))\} < d^{2p}(gz,z))$$

which gives that

$$z = gz. \tag{22}$$

Therefore, $fz = gz = hz = z$, hence z is a common fixed point of f, g and h.

Similarly, the result follows when condition (b) holds.

Now, suppose that the set of common fixed points of f, g and h in $\overline{O(x_0; f, g, h)}$ is well ordered. We claim that there is a unique common fixed point of f, g and h in $\overline{O(x_0; f, g, h)}$. Assume to the contrary that

$fu = gu = hu = u$ and $fv = gv = hv = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (3) to obtain

$$\begin{aligned}
&[d^p(fu, gv) + ad^p(hu, hv)]d^p(fu, gv)\\
&\quad \le a\max\{d^p(fu, hu)d^p(gv, hv), d^q(fu, hv)d^{q'}(gv, hu)\}\\
&\qquad + \max\Big\{\varphi_1(d^{2p}(hu, hv)), \varphi_2(d^r(fu, hu)d^{r'}(gv, hv)), \varphi_3(d^s(fu, hv)d^{s'}(gv, hu)),\\
&\qquad\quad \varphi_4\Big(\tfrac{1}{2}\Big[d^l(fu, hv)d^{l'}(fu, hu) + d^l(gv, hu)d^{l'}(gv, hv)\Big]\Big)\Big\},
\end{aligned}$$

that is

$$(1 + a)d^{2p}(u, v) \le ad^{q+q'}(u, v) + \max\{\varphi_1(d^{2p}(u, v)), \varphi_2(0), \varphi_3(d^{s+s'}(u, v)), \varphi_4(0)\}$$

i.e.,

$$d^{2p}(u, v) \le \frac{a}{1 + a}d^{q+q'}(u, v) + \frac{1}{1 + a}\varphi(d^{2p}(u, v)) < d^{2p}(u, v)$$

a contradiction. Hence, $u = v$. The converse is trivial. □

Remark 2.2. *It was shown by examples in [16] that (in similar situations):*
(1) if the contractive condition is satisfied just on $O(x_0; f, g, h)$, there might not exist a (common) fixed point;
(2) under the given hypotheses (common) fixed point might not be unique in the whole space $\mathcal{X}$.

2.1. Some special cases

The following are some consequences of the main result.

If h = identity mapping in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let g and f be self-maps on $\mathcal{X}$ satisfying*

$$\begin{aligned}
&[d^p(fx, gy) + ad^p(x, y)]d^p(fx, gy)\\
&\quad \le a\max\{d^p(fx, x)d^p(gy, y), d^q(fx, y)d^{q'}(gy, x)\}\\
&\qquad + \max\Big\{\varphi_1(d^{2p}(x, y)), \varphi_2(d^r(fx, x)d^{r'}(gy, y)), \varphi_3(d^s(fx, y)d^{s'}(gy, x)),\\
&\qquad\quad \varphi_4\Big(\tfrac{1}{2}\Big[d^l(fx, y)d^{l'}(fx, x) + d^l(gy, x)d^{l'}(gy, y)\Big]\Big)\Big\},
\end{aligned}$$

for all $x, y \in \overline{O(x_0; f, g)}$ (for some x_0) such that x and y are comparable, and some $\varphi_i \in \Phi$ ($i = 1, 2, 3, 4$), $a, p, q, q', r, r', s, s', l, l' \ge 0$ such that $2p = q + q' = r + r' = s + s' = l + l' \le 1$.

We assume the following hypotheses:

(i) (f, g) is a.r. at some point $x_0 \in \mathcal{X}$;

(ii) $\mathcal{X}$ is (f, g)-orbitally complete at x_0;

(iii) g and f are orbitally weakly increasing at x_0;

(iv) g and f are (f, g)-orbitally dominating maps at x_0.

Then g and f have a common fixed point. Moreover, the set of common fixed points of g and f in $\overline{O(x_0; f, g)}$ is well ordered if and only if it is a singleton.

If $f = g$ in Theorem 2.1, we have the following result.

Corollary 2.4. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let g and h be self-maps on $\mathcal{X}$ satisfying*

$$\begin{aligned}
&[d^p(gx, gy) + ad^p(hx, hy)]d^p(gx, gy)\\
&\quad\le a\max\{d^p(gx, hx)d^p(gy, hy), d^q(gx, hy)d^{q'}(gy, hx)\}\\
&\qquad+\max\Big\{\varphi_1(d^{2p}(hx, hy)), \varphi_2(d^r(gx, hx)d^{r'}(gy, hy)), \varphi_3(d^s(gx, hy)d^{s'}(gy, hx)),\\
&\qquad\quad\varphi_4\Big(\tfrac{1}{2}\Big[d^l(gx, hy)d^{l'}(gx, hx) + d^l(gy, hx)d^{l'}(gy, hy)\Big]\Big)\Big\},
\end{aligned}$$

for all $x, y \in \overline{\mathcal{O}(x_0; g, h)}$ (for some x_0) such that hx and hy are comparable, and some $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$, $a, p, q, q', r, r', s, s', l, l' \ge 0$ such that $2p = q + q' = r + r' = s + s' = l + l' \le 1$.

We assume the following hypotheses:

(i) g is a.r. with respect to h at $x_0 \in \mathcal{X}$;

(ii) $\mathcal{X}$ is (g, h)-orbitally complete at x_0;

(iii) g is orbitally weakly increasing with respect to h at x_0;

(iv) g is a (g, h)-orbitally dominating map at x_0;

(v) h is monotone and orbitally continuous at x_0.

Then g and h have a common fixed point. Moreover, the set of common fixed points of g and h in $\overline{\mathcal{O}(x_0; g, h)}$ is well ordered if and only if it is a singleton.

If h = identity mapping in the Corollary 2.4, we have the following result.

Corollary 2.5. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let g be a self-map on $\mathcal{X}$ satisfying for all $x, y \in \overline{\mathcal{O}(x_0; g)}$ such that x and y are comparable,*

$$\begin{aligned}
&[d^p(gx, gy) + ad^p(x, y)]d^p(gx, gy)\\
&\quad\le a\max\{d^p(gx, x)d^p(gy, y), d^q(gx, y)d^{q'}(gy, x)\}\\
&\qquad+\max\Big\{\varphi_1(d^{2p}(x, y)), \varphi_2(d^r(gx, x)d^{r'}(gy, y)), \varphi_3(d^s(gx, y)d^{s'}(gy, x)),\\
&\qquad\quad\varphi_4\Big(\tfrac{1}{2}\Big[d^l(gx, y)d^{l'}(gx, x) + d^l(gy, x)d^{l'}(gy, y)\Big]\Big)\Big\},
\end{aligned}$$

for some $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$, $a, p, q, q', r, r', s, s', l, l' \ge 0$ such that $2p = q + q' = r + r' = s + s' = l + l' \le 1$.

We assume the following hypotheses:

(i) g is a.r. at some point x_0 of $\mathcal{X}$;

(ii) $\mathcal{X}$ is g-orbitally complete at x_0;

(iii) $gx \preceq g(gx)$ for all $x \in \mathcal{O}(g; x_0)$;

(iv) g is an orbitally dominating map at x_0.

Then g has a fixed point. Moreover, the set of fixed points of g in $\overline{\mathcal{O}(x_0; g)}$ is well ordered if and only if it is a singleton.

If $a = 0$ in Theorem 2.1, we have the following consequence.

Theorem 2.6. *Let $(\mathcal{X}, d, \preceq)$ be a regular ordered metric space and let f, g and h be self-maps on $\mathcal{X}$ satisfying*

$$\begin{aligned}
d^{2p}(fx, gy) \le \max\Big\{&\varphi_1(d^{2p}(hx, hy)), \varphi_2(d^r(fx, hx)d^{r'}(gy, hy)), \varphi_3(d^s(fx, hy)d^{s'}(gy, hx)),\\
&\varphi_4\Big(\tfrac{1}{2}\Big[d^l(fx, hy)d^{l'}(fx, hx) + d^l(gy, hx)d^{l'}(gy, hy)\Big]\Big)\Big\}, \qquad (23)
\end{aligned}$$

for all $x, y \in \overline{O(x_0; f, g, h)}$ *(for some* x_0*) such that* hx *and* hy *are comparable, and some* $\varphi_i \in \Phi$ $(i = 1, 2, 3, 4)$*,* $p, r, r', s, s', l, l' \geq 0$ *such that* $2p = r + r' = s + s' = l + l' \leq 1$*. Suppose that the conditions (i)–(v) and (a) or (b) of Theorem 2.1 hold. Then* f, g *and* h *have a common fixed point. Moreover, the set of common fixed points of* f, g *and* h *in* $\overline{O(x_0; f, g, h)}$ *is well ordered if and only if it is a singleton.*

In Theorem 2.1, if we put $a = 0$ and $\varphi_i(t) = kt$ $(i = 1, 2, 3, 4)$, where $0 < k < 1$, we get the following consequence.

Theorem 2.7. *Let* $(\mathcal{X}, d, \preceq)$ *be a regular ordered metric space and let* f*,* g *and* h *be self-maps on* $\mathcal{X}$ *satisfying*

$$d^{2p}(fx, gy) \leq k \max\Big\{d^{2p}(hx, hy), d^r(fx, hx)d^{r'}(gy, hy), d^s(fx, hy)d^{s'}(gy, hx), \\ \tfrac{1}{2}\Big[d^l(fx, hy)d^{l'}(fx, hx) + d^l(gy, hx)d^{l'}(gy, hy)\Big]\Big\}, \tag{24}$$

for all $x, y \in \overline{O(x_0; f, g, h)}$ *(for some* x_0*) such that* hx *and* hy *are comparable, and some* k*,* $0 < k < 1$*,* $p, r, r', s, s', l, l' \geq 0$ *such that* $2p = r + r' = s + s' = l + l' \leq 1$*. Suppose that the conditions (i)–(v) and (a) or (b) of Theorem 2.1 hold. Then* f, g *and* h *have a common fixed point. Moreover, the set of common fixed points of* f, g *and* h *in* $\overline{O(x_0; f, g, h)}$ *is well ordered if and only if it is a singleton.*

Example 2.8. Let $\mathcal{X} = [0, +\infty)$ be equipped with standard metric and order. Consider the mappings $f, g, h : \mathcal{X} \to \mathcal{X}$ given by

$$fx = \begin{cases} \frac{3+x}{4}, & 0 \leq x \leq 1 \\ 4x - 3, & x > 1, \end{cases} \qquad gx = \begin{cases} \frac{2+x}{3}, & 0 \leq x \leq 1 \\ 3x - 2, & x > 1, \end{cases} \qquad hx = \begin{cases} \frac{1+x}{2}, & 0 \leq x \leq 1 \\ 5x - 4, & x > 1. \end{cases}$$

Conditions (i)–(v) and (a) (or (b)) of Theorem 2.1 are easy to check. For example, in order to check condition (iii), take $x_0 = 0$. Then $O(x_0; f, g, h) \subset [0, 1]$. If $x, y \in [0, 1]$ are such that $hy = gx$ then $y = \frac{2x+1}{3}$ and $gx = \frac{2+x}{3} \leq \frac{x+5}{6} = \frac{3+y}{4} = fy$; similarly, $hy = fx$ implies that $fx \leq gy$. Note, however, that conditions (iii) and (iv) are not fulfilled on the whole space $\mathcal{X}$.

We will prove now that condition (24) of Theorem 2.7 is fulfilled with $x_0 = 0$, $p = r = r' = s = s' = l = l' = \frac{1}{2}$, $k = \frac{2}{3}t$. Take $x, y \in O(0; f, g, h) \subset [0, 1]$. Then (24) becomes

$$\left|\frac{3+x}{4} - \frac{2+y}{3}\right| \leq \frac{2}{3}\max\left\{\left|\frac{x-y}{2}\right|, \sqrt{\frac{1-x}{4}\cdot\frac{1-y}{6}}, \sqrt{\left|\frac{3+x}{4} - \frac{1+y}{2}\right|\cdot\left|\frac{2+y}{3} - \frac{1+x}{2}\right|}, \\ \frac{1}{2}\left[\sqrt{\left|\frac{3+x}{4} - \frac{1+y}{2}\right|\cdot\frac{1-x}{4}} + \sqrt{\left|\frac{2+y}{3} - \frac{1+x}{2}\right|\cdot\frac{1-y}{6}}\right]\right\}.$$

Using the substitution $x = 1 - \xi$, $y = 1 - \xi t$, $0 \leq \xi \leq 1$, $t \geq 0$, the previous inequality becomes

$$\left|\frac{t}{3} - \frac{1}{4}\right| \leq \frac{2}{3}\max\left\{\left|\frac{1-t}{2}\right|, \sqrt{\frac{t}{24}}, \sqrt{\left|\frac{t}{2} - \frac{1}{4}\right|\cdot\left|\frac{1}{2} - \frac{t}{3}\right|}, \frac{1}{2}\left[\sqrt{\frac{1}{4}\left|\frac{t}{2} - \frac{1}{4}\right|} + \sqrt{\frac{1}{5}\left|\frac{1}{2} - \frac{t}{3}\right|}\right]\right\},$$

and can be checked by discussion on possible values of $t \geq 0$. Note, again, that condition (24) does not hold for all $x, y \in \mathcal{X}$.

Thus, f, g and h have a (unique) common fixed point (which is $z = 1$).

3. Application to Systems of Integral Equations

Consider the following system of integral equations:

$$\begin{cases} u(t) = \int_0^T K_1(t, s, u(s))\, ds + w(t), \\ u(t) = \int_0^T K_2(t, s, u(s))\, ds + w(t), \end{cases} \tag{25}$$

$t \in I = [0, T]$, where $T > 0$. The purpose of this section is to give an existence theorem for a solution of the system (25) using Theorem 2.7.

Let $\ll$ be an arbitrary partial order relation on $\mathbb{R}^n$. Let $\mathcal{X} := C(I, \mathbb{R}^n)$ with the usual supremum norm, i.e., $\|x\|_{\mathcal{X}} = \max_{t\in I} \|x(t)\|$, for $x \in C(I, \mathbb{R}^n)$. Consider on $\mathcal{X}$ the partial order defined by

$$x \preceq y \text{ if and only if } x(t) \ll y(t) \text{ for any } t \in [0, T].$$

Then $(\mathcal{X}, \preceq)$ is a partially ordered set. Also $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a complete metric space. Moreover for any increasing sequence $\{x_n\}$ in $\mathcal{X}$ converging to $x^* \in \mathcal{X}$, we have $x_n(t) \ll x^*(t)$ for any $t \in [0, T]$.

Define $f, g : \mathcal{X} \to \mathcal{X}$ by

$$fx(t) = \int_0^T K_1(t, s, x(s))ds + w(t),\ t \in [0, T],$$

and

$$gx(t) = \int_0^T K_2(t, s, x(s))ds + w(t),\ t \in [0, T].$$

Theorem 3.1. *Consider the integral equations (25). Assume the following hypotheses:*
(i) *$K_1, K_2 : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $w : \mathbb{R}^n \to \mathbb{R}^n$ are continuous,*
(ii) *for each $t, s \in [0, T]$,*

$$K_1(t, s, x(s)) \ll K_2(t, s, \int_0^T K_1(s, \tau, x(\tau))\,d\tau + w(s)),$$
$$K_2(t, s, x(s)) \ll K_1(t, s, \int_0^T K_2(s, \tau, x(\tau))\,d\tau + w(s)),$$

(iii) *there exists a continuous function $G : I \times I \to \mathbb{R}_+$ such that*

$$\begin{aligned}&\|K_1(t, s, u(t)) - K_2(t, s, v(t))\|^p\\ &\le G(t, s) \max\left\{\begin{array}{c}\|u(t) - v(t)\|^{2p}, \|fu(t) - u(t)\|^r \|gv(t) - v(t)\|^{r'},\\ \|fu(t) - v(t)\|^s \|gv(t) - u(t)\|^{s'},\\ \frac{1}{\sqrt{2}}[\ \|fu(t) - v(t)\|^l \|fu(t) - u(t)\|^{l'} + \|gv(t) - u(t)\|^l \|gv(t) - v(t)\|^{l'}\ ]\end{array}\right\}\end{aligned}$$

for each $t, s \in I$ and comparable $u, v \in \mathcal{X}$, where $p, r, r', s, s', l, l' \ge 0$ with $2p = r + r' = s + s' = l + l' \le 1$;
(iv) *$\max_{t\in I} \int_0^T G(t, s)\,ds = \alpha < T^{-p}$.*
(v) *$x \preceq fx$ and $x \preceq gx$ for all $x \in \mathcal{X}$.*
Then the system of integral equations (25) *has a solution u^* in $C(I, \mathbb{R}^n)$.*

Proof. By assumption **(ii)**, we have, for all $t \in [0, T]$,

$$\begin{aligned}fx(t) &= \int_0^T K_1(t, s, x(s))\,ds + w(t)\\ &\ll \int_0^T K_2(t, s, \int_0^T K_1(s, \tau, x(\tau))\,d\tau + w(s))\,ds + w(t)\\ &= \int_0^T K_2(t, s, fx(s))ds + w(t)\\ &= gfx(t)\end{aligned}$$

and

$$\begin{aligned}
gx(t) &= \int_0^T K_2(t,s,x(s))\,ds + w(t)\\
&\ll \int_0^T K_1(t,s,\int_0^T K_2(s,\tau x(\tau))\,d\tau + w(s))\,ds + w(t)\\
&= \int_0^T K_1(t,s,fx(s))\,ds + w(t)\\
&= fgx(t).
\end{aligned}$$

Thus, we have $gx \preceq fgx$ and $fx \preceq gfx$ for all $x \in \mathcal{X}$. This shows that g and f are weakly increasing. From assumption **(v)**, f and g are dominating maps.

Also, for each comparable $u, v \in \mathcal{X}$, by **(iii)** and **(iv)**, we have:

$$\begin{aligned}
\|fu(t) - gv(t)\| &\le \int_0^T \|K_1(t,s,u(s)) - K_2(t,s,v(s))\|\,ds\\
&\le T\left(\int_0^T G(t,s)\,ds\right)^{1/p} \max\left\{ \begin{array}{c} \|u(t)-v(t)\|^{2p}, \|fu(t)-u(t)\|^r \|gv(t)-v(t)\|^{r'},\\ \|fu(t)-v(t)\|^s \|gv(t)-u(t)\|^{s'},\\ \frac{1}{\sqrt{2}}[\,\|fu(t)-v(t)\|^l \|fu(t)-u(t)\|^{l'} + \|gv(t)-u(t)\|^l \|gv(t)-v(t)\|^{l'}\,] \end{array} \right\}^{1/p}\\
&\le T\alpha^{1/p} \max\left\{ \begin{array}{c} d^{2p}(u,v), d^r(fu,u)d^{r'}(gv,v), d^s(fu,v)d^{s'}(gv,u),\\ \frac{1}{\sqrt{2}}[d^l(fu,v)d^{l'}(fu,u) - d^l(gv,u)d^{l'}(gv,v)] \end{array} \right\}^{1/2p}.
\end{aligned}$$

On routine calculations, we get

$$d^{2p}(fu,gv) \le \alpha^2 T^{2p} \max\left\{ \begin{array}{c} d^{2p}(u,v), d^r(fu,u)d^{r'}(gv,v), d^s(fu,v)d^{s'}(gv,u),\\ \frac{1}{2}[d^l(fu,v)d^{l'}(fu,u) + d^l(gv,u)d^{l'}(gv,v)] \end{array} \right\}$$

for each comparable $u, v \in \mathcal{X}$. Then, Theorem 2.7 is applicable, where h is the identity mapping and $k = \alpha^2 T^{2p} \in (0,1)$. So f and g have a common fixed point. Thus, there exists a $u^* \in C(I)$, a common fixed point of f and g, that is u^* is a solution to (25). □

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