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On Zweier I-Convergent Double Sequence Spaces

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Abstract. In this article we introduce the Zweier I-convergent double sequence spaces ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}^I_0$ and ${}_2\mathcal{Z}^I_\infty$. We prove the decomposition theorem and study topological properties, algebraic properties and some inclusion relations on these spaces.

1. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively. We write

$${}_2\omega = \{x = (x_{ij}) : x_{ij} \in \mathbb{R} \text{ or } \mathbb{C}\},$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $\|x\|_\infty = \sup_k |x_k|$.

At the initial stage the notion of I-convergence was introduced by Kostyrko,Šalát and Wilczyński[1]. Later on it was studied by Šalát, Tripathy and Ziman[2], Demirci [3] and many others. I-convergence is a generalization of Statistical Convergence.

Now we have a list of some basic definitions used in the paper .

Definition 1.1. [4,5] Let X be a non empty set. Then a family of sets I$\subseteq 2^X$(2^X denoting the power set of X) is said to be an ideal in X if
(i) $\emptyset \in$ I
(ii) I is additive i.e A,B∈I ⇒ A ∪ B∈I.
(iii) I is hereditary i.e A∈I, B⊆ A ⇒B∈I.

An Ideal I$\subseteq 2^X$ is called non-trivial if I$\neq 2^X$. A non-trivial ideal I$\subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\}$ ⊆I. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J≠I containing I as a subset. For each ideal I, there is a filter £(I) corresponding to I. i.e

$$£(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where } \; K^c = N - K.$$

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Definition 1.2. A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}), where the two subscripts run through the sequence of natural numbers independent of each other. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$|(x_{ij}) - a| < \epsilon, \qquad for\ all\ i, j \geq N \ \ (see\ [6,7,8])$$

Definition 1.3.[7] A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$,

$$\{i, j \in \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I.$$

In this case we write $I - \lim x_{ij} = L$.

Definition 1.4.[7] A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-null if L = 0. In this case we write

$$I - \lim x_{ij} = 0.$$

Definition 1.5. [7] A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exist numbers m = m(ϵ), n= n(ϵ) such that

$$\{i, j \in \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I.$$

Definition 1.6.[7] A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{i, j \in \mathbb{N} : |x_{ij}| > M\}.$$

Definition 1.7.[7] A double sequence space E is said to be solid or normal if $(x_{ij}) \in$ E implies $(\alpha_{ij}x_{ij}) \in$ E for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all i,j $\in \mathbb{N}$.

Definition 1.8.[7] A double sequence space E is said to be monotone if it contains the canonical preimages of its stepspaces.

Definition 1.9.[7] A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$.

Definition 1.10.[7] A double sequence space E is said to be a sequence algebra if $(x_{ij}.y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.11.[7] A double sequence space E is said to be symmetric if $(x_{ij}) \in E$ implies $(x_{\pi(ij)}) \in E$, where π is a permutation on N.

A sequence space $\lambda \subset \omega$ with linear topology is called a K-space provided each of maps $p_i \longrightarrow C$ defined by $p_i(x) = x_i$ is continuous for all $i \in N$.
Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N$. Then we say that A defines a matrix mapping from λ to μ, and we denote it by writing $A : \lambda \longrightarrow \mu$.
If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in N). \qquad (1)$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$.
Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1) converges for each $n \in N$ and every $x \in \lambda$. (see[9]).
The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently studied by Başar and Altay[10], Malkowsky[11], Ng and Lee[12] and

Wang[13], Başar, Altay and Mursaleen[14]. For more information one can refer [15,16,17] Şengönül[18] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1-p, (i-1 = k); (i, k \in N), \\ 0, \text{otherwise}. \end{cases}$$

Following Basar,and Altay [10], Şengönül[18] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

The following Lemmas will be used for establishing some results of this article.

Lemma 1.12. A sequence space E is solid implies that E is monotone.(See[19,20])
Lemma 1.13. Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.(See[19,20])
Lemma 1.14. If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.(See[21,22,23])

Main Results

In this article we introduce the following classes of sequence spaces.

$$_2\mathcal{Z}^I = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in N \times N : I - \lim Z^p x = L \text{ for some L}\}\} \in I$$

$$_2\mathcal{Z}^I_0 = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in N \times N : I - \lim Z^p x = 0\}\} \in I$$

$$_2\mathcal{Z}^I_\infty = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in N \times N : \sup_{i,j} |Z^p x| < \infty\}\} \in I$$

We also denote by

$$_2m^I_{\mathcal{Z}} = {}_2\mathcal{Z}_\infty \cap {}_2\mathcal{Z}^I \quad and \quad {}_2m^I_{\mathcal{Z}_0} = {}_2\mathcal{Z}_\infty \cap {}_2\mathcal{Z}^I_0$$

Theorem 2.1. The classes of sequences $_2\mathcal{Z}^I$, $_2\mathcal{Z}^I_0$, $_2m^I_{\mathcal{Z}}$ and $_2m^I_{\mathcal{Z}_0}$ are linear spaces.

Proof. We shall prove the result for the space $_2\mathcal{Z}^I$.
The proof for the other spaces will follow similarly.
Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I$ and let α, β be scalars. Then

$$I - \lim |x_{ij} - L_1| = 0, \text{ for some } L_1 \in C \ ;$$

$$I - \lim |y_{ij} - L_2| = 0, \text{ for some } L_2 \in C \ ;$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{(i,j) \in N \times N : |x_{ij} - L_1| > \frac{\epsilon}{2}\} \in I, \tag{1}$$

$$A_2 = \{i, j \in N \times N : |y_{ij} - L_2| > \frac{\epsilon}{2}\} \in I. \tag{2}$$

we have

$$|(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)| \leq |\alpha|(|x_{ij} - L_1|) + |\beta|(|y_{ij} - L_2|)$$
$$\leq |x_{ij} - L_1| + |y_{ij} - L_2|.$$

Now, by (1) and (2),

$$\{(i,j) \in N \times N : |(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2.$$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in {}_2\mathcal{Z}^I$. Hence ${}_2\mathcal{Z}^I$ is a linear space.

We state the following result without proof in view of Theorem 2.1.

Theorem 2.2. The spaces ${}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$ are normed linear spaces, normed by

$$\|x_{ij}\|_* = \sup_{i,j} |x_{ij}|. \tag{3}$$

Theorem 2.3. A sequence $x = (x_{ij}) \in {}_2m^I_{\mathcal{Z}}$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon = (m,n) \in N \times N$ such that

$$\{(i,j) \in N \times N : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}} \tag{4}$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{(i,j) \in N \times N : |x_{ij} - L| < \frac{\epsilon}{2}\} \in {}_2m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon = (m,n) \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_{ij}| \leq |x_{N_\epsilon} - L| + |L - x_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i,j) \in B_\epsilon$.
Hence $\{(i,j) \in N \times N : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$.

Conversely, suppose that $\{(i,j) \in N \times N : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$.
That is

$$\{(i,j) \in N \times N : |x_k - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$$

for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i,j) \in N \times N : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in {}_2m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$.
If we fix an $\epsilon > 0$ then we have $C_\epsilon \in {}_2m^I_{\mathcal{Z}}$ as well as $C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}$.
Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}$.
This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in N \times N : x_{ij} \in J\} \in {}_2m^I_{\mathcal{Z}}$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in N \times N : x_{ij} \in I_k\} \in m^I_{\mathcal{Z}}$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $(i,j) \in N \times N$ such that $\xi = I - \lim x$, that is $L = I - \lim x$.

Theorem 2.4. Let I be an admissible ideal. Then the following are equivalent.
(a) $(x_{ij}) \in {}_2\mathcal{Z}^I$;
(b) there exists $(y_{ij}) \in {}_2\mathcal{Z}$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I;
(c) there exists$(y_{ij}) \in {}_2\mathcal{Z}$ and $(z_{ij}) \in {}_2\mathcal{Z}^I_0$ such that $x_{ij} = y_{ij} + z_{ij}$ for all $(i,j) \in N \times N$ and

$$\{(i,j) \in N \times N : |y_{ij} - L| \geq \epsilon\} \in I;$$

(d) there exists a subset $K = \{k_1 < k_2 ....\}$ of N such that $K \in \pounds(I)$
and $\lim_{n\to\infty} |x_{k_n} - L| = 0$.

Proof. (a) implies (b). Let $(x_{ij}) \in {}_2\mathcal{Z}^I$. Then there exists $L \in C$ such that

$$\{(i,j) \in N \times N : |x_{ij} - L| \geq \epsilon\} \in I.$$

Let (m_t, n_t) be an increasing sequence with $(m_t, n_t) \in N \times N$ such that

$$\{(i,j) \leq (m_t, n_t) : |x_{ij} - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_{ij}) as

$$y_{ij} = x_{ij}, \text{ for all } (i,j) \leq (m_1, n_1).$$

For $(m_t, n_t) < (i,j) \leq (m_{t+1}, n_{t+1})$ for $t \in N$.

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } |x_{ij} - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_{ij}) \in {}_2\mathcal{Z}$ and form the following inclusion

$$\{(i,j) \leq (m_t, n_t) : x_{ij} \neq y_{ij}\} \subseteq \{(i,j) \leq (m_t, n_t) : |x_{ij} - L| \geq \epsilon\} \in I.$$

We get $x_{ij} = y_{ij}$, for a.a.k.r.I.

(b) implies (c). For $(x_{ij}) \in {}_2\mathcal{Z}^I$, there exists $(y_{ij}) \in {}_2\mathcal{Z}$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I.
Let $K = \{(i,j) \in N \times N : x_{ij} \neq y_{ij}\}$, then $K \in I$.
Define a sequence (z_{ij}) as

$$z_{ij} = \begin{cases} x_{ij} - y_{ij}, & \text{if } (i,j) \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_{ij} \in {}_2\mathcal{Z}^I_0$ and $y_{ij} \in {}_2\mathcal{Z}$.

(c)implies (d). Let $P_1 = \{(i,j) \in N \times N : |z_{ij}| \geq \epsilon\} \in I$ and

$$K = P^c_1 = \{(i_1, j_1) < (i_2, j_2) < ...\} \in \pounds(I).$$

Then we have $\lim_{n\to\infty} |x_{(i_n,j_n)} - L| = 0$.

(d) implies (a). Let $K = \{(i_1, j_1) < (i_2, j_2) < ...\} \in \pounds(I)$ and $\lim_{n\to\infty} |x_{(i_n,j_n)} - L| = 0$.
Then for any $\epsilon > 0$, and Lemma 1.17, we have

$$\{(i,j) \in N \times N : |x_{ij} - L| \geq \epsilon\} \subseteq K^c \cup \{(i,j) \in K : |x_{ij} - L| \geq \epsilon\}.$$

Thus $(x_{ij}) \in {}_2\mathcal{Z}^I$.

Theorem 2.5. The inclusions ${}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I \subset {}_2\mathcal{Z}_\infty^I$ hold and are proper.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I$. Then there exists L ∈ C such that

$$I - \lim |x_{ij} - L| = 0$$

We have $|x_{ij}| \leq \frac{1}{2}|x_{ij} - L| + \frac{1}{2}|L|$.
Taking the supremum over (i, j) on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I$.
The inclusion ${}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I$ is obvious. The strict inclusion is also trivial.

Theorem 2.6. The function $\hbar : {}_2m_{\mathcal{Z}}^I \to R$ is the Lipschitz function,where ${}_2m_{\mathcal{Z}}^I = {}_2\mathcal{Z}^I \cap {}_2\mathcal{Z}_\infty$, and hence uniformly continuous.

Proof. Let $x, y \in {}_2m_{\mathcal{Z}}^I$, $x \neq y$. Then the sets

$$A_x = \{(i, j) \in N \times N : |x_{ij} - \hbar(x)| \geq \|x - y\|_*\} \in I,$$

$$A_y = \{(i, j) \in N \times N : |y_{ij} - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{(i, j) \in N \times N : |x_{ij} - \hbar(x)| < \|x - y\|_*\} \in £(I)$$

$$B_y = \{(i, j) \in N \times N : |y_{ij} - \hbar(y)| < \|x - y\|_*\} \in £(I).$$

Hence also $B = B_x \cap B_y \in £(I)$, so that $B \neq \phi$.
Now taking (i,j) in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus $\hbar$ is a Lipschitz function.
For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.7. If $x, y \in {}_2m_{\mathcal{Z}}^I$, then $(x.y) \in {}_2m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{(i, j) \in N \times N : |x - \hbar(x)| < \epsilon\} \in £(I),$$

$$B_y = \{(i, j) \in N \times N : |y - \hbar(y)| < \epsilon\} \in £(I).$$

Now,

$$\begin{aligned} |x.y - \hbar(x)\hbar(y)| &= |x.y - x\hbar(y) + x\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x||y - \hbar(y)| + |\hbar(y)||x - \hbar(x)| \end{aligned} \quad (5)$$

As ${}_2m_{\mathcal{Z}}^I \subseteq \mathcal{Z}_\infty$, there exists an $M \in R$ such that $\hbar|x| < M$ and $|\hbar(y)| < M$.
Using eqn(5) we get

$$|x.y - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(i, j) \in B_x \cap B_y \in £(I)$.
Hence $(x.y) \in {}_2m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.
For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.8. The spaces ${}_2\mathcal{Z}_0^I$ and ${}_2m_{\mathcal{Z}_0}^I$ are solid and monotone .

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I$.
Let $(x_{ij}) \in \mathcal{Z}_0^I$. Then

$$I - \lim_k |x_{ij}| = 0 \qquad (6)$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i, j) \in N \times N$. Then the result follows from (6) and the following inequality
$|\alpha_{ij}x_{ij}| \leq |\alpha_{ij}||x_{ij}| \leq |x_{ij}|$ for all $(i, j) \in N \times N$.
That the space ${}_2\mathcal{Z}_0^I$ is monotone follows from the Lemma 1.16.
For ${}_2m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.9. If I is not maximal, then the space ${}_2\mathcal{Z}^I$ is neither solid nor monotone.
Proof. Here we give a counter example. Let $(x_{ij}) = 1$ for all $(i, j) \in N \times N$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I$. Let $K \subseteq N \times N$ be such that $K \notin I$ and $N \times N - K \notin I$
Define the sequence

$$(y_{ij}) = \begin{cases} (x_{ij}), & \text{if } (i, j) \in K, \\ 0, & \textit{otherwise.} \end{cases}$$

Then (y_{ij}) belongs to the canonical preimage of K-step space of ${}_2\mathcal{Z}^I$ but $(y_{ij}) \notin {}_2\mathcal{Z}^I$.
Hence ${}_2\mathcal{Z}^I$ is not monotone.

Theorem 2.10. The spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are sequence algebras.

Proof. We prove that ${}_2\mathcal{Z}_0^I$ is a sequence algebra.
Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I$. Then

$$I - \lim |x_{ij}| = 0 \quad \textit{and} \quad I - \lim |y_{ij}| = 0$$

Then we have $I - \lim |(x_{ij}.y_{ij})| = 0$. Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I$
Hence ${}_2\mathcal{Z}_0^I$ is a sequence algebra.
For the space ${}_2\mathcal{Z}^I$, the result can be proved similarly.

Theorem 2.11. The spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$. Consider the sequence (x_{ij}) and (y_{ij}) defined by

$$x_{ij} = \frac{1}{i.j} \text{ and } y_{ij} = i.j \text{ for all (i,j)} \in N \times N$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$.
Hence the spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not convergence free.

Theorem 2.12. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not symmetric.

Proof. Let $A \in I$ be infinite. If

$$x_{ij} = \begin{cases} 1, & \text{for } i\,,\ j \in A, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $x_{ij} \in {}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I$
let $K \subset N$ be such that $K \notin I$ and $N - K \notin I$. Let $\phi : K \to A$ and $\psi : N - K \to N - A$ be bijections, then the map $\pi : N \to N$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \textit{otherwise.} \end{cases}$$

is a permutation on N, but $x_{(\pi(m)\pi(n))} \notin \mathcal{Z}^I$ and $x_{(\pi(m)\pi(n))} \notin {}_2\mathcal{Z}_0^I$. Hence ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not symmetric.

Theorem 2.13. The sequence spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are linearly isomorphic to the spaces ${}_2c^I$ and ${}_2c_0^I$ respectively, i.e ${}_2\mathcal{Z}^I \cong {}_2c^I$ and ${}_2\mathcal{Z}_0^I \cong {}_2c_0^I$.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I$ and ${}_2c^I$. The proof for the other spaces will follow similarly. We need to show that there exists a linear bijection between the spaces ${}_2\mathcal{Z}^I$ and c^I. Define a map $T: {}_2\mathcal{Z}^I \longrightarrow {}_2c^I$ such that $x \to x' = Tx$

$$T(x_{ij}) = px_{ij} + (1-p)x_{(i-1)(j-1)} = x'_{ij}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$. Clearly T is linear. Further, it is trivial that $x = 0 = (0, 0, 0, ...)$ whenever $Tx = 0$ and hence injective. Let $x'_{ij} \in {}_2c^I$ and define the sequence $x = x_{ij}$ by

$$x_{ij} = M \sum_{r=0}^{i} \sum_{s=0}^{j} (-1)^{(i-r)(j-s)} N^{(i-r)(j-s)} x'_{ij}$$

for $(i, j) \in N \times N$ and where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$. Then we have

$$\lim_{(i,j)\to\infty} px_{ij} + (1-p)x_{(i-1)(j-1)} = p \lim_{(i,j)\to\infty} M \sum_{r=0}^{i} \sum_{s=0}^{j} (-1)^{(i-r)(j-s)} N^{(i-r)(j-s)} x'_{ij}$$

$$+ (1-p) \lim_{(i,j)\to\infty} M \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} (-1)^{(i-1-r)(j-1-s)} N^{(i-1-r)(j-1-s)} x'_{(i-1)(j-1)}$$

$$= \lim_{(i,j)\to\infty} x'_{ij}$$

which shows that $x \in {}_2\mathcal{Z}^I$. Hence T is a linear bijection. Also we have $\|x\|_* = \|Z^p x\|_c$
Therefore

$$\|x\|_* = \sup_{(i,j)\in N\times N} |px_{ij} + (1-p)x_{(i-1)(j-1)}|$$

$$= \sup_{(i,j)\in N\times N} |pM \sum_{r=0}^{i} \sum_{s=0}^{j} (-1)^{(i-r)(j-s)} N^{(i-r)(j-s)} x'_{ij}$$

$$+ (1-p)M \sum_{r=0}^{i-1} \sum_{s=0}^{j-1} (-1)^{(i-1-r)(j-1-s)} N^{(i-1-r)(j-1-s)} x'_{(i-1)(j-1)}$$

$$= \sup_{(i,j)\in N\times N} |x'_{ij}| = \|x'\|_{{}_2c^I}.$$

Hence ${}_2\mathcal{Z}^I \cong {}_2c^I$.

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References

[1] P.Kostyrko,T.Šalát,W.Wilczyński, I-convergence, Real Anal. Exch.26(2) 669-686(2000).

[2] T.Šalát, B.C.Tripathy, M.Ziman, On some properties of I-convergence, Tatra Mt. Math. Publ.,(28)279-286(2004).

[3] K.Demirci, I-limit superior and limit inferior, Math. Commun.,6: 165-172(2001).

[4] P.Das, P.Kostyrko,P.Malik,W. Wilczyński, I and I^*-Convergence of Double Sequences, Mathematica Slovaca, 58,605-620,(2008).

[5] B.C.Tripathy and B.Hazarika, Paranorm I-Convergent sequence spaces, Math. Slovaca,59(4)485-494.(2009).

[6] M.Gurdal, S.Ahmet, Extremal I-Limit Points of Double sequences, Applied Mathematics E-Notes,8,131-137,(2008).

[7] V.A.Khan and N.Khan, On a new I-convergent double sequence spaces, International Journal of Analysis,Hindawi Publishing Corporation, Article ID-126163 Vol 2013,1-7(2013).

[8] V.A.Khan and S.Tabassum, On Some New double sequence spaces of Invariant Means defined by Orlicz function, Communications Fac. Sci. 60,11-21 (2011).

[9] V.A.Khan and K. Ebadullah, On Zweier I-convergent sequence spaces defined by Orlicz functions, Proyecciones Journal of Mathematics, Universidad Catolica del Norte Antofagasta-Chile(CHILE),Vol. 33, No.3, 259-276 (2014).

[10] F.Başar and B.Altay, On the spaces of sequences of p-bounded variation and related matrix mappings,Ukrainian Math.J.55.(2003).

[11] E.Malkowsky, Recent results in the theory of matrix transformation in sequence spaces, Math.Vesnik.(49)187-196(1997).

[12] P.N.Ng and P.Y.Lee, Cesaro sequence spaces of non-absolute type,Comment.Math. Prace.Mat.20(2)429-433(1978).

[13] C.S.Wang, On Nörlund sequence spaces,Tamkang J.Math.(9)269-274(1978).

[14] B.Altay,F.Başar, and M.Mursaleen, On the Euler sequence space which include the spaces l_p and l_∞, Inform.Sci., 176(10)(2006),1450-1462.

[15] N.L. Braha, H.M.Srivastava and S.A.Mohiuddine, A Korovin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean, Appl. Math. Comput., 228 (2014), 162-169.

[16] M. Mursaleen, A.Khan and H.M.Srivastava, Operators constructed by means of q-Lagrange polynomials and A-statistical approximation, Appl. Math. Comput. 219 (2013), 6911-6918.

[17] H.M.Srivastava, M. Mursaleen and A.Khan, Generalized equi-statistical convergence of positive linear operators and associated approximation theorems, Math. Comput. Modelling, 55 (2012) 2040-2051.

[18] M.Şengönül, On The Zweier Sequence Space, Demonstratio Math. Vol.XL No.(1)181-196(2007).

[19] V.A.Khan and N.Khan, On some I- Convergent double sequence spaces defined by a sequence of modulii, Journal of Mathematical Analysis, 4(2), 1-8, (2013).

[20] V.A.Khan and N.Khan, On some I- Convergent double sequence spaces defined by a modulus function, Engineering, Scientific Research, 5, 35-40, (2013).

[21] V.A.Khan and N.Khan, I-Pre-Cauchy Double Sequences and Orlicz Functions, Engineering,Scientific Research, 5, 52-56, (2013).

[22] V.A.Khan and K.Ebadullah, Ayhan Esi, N. Khan, M.Shafiq, On Paranorm Zweier I-Convergent Sequence Spaces, Journal of Mathematics,Hindawi Publishing Corporation,Vol 2013, Article ID 613501, 1-6(2013).

[23] V.A.Khan and K.Ebadullah, The Sequence Space BV_σ^I (p),Filomat(SERBIA) DOI 10.2298/FIL1404829K 28:4 (2014),829 - 838.