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Left Big Subsets of Topological Polygroups

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Abstract. We define and study the properties of left big subsets of a topological polygroup and give some new results.

1. Introduction

Applications of hypergroups have mainly appeared in special subclasses. For example polygroups, which are a certain subclass of hypergroups, are used to study color algebra and combinatorics [3, 4]. A polygroup is completely regular, reversible in its hypergroup. A new monograph [7] is devoted to the study polygroup theory. It contains all the previously known results in polygroup theory.

Complete parts were introduced and studied for the first time by Koskas [16]. Later, this topic was analyzed by Corsini [5] and Sureau [18] mostly in the general theory of hypergroups. This concept was studied by mostly authors, for example see [5–7, 10, 11, 13]. In [10], Heidari et al. introduced and studied the concept of topological hypergroups as a generalization of topological groups. A topological hypergroup is a non-empty set endowed with two structures: that of a topological space and that of a hypergroup. These structures are connected in a way that algebraic properties of the hypergroup affect topological properties of the space and vice versa [10]. It is important to mention that this topological hypergroups are different from topological hypergroups which were initiated by Dunkl [8] and Jewett [15].

By using complete parts in topological polygroups, some interesting results were obtained by Heidari et al. [11]. In this paper, we establish some simple facts about complete parts in polygroups and we use these facts to obtain some new results in topological polygroups.

2. Preliminaries and elementary observations

A *hyperstructure* is a non-empty set H together with a *hyperoperation* $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ A = \{x\} \circ A$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, we have $x \circ H = H = H \circ x$. This condition is called the *reproduction axiom*. A quasihypergroup $(H, \circ)$ which is a semihypergroup is called a *hypergroup*, see [5, 6].

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The notion of a hypergroup was introduced for the first time by Marty [17]. For all $n > 1$, we define the relation β_n on a semihypergroup H as follows:

$$a\beta_n b \Leftrightarrow \exists(x_1, x_2, \ldots, x_n) \in H^n \text{ such that } \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and $\beta = \bigcup_{i=1}^{\infty} \beta_i$, where $\beta_1 = \{(x,x) | x \in H\}$. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Theorem 2.1. ([9]) *If $(H, \circ)$ is a hypergroup, then $\beta = \beta^*$.*

The relation β^* is called the *fundamental relation* on H and H/β^* is called the *fundamental group*. The fundamental relation β^* was introduced on hypergroups by Koskas [16] for the first time and studied by many authors, for example see [5–7, 19]. The fundamental relation is defined on hypergroups as the smallest equivalence relation so that the quotient would be a group.

Let $(H, \circ)$ be a semihypergroup and A be a non-empty subset of H. We say that A is a *complete part* of H if for any natural number n and for all $a_1, a_2, \ldots, a_n \in H$, the following implication holds:

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A.$$

Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two hypergroups. A map $f : H_1 \to H_2$, is called (1) a *homomorphism* if for every $x, y \in H$, we have $f(x \circ_1 y) \subseteq f(x) \circ_2 f(y)$; (2) a *good homomorphism* if for all $x, y \in H$, we have $f(x \circ_1 y) = f(x) \circ_2 f(y)$; (3) an *isomorphism* if it is a homomorphism, and f^{-1} is a homomorphism, too. A special subclass of hypergroups is the class of polygroups. A *polygroup* [3] is a system $P = \langle P, \circ, e, ^{-1} \rangle$, where $\circ : P \times P \to \mathcal{P}^*(P)$, $e \in P$, $^{-1}$ is a unitary operation on P and the following axioms hold for all $x, y, z \in P$: (1) $(x \circ y) \circ z = x \circ (y \circ z)$; (2) $e \circ x = x \circ e = x$; (3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$. The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. A non-empty subset K of a polygroup P is a *subpolygroup* of P if and only if $a, b \in K$ implies $a \circ b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. The subpolygroup N of P is *normal* in P if and only if $a^{-1} \circ N \circ a \subseteq N$ for every $a \in P$. An immediate consequence of the definition of a polygroup is the following result:

Proposition 2.2. *For any subsets A, B and C of a polygroup $P = \langle P, \circ, e, ^{-1} \rangle$, $(A \circ B) \cap C = \emptyset$ if and only if $A \cap (C \circ B^{-1}) = \emptyset$.*

The following proposition is surely known in folklore, but we give its proof for the sake of completeness.

Proposition 2.3. *Let A and B be non-empty subsets of a polygroup $P = \langle P, \circ, e, ^{-1} \rangle$ such that A is a complete part and $x \in P$. Then,*

(1) $x^{-1} \circ x \circ A = x \circ x^{-1} \circ A = A$;

(2) A^{-1} is a complete part;

(3) $x \circ A$ and $A \circ x$ are complete parts;

(4) $B \subseteq x^{-1} \circ A$ if and only if $x \circ B \subseteq A$.

Proof. (1) Clearly, $A \subseteq x^{-1} \circ x \circ A$. If $t \in A$, then $t \in ((x^{-1} \circ x) \circ t) \cap A$, and so $(x^{-1} \circ x) \circ t \subseteq A$ since A is a complete part. Thus, $(x^{-1} \circ x) \circ A = \bigcup_{t \in A} (x^{-1} \circ x) \circ t \subseteq A$. Hence, $(x^{-1} \circ x) \circ A = A$. Similarly, $x \circ x^{-1} \circ A = A$

(2) The proof is straightforward.

(3) If $\emptyset \neq (x \circ A) \cap \prod_{i=1}^{n} a_i$, where $n \in \mathbb{N}$ and $(a_1, \ldots, a_n) \in P^n$, then $\emptyset \neq x^{-1} \circ ((x \circ A) \cap \prod_{i=1}^{n} a_i) \subseteq (x^{-1} \circ x \circ A) \cap (x^{-1} \circ \prod_{i=1}^{n} a_i) = A \cap (x^{-1} \circ \prod_{i=1}^{n} a_i)$ by (1). Since A is a complete part we have $x^{-1} \circ \prod_{i=1}^{n} a_i \subseteq A$ and so $\prod_{i=1}^{n} a_i \subseteq x \circ A$. Thus, $x \circ A$ is a complete part. Similarly, $A \circ x$ is a complete part.

(4) Let $B \subseteq x^{-1} \circ A$, then $x \circ B \subseteq x \circ x^{-1} \circ A = A$ by (1). The converse is obvious. □

3. Complete Parts in Topological Polygroups

Until now, only a few papers treated the notion of topological hyperstructures, in the classical case, see [2, 10–12]. Topological polygroups are studied in [11] and by considering the relative topology on subpolygroups, the authors studied some properties of them. A topological polygroup is a polygroup P together with a topology on P such that the polygroup's binary hyperoperation and the polygroup's inverse function are continuous functions with respect to the topology.

Thus, one may perform algebraic hyperoperations, because of the polygroup structure, and one may talk about continuous functions, because of the topology.

In this section, we study the concept of topological polygroups and we prove some new results in this respect. Some applications of complete parts in topological polygroups are investigated.

Lemma 3.1. ([12]) *Let (H, τ) be a topological space. Then, the family $\mathcal{B} = \{S_V |\ V \in \tau\}$, where $S_V = \{U \in \mathcal{P}^*(H) |\ U \subseteq V\}$ is a base for a topology on $\mathcal{P}^*(H)$. This topology is denoted by τ^*.*

Let (H, τ) be a topological space. We consider the product topology on $H \times H$ and the topology τ^* on $\mathcal{P}^*(H)$.

Theorem 3.2. *Let $(H, \circ)$ be a hypergroupoid, τ be a topology on H and $A, B \subseteq H$. If the hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$ is continuous, then the following conditions hold.*

(1) The subspace $\{\{x\} \mid x \in H\}$ of $\mathcal{P}^*(H)$ and H are homeomorphic;

(2) If A and B are compact (respectively, countably compact, separable) subsets of H, then $\circ(A \times B) = \{\circ(a, b) | a \in A, b \in B\}$ is a compact (respectively, countably compact, separable) subset of $\mathcal{P}^*(H)$.

Proof. (1) It is easily seen that the mapping ϕ from H to $\{\{x\} | x \in H\}$ defined by $\phi(x) = \{x\}$ is a homeomorphism.

(2) Let A and B be compact (respectively, countably compact, separable) subsets of H. Then, $A \times B$ is compact (respectively, countably compact, separable). Since the continuous image of a compact (respectively, countably compact, separable) subset is compact (respectively, countably compact, separable), the set $\circ(A \times B)$ is compact (respectively, countably compact, separable). □

Definition 3.3. ([11]) Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup and (P, τ) be a topological space. Then, the system $P = (P, \circ, e, ^{-1}, \tau)$ is called a *topological polygroup* if the mappings $\mu : P \times P \to \mathcal{P}^*(P)$ and $\iota : P \to P$ defined by $\mu(x, y) = x \circ y$ and $\iota(x) = x^{-1}$ are continuous.

Lemma 3.4. ([11]) *Let P be a topological polygroup. Then, the hyperoperation $\circ : P \times P \to \mathcal{P}^*(P)$ is continuous if and only if for every $x, y \in P$ and $U \in \tau$ such that $x \circ y \subseteq U$ there exist $V, W \in \tau$ such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$.*

Lemma 3.5. ([11]) *Let P be a topological polygroup. Then, the mappings*

(1) ${}_a\varphi : P \to \mathcal{P}^*(P)$ defined by ${}_a\varphi(x) = a \circ x$;

(2) $\varphi_a : P \to \mathcal{P}^*(P)$ defined by $\varphi_a(x) = x \circ a$;

are continuous for every $a \in P$.

Lemma 3.6. ([11]) *Let U be an open subset of a topological polygroup P such that U is a complete part. Then, $a \circ U$ and $U \circ a$ are open subsets of P for every $a \in P$.*

Lemma 3.7. ([11]) *Let P be a topological polygroup such that every open subset of P is a complete part. Let $\mathcal{U}$ be an open base at e. Then, the families $\{x \circ U |\ x \in P, U \in \mathcal{U}\}$ and $\{U \circ x |\ x \in P, U \in \mathcal{U}\}$ are open bases for P.*

Lemma 3.8. *Let every open subset of a topological polygroup P be a complete part. If F is a compact subset of P, then for every $a \in P$, $a \circ F$ and $F \circ a$ are compact.*

Proof. Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $a \circ F$. Then, $a \circ F \subseteq \bigcup_{\alpha \in A} U_\alpha$ and so

$$F \subseteq a^{-1} \circ a \circ F \subseteq a^{-1} \circ \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} a^{-1} \circ U_\alpha.$$

By Lemma 3.6, every $a^{-1} \circ U_\alpha$ is open. Since F is compact, there exist $\alpha_1, \ldots, \alpha_n \in A$ such that $F \subseteq \bigcup_{i=1}^{n} a^{-1} \circ U_{\alpha_i}$. Therefore, by Proposition 2.3, $a \circ F \subseteq \bigcup_{i=1}^{n} a \circ a^{-1} \circ U_{\alpha_i} = \bigcup_{i=1}^{n} U_{\alpha_i}$, i.e., $a \circ F$ is compact. Similarly, $F \circ a$ is compact. □

Corollary 3.9. *Let F be a compact subset of a topological polygroup P and E be a finite subset of P. If every open subset of P is a complete part, then $E \circ F$ and $F \circ E$ are compact.*

Proof. $F \circ E = \bigcup_{a \in E} F \circ a$ and every $F \circ a$ is compact by Lemma 3.8. Since every finite union of compact subsets is compact, it follows that $F \circ E$ is compact. Similarly, $E \circ F$ is compact. □

Remark 3.10. Being complete part is necessary in Lemma 3.8 as it is illustrated in the following example.

Example 3.11. Consider the set of integer numbers $\mathbb{Z}$ and define the hyperoperation $\circ$ on it as follows: for every $m \in \mathbb{Z}$, $m \circ 0 = m$ and if $m, n \in \mathbb{Z} \setminus \{0\}$, then

$$m \circ n = \begin{cases} \mathbb{E} & \text{if } m + n \in \mathbb{E} \\ \mathbb{E}^c & \text{if } m + n \in \mathbb{E}^c \end{cases}$$

where $\mathbb{E} = 2\mathbb{Z}$. Let τ be discrete topology on $\mathbb{Z}$. Then, $(\mathbb{Z}, \circ, 0, -)$ is a topological polygroup, where the unitary operation $-$ is the ordinary negation. Thus, $F = \{1\}$ is compact, but $1 \circ F = \mathbb{E}$ is not compact.

Theorem 3.12. ([11]) *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and $\mathcal{U}$ be a basis at e. Then, the following assertions hold:*

(1) for every $U \in \mathcal{U}$ and $x \in U$ there exists $V \in \mathcal{U}$ such that $x \circ V \subseteq U$;

(2) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;

(3) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.

Theorem 3.13. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup, A be a non-empty subset of P and $a \in A$. Then the function f_a from $\mathcal{P}^*(P)$ to itself defined by $f_a(A) = a \circ A$ is continuous.*

Proof. Suppose that $U \in \tau$ and $f(A) \in S_U$. Then, $a \circ A \subseteq U$. Thus, $a \circ b \subseteq U$ for every $b \in A$. By Lemma 3.4 there exist $V_b, W_b \in \tau$ such that $a \in V_b, b \in W_b$ and $V_b \circ W_b \subseteq U$. Therefore, for every $b \in A$, we have $a \circ W_b \subseteq U$. Now, suppose that $W = \bigcup_{b \in A} W_b$. Thus, $f_a(S_W) \subseteq S_U$ and $A \in S_W$. Therefore, f_a is continuous. □

Example 3.14. Suppose that the multiplication table for polygroup $P = \langle P, \circ, 1, ^{-1} \rangle$, where $P = \{1, 2\}$, $1^{-1} = 1$ and $2^{-1} = 2$ is

$\circ$	1	2
1	$\{1\}$	$\{2\}$
2	$\{2\}$	$\{1, 2\}$

If $\tau = \{\emptyset, \{1\}, \{1, 2\}\}$, then $P = (P, \circ, ^{-1}, \tau)$ is a topological polygroup. We note that the open subset $\{1\}$ of P is not a complete part. The function f_2 from $\mathcal{P}^*(P)$ to itself defined by $f_2(A) = 2 \circ A$ is continuous by Theorem 3.13. But the function f_2 is not open since $\{\{1\}\} \in \tau^*$ but $f_2(\{\{1\}\}) = \{\{2\}\}$ is not open in $\mathcal{P}^*(P)$.

Proposition 3.15. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and $a \in P$. Then, the map $g_a : P \to \mathcal{P}^*(P)$ defined by $g_a(x) = a \circ x \circ a^{-1}$ is continuous.*

Proof. By Theorem 3.13, the function f_a is continuous and by Lemma 3.5, the function $\varphi_{a^{-1}}$ is continuous. Since g_a is the composite $f_a \circ \varphi_{a^{-1}}$ of f_a and $\varphi_{a^{-1}}$, the function g_a is continuous. □

Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup. We denote by $\nu_P(e)$ the set of all neighborhoods of e.

Proposition 3.16. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup, every open subset of P is a complete part and $a \in P$. Then, for every $U \in \nu_P(e)$ there exists a $V \in \nu_P(e)$ such that $a \circ V \circ a^{-1} \subseteq U$.*

Proof. By Proposition 3.15, the function g_a is continuous and so $g_a^{-1}(S_U)$ is open in P. Since U is a complete part, it follows that $e \in a^{-1} \circ U \circ a = g_a^{-1}(S_U)$. Thus, there exists a $V \in \nu_P(e)$ such that $V \subseteq a^{-1} \circ U \circ a$, and so $a \circ V \circ a^{-1} \subseteq U$ by Proposition 2.3. □

Theorem 3.17. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part, F be a compact subsets of P and G be a closed subset of P such that $F \cap G = \emptyset$. Then, there exists an open neighbourhood V of e such that $(F \circ V) \cap G = \emptyset$ and $(V \circ F) \cap G = \emptyset$.*

Proof. If $x \in F$, then $x \in P \setminus G$. Since $P \setminus G$ is an open subset of P by Lemma 3.7, there exists an open neighbourhood V_x of e such that $x \in x \circ V_x \subseteq P \setminus G$. Then, $x \circ V_x \cap G = \emptyset$. By Theorem 3.12 we can also take an open neighbourhood W_x of e such that $W_x \circ W_x \subseteq V_x$. So, $F \subseteq \bigcup_{x \in F} x \circ W_x$. Hence, there exists a finite set $C \subseteq F$ such that $F \subseteq \cup_{x \in C} x \circ W_x$. Put $V_1 = \bigcap_{x \in C} W_x$. We claim that $(F \circ V_1) \cap G = \emptyset$. That is for each $t \in F$, $t \circ V_1 \cap G = \emptyset$. Given a $t \in F$. Then, there exists x in C such that $t \in x \circ W_x$. Thus,

$$t \circ V_1 \subseteq x \circ W_x \circ V_1 \subseteq x \circ W_x \circ W_x \subseteq x \circ V_x \subseteq P \setminus G,$$

and so $(t \circ V_1) \cap G = \emptyset$. Thus, $(F \circ V_1) \cap G = \emptyset$.

Similarly, one can find an open neighbourhood V_2 of e in P satisfying $(V_2 \circ F) \cap G = \emptyset$. Then, the set $V = V_1 \cap V_2$ is as required. □

Theorem 3.18. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup such that every open subset of P is a complete part, F be a compact subset of P and G be a closed subset of P. Then, the sets $F \circ G$ and $G \circ F$ are closed in P.*

Proof. We prove that $P \setminus (F \circ G)$ is open. If $a \in P \setminus (F \circ G)$ then we claim that $(F^{-1} \circ a) \cap G = \emptyset$. Indeed, if $x \in (F^{-1} \circ a) \cap G$ then $x \in F^{-1} \circ a$ and $x \in G$. So, there exists an $f \in F$ such that $x \in f^{-1} \circ a$. Thus, $a \in F \circ x \subseteq F \circ G$, which is a contradiction. Since the unitary operation is continuous and F is compact, then F^{-1} is compact. By Lemma 3.8, $F^{-1} \circ a$ is compact. Hence, by Theorem 3.17 there exists an open neighborhood U of e such that $(F^{-1} \circ a \circ U) \cap G = \emptyset$. Therefore, by Proposition 2.2, $(a \circ U) \cap (F \circ G) = \emptyset$. Thus, $a \in a \circ U \subseteq P \setminus F \circ G$. That is $P \setminus (F \circ G)$ is open. Hence, $F \circ G$ is closed. Analogously, it is possible to prove that $G \circ F$ is closed. □

Lemma 3.19. ([11]) *If H is a subpolygroup of a polygroup $P = (P, \circ, e, ^{-1}, \tau)$ and every open subset of P is a complete part, then $\overline{H}$ is a subpolygroup of P.*

Theorem 3.20. *Let H be a non-empty subset of a topological polygroup $P = (P, \circ, e, ^{-1}, \tau)$ and every open subset of P is a complete part. Then*

(1) $\overline{H} = \bigcap_{U \in \nu_P(e)} U \circ H = \bigcap_{U \in \nu_P(e)} H \circ U = \bigcap_{U \in \nu_P(e), V \in \nu_P(e)} U \circ H \circ V$;

(2) if H is a normal subpolygroup, then $\overline{H}$ is a normal subpolygroup.

Proof. (1) Suppose that $x \in \overline{H}$ and $U^{-1} \in \nu_P(e)$. Since $U^{-1} \circ x$ is open and $x \in U^{-1} \circ x$, there exists a $t \in U^{-1} \circ x \cap H$. Hence, there exists a $u \in U$ such that $t \in u^{-1} \circ x$, and so $x \in u \circ t \subseteq U \circ H$. Thus, $x \in \bigcap_{U \in \nu_P(e)} U \circ H$. Now, suppose that $x \in \bigcap_{U \in \nu_P(e)} U \circ H$ and $x \notin \overline{H}$, then there exists a symmetric $U \in \nu_P(e)$ such that $U \circ x \cap H = U^{-1} \circ x \cap H = \emptyset$. Hence, $x \notin U \circ H$ which is contradiction.

Let $V \in \nu_P(e)$. Since $e \in V$, we have $U \circ H \subseteq U \circ H \circ V$ for every $U \in \nu_P(e)$. Therefore,

$$\begin{aligned}
\overline{H} &= \textstyle\bigcap_{U \in \nu_P(e)} U \circ H \\
&\subseteq \textstyle\bigcap_{U \in \nu_P(e), V \in \nu_P(e)} U \circ H \circ V \\
&= \textstyle\bigcap_{U \in \nu_P(e)} \bigcap_{V \in \nu_P(e)} U \circ H \circ V \\
&= \textstyle\bigcap_{U \in \nu_P(e)} \overline{U \circ H} \\
&\subseteq \textstyle\bigcap_{U \in \nu_P(e)} U \circ U \circ H \\
&= \textstyle\bigcap_{W \in \nu_P(e)} W \circ H.
\end{aligned}$$

(2) Since H is a subpolygroup of P, $\overline{H}$ is a subpolygroup of P. Let x be in $\overline{H}$. We prove that $a \circ x \circ a^{-1} \subseteq \overline{H}$ for every $a \in P$. Let $U, V \in \nu_P(e)$, then by Lemma 3.16, there exist $U_1, V_1 \in \nu_P(e)$ such that $a \circ U_1 \subseteq U \circ a$, $V_1 \circ a^{-1} \subseteq a^{-1} \circ V$. Since $x \in \overline{H}$, it follows from (1) that $x \in U_1 \circ H \circ V_1$. Therefore,

$$a \circ x \circ a^{-1} \subseteq a \circ U_1 \circ H \circ V_1 \circ a^{-1} \subseteq U \circ a \circ H \circ a^{-1} \circ V \subseteq U \circ H \circ V.$$

Thus, by (1), $a \circ x \circ a^{-1} \subseteq \overline{H}$. □

4. Left Big Subsets of Polygroups

In this section, we define and study the concept of left big subsets of polygroups. Applications of complete parts in topological polygroups in which every non-empty open subset is left big are investigated.

Definition 4.1. A non-empty subset B of a hypergroupoid $(H, \circ)$ is *left (right) big* if there exists a finite subset F of H such that $H = F \circ B$ ($H = B \circ F$). If $F \circ B = H = B \circ F$, then we say that B is *big*.

Remark 4.2. If $(H, \circ)$ is a quasihypergroup, then it is evident that H is big. The converse of this implication is not true in general as shown in the following example.

Example 4.3. Suppose that $H = \{a, b\}$ and the hyperoperation $\circ$ on H is defined as follows:

$\circ$	a	b
a	$\{a, b\}$	$\{a\}$
b	$\{a\}$	$\{a\}$

It is easy to see that H is big but the hypergroupoid $(H, \circ)$ is not a quasihypergroup.

Example 4.4. Let H be a set and S be a non-empty subset of H. For every $a, b \in H$ we define $a \circ b = S$. Then, H is left big in the hypergroupoid $(H, \circ)$ if and only if $H = S$.

Example 4.5. Let $(H, \circ)$ be the polygroup defined in Example 3.11. Then, $S = \{0\}$ is not big. It is evident that any non-zero subset of H which is non-empty is big.

Proposition 4.6. *Let $P = \langle P, \circ, e, ^{-1} \rangle$ be a polygroup. Then the following conditions hold:*

(1) If $B \subseteq P$ is left big, then B^{-1} is right big;

(2) If $A \subseteq B \subseteq P$ and A is left big, then B is left big.

Proof. (1) Let F be a finite subset of P such that $F \circ B = P$. Then, $B^{-1} \circ F^{-1} = P$, and so B^{-1} is right big.
(2) It is evident. □

Proposition 4.7. *Let B be a left big subset of a polygroup $P = \langle P, \circ, e, ^{-1} \rangle$. If B is a complete part and H is a subpolygroup of P, then $(B^{-1} \circ B) \cap H$ is a left big subset of H.*

Proof. Since B is a left big subset of P, there exists a finite subset F such that $F \circ B = P$. For every $f \in F$, if $(f \circ B) \cap H \neq \emptyset$, choose an a_f in $(f \circ B) \cap H$ and if $(f \circ B) \cap H = \emptyset$, choose an arbitrary element a_f in H. Thus, $F_1 = \{a_f | f \in F\}$ is a finite subset of H. If $x \in H$, then there exists an $f \in F$ such that $x \in f \circ B$ since the equality $P = F \circ B$ holds. Thus, $a_f \in (f \circ B) \cap H$ by definition of F_1, and so $a_f^{-1} \circ x \subseteq B^{-1} \circ f^{-1} \circ f \circ B = B^{-1} \circ B$ since B is a complete part. Thus, $H \subseteq F_1 \circ ((B^{-1} \circ B) \cap H)$ and so $H = F_1 \circ ((B^{-1} \circ B) \cap H)$ which completes the proof. □

Theorem 4.8. *Let $P_1 = \langle P_1, \circ_1, e_1, ^{-1} \rangle$ and $\langle P_2, \circ_2, e_2, ^{-1} \rangle$ be two polygroups and $f : P_1 \to P_2$ be a good homomorphism which is surjective. Then*

(1) if B is a left big subset of P_2, then $f^{-1}(B)$ is a left big subset of P_1;

(2) if A is a left big subset of P_1, then $f(A)$ is a left big subset of P_2.

Proof. (1) Let B be a left big subset of P_2, then there exists an n-element subset $F = \{x_1, \dots, x_n\}$ of P_2 such that $F \circ_2 B = P_2$. For every $1 \leq i \leq n$ choose a t_i in $f^{-1}(x_i)$. Now, suppose that $F_1 = \{t_1, \dots, t_n\}$. Then, $f(F_1 \circ_1 f^{-1}(B)) = f(F_1) \circ_2 f(f^{-1}(B)) = F \circ_2 B = P_2$. Hence, $F_1 \circ_1 f^{-1}(B) = P_1$. which shows that $f^{-1}(B)$ is a left big subset of P_1.

(2) Let A be a left big subset of P_1, then there exists a finite subset F of P_1 such that $F \circ_1 A = P_1$. Thus, $P_2 = f(F \circ_1 A) = f(F) \circ_2 f(A)$ which shows that $f(A)$ is a left big subset of P_2 . □

Definition 4.9. A topological polygroup P is called *totally bounded* if every open non-empty subset U of P is left big.

Clearly, a topological polygroup P is totally bounded if and only if every open non-empty subset U of P is right big.

Theorem 4.10. *If $P = (P, \circ, e, ^{-1}, \tau)$ is a totally bounded topological polygroup such that every open subset of P is a complete part, then for every $U \in \nu_P(e)$ there exists a $V \in \nu_P(e)$ such that for every $g \in P$, $g \circ V \circ g^{-1} \subseteq U$.*

Proof. Let $U \in \nu_P(e)$, then there exist $V, W \in \nu_P(e)$ such that $V \circ V \subseteq U$, $W \circ W \subseteq V$ and $W = W^{-1}$ by Theorem 3.12. Therefore, $W \circ W \circ W \subseteq U$. By hypothesis, there exists a finite subset F of P such that $W \circ F = P$, and so $F \subseteq \cup_{x \in F} W \circ x$. Let $V = \cap_{x \in F} x^{-1} \circ W \circ x$. For every y in F: $y \circ V \circ y^{-1} = y \circ (\cap_{x \in F} x^{-1} \circ W \circ x) \circ y \subseteq y \circ y^{-1} \circ W \circ y \circ y^{-1} = W$ by Proposition 2.3. If $g \in P$, then there exists a $z \in F$ and a $w \in W$ such that $g \in w \circ z$. Hence, $g \circ V \circ g^{-1} \subseteq (w \circ z) \circ V \circ (z^{-1} \circ w^{-1}) \subseteq w \circ (z \circ V \circ z^{-1}) \circ w^{-1} \subseteq W \circ W \circ W \subseteq U$ and this completes the proof. □

Lemma 4.11. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup, every open subset of P is a complete part and $\mathcal{U}$ be a base at e. Then, P is totally bounded if and only if for every $U \in \mathcal{U}$, U is left big.*

Proof. Obviously, if P is totally bounded, then every member of $\mathcal{U}$ is left big. Conversely, suppose that every member of $\mathcal{U}$ is left big. Let V be an open subset of P and $x \in V$. Then, by Lemma 3.7, there exists a $W \in \mathcal{U}$ such that $W \circ x \subseteq V$. Since W is left big, there exists a finite subset F of P such that $P = F \circ W$. Thus, $P = P \circ x = (F \circ W) \circ x = F \circ (W \circ x) \subseteq F \circ V$. Hence, $P = F \circ V$, i.e., V is left big. Therefore, P is totally bounded. □

Proposition 4.12. *If every open subset of a totally bounded topological polygroup $P = (P, \circ, e, ^{-1}, \tau)$ is a complete part and H is a subpolygrop of P, then H is totally bounded.*

Proof. Let $U \in \nu_H(e)$. Then, there exists a $W \in \nu_P(e)$ such that $U = H \cap W$. By Theorem 3.12, pick a $V \in \nu_P(e)$ such that $V^{-1} \circ V \subseteq W$. Then, $(V^{-1} \circ V) \cap H \subseteq W \cap H = U$. By Proposition 4.7, $(V^{-1} \circ V) \cap H$ is big in H, and so U is big in H. Thus, H is totally bounded by Lemma 4.11. □

Theorem 4.13. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup and H be a dense subpolygroup of P. If H is totally bounded and every open subset of P is a complete part, then P is totally bounded.*

Proof. Let U be in $\nu_P(e)$, then there exits a $V \in \nu_P(e)$ such that $V \circ V \subseteq U$ by Theorem 3.12. Since $V \cap H$ is open in H and H is totally bounded, there exists a finite set $F \subseteq H$ such that $F \circ (V \cap H) = H$. Hence, by Theorem 3.20:

$$P = \overline{H} = \bigcap_{W \in \nu_P(e)} H \circ W \subseteq H \circ V = F \circ (V \cap H) \circ V \subseteq F \circ V \circ V \subseteq F \circ U,$$

and so $P = F \circ U$. Thus, by Lemma 4.11, P is totally bounded. □

Theorem 4.14. *Let $P = (P, \circ, e, ^{-1}, \tau)$ be a topological polygroup. Then, P is not totally bounded if and only if there exist a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of P and a non-empty open set V such that $x_n \circ V \cap x_m \circ V = \emptyset$ for every $m \neq n \in \mathbb{N}$.*

Proof. If P is not totally bounded, then there is a $U \in \nu_P(e)$ such that U is not left big by Lemma 4.11. By Theorem 3.12, there exists a $V \in \nu_P(e)$ such that $V \circ V^{-1} \subseteq U$. The subset $V \circ V^{-1}$ is not left big since U is not left big. Thus, there is a g_1 in $P \setminus (V \circ V^{-1})$. Similarly, there is a g_2 in $P \setminus (V \circ V^{-1}) \cup (g_1 \circ (V \circ V^{-1}))$. By a simple induction a sequence $(g_n)_{n\in\mathbb{N}}$ can be defined such that for every $n > 1$, $g_n \in P \setminus ((V \circ V^{-1}) \bigcup \cup_{i=1}^{n-1} g_i \circ (V \circ V^{-1}))$. If $n > m$ and $x \in (g_n \circ V) \cap (g_m \circ V)$, then there exist v_1 and v_2 in V such that $x \in g_n \circ v_1$ and $x \in g_m \circ v_2$. Thus, $g_n \in x \circ v_1^{-1}$ and $g_m \in x \circ v_2^{-1}$ so $g_n \in x \circ v_1^{-1} \subseteq g_m \circ v_2 \circ v_1^{-1} \subseteq g_m \circ (V \circ V^{-1})$, which is a contradiction.

Conversely, suppose that there exist a $W \in \nu_P(e)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of P such that for every distinct elements m, n of $\mathbb{N}$ we have $x_n \circ W \cap x_m \circ W = \emptyset$. By Theorem 3.12, there exists a $V \in \nu_P(e)$ such that $V^{-1} \subseteq W$. Thus, for every distinct elements m, n of $\mathbb{N}$ we have $x_n \circ V^{-1} \cap x_m \circ V^{-1} = \emptyset$. We claim that V is not left big. Otherwise, suppose that there exists a finite set F such that $F \circ V = P$. Thus, there are distinct natural numbers m, n and an $x \in F$ such that $\{x_m, x_n\} \subseteq x \circ V$. Therefore, $x \in x_n \circ V^{-1} \cap x_m \circ V^{-1}$ and this is a contradiction. □

Theorem 4.15. *Let $(P_1, \circ_1, e_1, ^{-1}, \tau_1)$ and $(P_2, \circ_2, e_2, ^{-1}, \tau_2)$ be two topological polygroups and $f : P_1 \to P_2$ be a continuous surjective good homomorphism. Then the following assertions hold:*

(1) if P_1 is totally bounded, then P_2 is totally bounded;

(2) if $\beta = \{f^{-1}(U) \mid U \in \tau_2\}$ is a base for the topological space (P_1, τ_1) and P_2 is totally bounded, then P_1 is totally bounded.

Proof. (1) Let P_1 be totally bounded and U be open in P_2. Then, $f^{-1}(U)$ is an open subset of P_1 which is left big since P_1 is totally bounded. Now, by Theorem 4.8, $U = f(f^{-1}(U))$ is left big. Thus, P_2 is totally bounded.

(2) Let $\beta = \{f^{-1}(U) \mid U \in \tau_2\}$ is a base for the topological space (P_1, τ_1), P_2 is totally bounded and $V \in \nu_{P_1}(e_1)$. Then, there exists $U \in \nu_{P_2}(e_2)$ such that $f^{-1}(U) \subseteq V$ since β is a base. Since P_2 is totally bounded, U is left big. Hence, by Theorem 4.8, $f^{-1}(U)$ is left big, and so V is left big. Thus, by Lemma 4.11, P_1 is totally bounded. □

5. Conclusion

A topological polygroup is a polygroup P together with a topology on P such that the hyperoperation and the polygroup's inverse function are continuous functions with respect to the topology. A subset A is a complete part of P if for any natural number n and for all $a_1, a_2, \ldots, a_n \in P$, the following implication holds: $A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A$. In this paper, we investigated the properties of complete parts and open sets in a topological polygroup.

Also it might be interesting to study topological polygroups in the fuzzy case.

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