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Lie Higher Derivations on Triangular Algebras Revisited

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Abstract. Motivated by the extensive works of W.-S. Cheung [Linear Multilinear Algebra, **51** (2003), 299-310] and X.F. Qi [Acta Math. Sinica, English Series, **29** (2013), 1007-1018], we present the structure of Lie higher derivations on a triangular algebra explicitly. We then study those conditions under which a Lie higher derivation on a triangular algebra is proper. Our approach provides a direct proof for some known results concerning to the properness of Lie higher derivations on triangular algebras.

1. Introduction

Suppose that $\mathfrak{A}$ is an associative algebra over a commutative unital ring. Let $[x, y] = xy - yx$ denote the commutator of $x, y \in \mathfrak{A}$, and let $\mathbb{N}$ denote the set of non-negative integers. A sequence $\mathcal{L} = \{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of linear mappings on $\mathfrak{A}$ (with $\mathcal{L}_0 = id_{\mathfrak{A}}$) is said to be

- a higher derivation if for each $n \in \mathbb{N}$,

$$\mathcal{L}_n(xy) = \sum_{i+j=n} \mathcal{L}_i(x)\mathcal{L}_j(y) \qquad (x, y \in \mathfrak{A}); \tag{1.1}$$

- a Lie higher derivation if for each $n \in \mathbb{N}$,

$$\mathcal{L}_n([x, y]) = \sum_{i+j=n} [\mathcal{L}_i(x), \mathcal{L}_j(y)] \qquad (x, y \in \mathfrak{A}). \tag{1.2}$$

For a standard example of a (Lie) higher derivation, one may consider $\mathcal{L} = \{\frac{L^n}{n!}\}_{n\in\mathbb{N}}$, where $L : \mathfrak{A} \to \mathfrak{A}$ is a (Lie) derivation. These kinds of (Lie) higher derivations are called ordinary (Lie) higher derivations. It is obvious that if $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ is a (Lie) higher derivation then $\mathcal{L}_1$ is a (Lie) derivation. Further, every higher derivation is clearly a Lie higher derivation, however the converse is not true, in general.

If $\mathscr{D} = \{\mathscr{D}_n\}_{n\in\mathbb{N}}$ is a higher derivation and $\mathscr{F} = \{f_n\}_{n\in\mathbb{N}}$ is a sequence of linear maps from $\mathfrak{A}$ to $Z(\mathfrak{A})$, the center of $\mathfrak{A}$, then $\mathscr{D} + \mathscr{F}$ is a Lie higher derivation if and only if f_n annihilates all commutators $[x, y]$ for all

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$n \in \mathbb{N}$ and $x, y \in \mathfrak{A}$. Lie higher derivations of the above form are called proper. The fundamental question in the realm of Lie higher derivations is that under what conditions a Lie higher derivation on an algebra is proper. Many authors have studied the question for various algebras; see for example [4–6, 8–14] and references therein.

One of the most studied algebra in this context is the triangular algebra $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$, which was first introduced by Chase [1]. Let $\mathscr{A}$ and $\mathscr{B}$ be unital algebras and $\mathscr{M}$ be a left $\mathscr{A}$-module and right $\mathscr{B}$-module, then

$$\mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B}) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : \ a \in \mathscr{A}, m \in \mathscr{M}, b \in \mathscr{B} \right\}$$

is an algebra under the usual matrix operations, which is called a triangular algebra. Upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras are standard examples of triangular algebras. Many properties of some mappings, such as automorphisms, derivations, Lie derivations and commuting mappings on a triangular algebra are explored in Cheung's dissertation [2]. Wei and Xiao [12] have examined innerness of higher derivations on triangular algebras. They have also discussed Jordan higher derivations and nonlinear Lie higher derivations on a triangular algebra in [13] and [14], respectively. Qi and Hou [11] showed that each Lie higher derivation on a nest algebra is proper. Li and Shen [5] and also Qi [10] have extended the main result of [11] for a triangular algebra by providing some sufficient conditions under which a Lie higher derivation on a triangular algebra is proper.

In this paper we characterize the construction of Lie higher derivations on a triangular algebra $\mathfrak{A}$. In this respect, we give the "higher" version of the constructions given by Cheung [3]. Then we study those conditions under which a Lie higher derivation on $\mathfrak{A}$ is proper. In the first section our constructions are for a general $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ without no restriction on $\mathscr{M}$. In the second section we have focused on the case when $\mathscr{M}$ is faithful. We then apply our characterizations for recapturing the main results of [5], [10] and also [14].

2. The General Case

For a triangular algebra $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ it can be readily verified that

$$Z(\mathfrak{A}) = \{a \oplus b = \begin{pmatrix} a & 0 \\ & b \end{pmatrix} \mid a \in Z(\mathscr{A}), b \in Z(\mathscr{B}),\ am = mb \ for\, all\ m \in \mathscr{M}\}. \tag{2.1}$$

It follows that $\pi_{\mathscr{A}}(Z(\mathfrak{A})) \subseteq Z(\mathscr{A})$ and $\pi_{\mathscr{B}}(Z(\mathfrak{A})) \subseteq Z(\mathscr{B})$, where $\pi_{\mathscr{A}} : \mathfrak{A} \longrightarrow \mathscr{A}$ and $\pi_{\mathscr{B}} : \mathfrak{A} \longrightarrow \mathscr{B}$ are the natural projections defined by $\pi_{\mathscr{A}} : \begin{pmatrix} a & m \\ & b \end{pmatrix} \longmapsto a$ and $\pi_{\mathscr{B}} : \begin{pmatrix} a & m \\ & b \end{pmatrix} \longmapsto b$, respectively. We also usually identify $a \in \mathscr{A}$ with $a \oplus 0 \in \mathfrak{A}$. Similar identifications can be made for the other entries by $b \mapsto 0 \oplus b$ and $m \mapsto \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}$.

We begin with the following result that describes the construction of a Lie higher derivation on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$.

Theorem 2.1. *Let $\mathcal{L} = \{\mathcal{L}_n\}_{n\in\mathbb{N}}$ be a sequence of linear maps on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$. Then $\mathcal{L}$ is a Lie higher derivation if and only if, $\mathcal{L}_n$ can be presented in the form*

$$\mathcal{L}_n \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_n(a) + h_n(b) & \sum_{i+j=n, i\neq n} \big((p_i(a) + h_i(b)) m_j - m_j (q_i(b) + h'_i(a)) \big) + S_n(m) \\ & q_n(b) + h'_n(a) \end{pmatrix}$$

where $\{m_j\}_{j\in\mathbb{N}} \subseteq \mathscr{M}$, and for each $n \in \mathbb{N}$, $h'_n : \mathscr{A} \longrightarrow Z(\mathscr{B})$, $h_n : \mathscr{B} \longrightarrow Z(\mathscr{A})$, $S_n : \mathscr{M} \longrightarrow \mathscr{M}$ are linear maps satisfying:

(1) $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ *are Lie higher derivations on $\mathscr{A}, \mathscr{B}$, respectively,*

(2) $h'_n[a,a'] = 0$ *and* $h_n[b,b'] = 0$ *for all* $a,a' \in \mathscr{A}, b,b' \in \mathscr{B}$, *and*

(3) $S_n(am) = \sum_{i+j=n}\left(p_i(a)S_j(m) - S_j(m)h'_i(a)\right)$ *and* $S_n(mb) = \sum_{i+j=n}\left(S_j(m)q_i(b) - h_i(b))S_j(m)\right)$, *for all* $a \in \mathscr{A}, b \in \mathscr{B}, m \in \mathscr{M}$.

Proof. We proceed the proof by induction on n. The case $n = 1$ follows from Cheung's result [3, Proposition 4]. Suppose that the conclusion is true for any integer less than n. Then $\mathcal{L}_n$ has the presentation

$$\mathcal{L}_n\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_n(a)+h_n(b)+k_n(m) & r_n(a)-r'_n(b)+S_n(m) \\ & q_n(b)+h'_n(a)+k'_n(m) \end{pmatrix}, \ (a \in \mathscr{A}, b \in \mathscr{B}, m \in \mathscr{M}),$$

in which the maps appeared in the entries are linear. As for each $n \in \mathbb{N}, a \in \mathscr{A}$,

$$\begin{aligned}
0 &= \mathcal{L}_n[(a\oplus 0),(1\oplus 0)] \\
&= \sum_{i+j=n}[\mathcal{L}_i(a\oplus 0),\mathcal{L}_j(1\oplus 0)] \\
&= \left[\begin{pmatrix} p_n(a) & r_n(a) \\ & h'_n(a) \end{pmatrix},\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix},\begin{pmatrix} p_n(1) & r_n(1) \\ & h'_n(1) \end{pmatrix}\right] \\
&\quad + \sum_{i+j=n,i,j\neq n}\left[\begin{pmatrix} p_i(a) & am_i \\ & h'_i(a) \end{pmatrix},\begin{pmatrix} p_j(1) & m_j \\ & h'_j(1) \end{pmatrix}\right] \\
&= \begin{pmatrix} \sum_{i+j=n}[p_i(a),p_j(1)] & -r_n(a)+ar_n(1)+\sum_{i+j=n,i,j\neq n}p_i(a)m_j - m_jh'_i(a) \\ & \sum_{i+j=n}[h'_i(a),h'_j(1)] \end{pmatrix}.
\end{aligned}$$

It follows that $r_n(a) = \sum_{i+j=n,i\neq n}(p_i(a)m_j - m_jh'_i(a))$.
For each $n \in \mathscr{M}$ we have $k_n = 0$ and $k'_n = 0$, as

$$\begin{aligned}
\begin{pmatrix} k_n(m) & S_n(m) \\ & k'_n(m) \end{pmatrix} &= \mathcal{L}_n\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \\
&= \mathcal{L}_n\left[\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix},\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] \\
&= \left[\mathcal{L}_n\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix},\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix},\mathcal{L}_n\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] \\
&\quad + \sum_{i+j=n,i,j\neq}\left[\mathcal{L}_i\begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix},\mathcal{L}_j\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & * \\ & 0 \end{pmatrix}.
\end{aligned}$$

Since for each $n \in \mathbb{N}, a, a' \in \mathscr{A}$,

$$\begin{aligned}
\begin{pmatrix} p_n[a,a'] & * \\ & h'_n[a,a'] \end{pmatrix} &= \mathcal{L}_n([a,a'] \oplus 0) \\
&= \mathcal{L}_n([a \oplus 0, a' \oplus 0]) \\
&= [\mathcal{L}_n(a \oplus 0), a' \oplus 0] + [a \oplus 0, \mathcal{L}_n(a' \oplus 0)] + \sum_{i+j=n, i, j \neq n} [\mathcal{L}_i(a \oplus 0), \mathcal{L}_j(a' \oplus 0)] \\
&= \left[\begin{pmatrix} p_n(a) & * \\ 0 & h'_n(a) \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p_n(a') & * \\ 0 & h'_n(a') \end{pmatrix} \right] \\
&\quad + \sum_{i+j=n, i, j \neq n} \left[\begin{pmatrix} p_i(a) & * \\ 0 & h'_i(a) \end{pmatrix}, \begin{pmatrix} p_j(a') & * \\ 0 & h'_j(a') \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} \sum_{i+j=n} [p_i(a), p_j(a')] & * \\ 0 & \sum_{i+j=n, i, j \neq n} [h'_i(a), h'_j(a')] \end{pmatrix} \right].
\end{aligned}$$

By the hypothesis of induction, $\sum_{i+j=n,i,j\neq n} [h'_i(a), h'_j(a')] = 0$, therefore $h'_n[a,a'] = 0$ and $p_n[a,a'] = \sum_{i+j=n} [p_i(a), p_j(a')]$. Since for each $n \in \mathbb{N}, a \in \mathscr{A}, m \in \mathscr{M}$,

$$\begin{aligned}
\begin{pmatrix} 0 & S_n(am) \\ & 0 \end{pmatrix} &= \mathcal{L}_n \begin{pmatrix} 0 & am \\ & 0 \end{pmatrix} \\
&= \mathcal{L}_n \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \left[\mathcal{L}_n \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \mathcal{L}_n \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&\quad + \sum_{i+j=n, i, j \neq n} \left[\mathcal{L}_i \begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \mathcal{L}_j \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} p_n(a) & \sum_{i+j=n, i\neq n} (p_i(a) m_j - m_j h'_i(a)) \\ 0 & h'_n(a) \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right] \\
&\quad + \left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & S_n(m) \\ 0 & 0 \end{pmatrix} \right] \\
&\quad + \sum_{i+j=n, i, j \neq n} \left[\begin{pmatrix} p_i(a) & \sum_{s+t=i, s\neq i} (p_s(a) m_t - m_t h'_s(a)) \\ 0 & h'_i(a) \end{pmatrix}, \begin{pmatrix} 0 & S_j(m) \\ 0 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} * & \sum_{i+j=n, j\neq n} (p_i(a) S_j(m) - S_j(m) h'_i(a)) + a S_n(m) \\ 0 & * \end{pmatrix},
\end{aligned}$$

we have

$$S_n(am) = \sum_{i+j=n} \Big(p_i(a) S_j(m) - S_j(m) h'_i(a) \Big).$$

From $[(a\oplus 0),(0\oplus b)]=0$, we get

$$\begin{aligned} 0 &= \mathcal{L}_n[(a\oplus 0),(0\oplus b)] \\ &= \left[\begin{pmatrix} p_n(a) & * \\ & h'_n(a)\end{pmatrix},\begin{pmatrix} 0 & 0 \\ & b\end{pmatrix}\right]+\left[\begin{pmatrix} a & 0 \\ & 0\end{pmatrix},\begin{pmatrix} h_n(b) & * \\ & q_n(b)\end{pmatrix}\right] \\ &\quad + \sum_{i+j=n,i,j\neq n}\left[\begin{pmatrix} p_i(a) & * \\ & h'_i(a)\end{pmatrix},\begin{pmatrix} h_j(b) & * \\ & q_j(b)\end{pmatrix}\right] \\ &= \begin{pmatrix} ah_n(b)-h_n(b)a & * \\ & h'_n(a)b-bh'_n(a)\end{pmatrix}. \end{aligned}$$

Again by the hypothesis, $\sum_{i+j=n,i,j\neq n}[p_i(a),h_j(b)]=0$ and $\sum_{i+j=n,i,j\neq n}[h'_i(a),q_j(b)]=0$ which imply $[a,h_n(b)]=0$ and $[h'_n(a),b]=0$ for all $a\in\mathscr{A}$, $b\in\mathscr{B}$. Thus we have $h'_n(\mathscr{A})\subseteq Z(\mathscr{B})$ and $h_n(\mathscr{B})\subseteq Z(\mathscr{A})$, as required. Similarly for each $b\in\mathscr{B}, n\in\mathbb{N}$, $r'_n(b)=\sum_{i+j=n,i\neq n}(h_i(b)m_j-m_jq_i(b))$. By a similar way one can prove the required assertions for $\mathscr{B}$. □

In the next result we present the construction of a higher derivation on $\mathfrak{A}=\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$.

Theorem 2.2. *Let $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ be a sequence of linear maps on $\mathfrak{A}=\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$. Then $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ is a higher derivation if and only if $\mathscr{D}_n$ can be written as*

$$\mathscr{D}_n\begin{pmatrix} a & m \\ & b\end{pmatrix}=\begin{pmatrix} p_n(a) & \sum_{i+j=n,i\neq n}\big(p_i(a)m_j-m_jq_i(b)\big)+S_n(m) \\ & q_n(b)\end{pmatrix}$$

where $\{m_j\}_{j\in\mathbb{N}}\subseteq\mathscr{M}$, and for each $n\in\mathbb{N}$, $p_n:\mathscr{A}\longrightarrow\mathscr{A}$, $q_n:\mathscr{B}\longrightarrow\mathscr{B}$, $S_n:\mathscr{M}\longrightarrow\mathscr{M}$ are linear maps satisfying:

(1) *$\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ are higher derivations on $\mathscr{A},\mathscr{B}$, respectively, and*

(2) *$S_n(am)=\sum_{i+j=n}p_i(a)S_j(m)$ and $S_n(mb)=\sum_{i+j=n}S_j(m)p_i(b)$, for all $a\in\mathscr{A}, b\in\mathscr{B}, m\in\mathscr{M}$.*

Proof. As every higher derivations is a Lie higher derivation, $\mathcal{L}_n$ has the presentation as in Theorem 2.1. Following the proof of Theorem 2.1, that $\{p_n\}_{n\in\mathbb{N}}$ is a higher derivation follows from the fact that for every $a,a'\in\mathscr{A}$, $(a\oplus 0)(a'\oplus 0)=(aa'\oplus 0)$. Further, the equality $(a\oplus 0)(1\oplus 0)=(a\oplus 0)$, implies that $h'_n=0$. A similar argument prove that $\{q_n\}_{n\in\mathbb{N}}$ is a higher derivation and $h_n=0$, as required. □

We are ready to present the main result of the paper which gives a necessary and sufficient condition for properness of a Lie higher derivation on $\mathfrak{A}=\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$.

Theorem 2.3. *A Lie higher derivation $\mathcal{L}=\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ on $\mathfrak{A}=\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ with the presentation (as given in Theorem 2.1)*

$$\mathcal{L}_n\begin{pmatrix} a & m \\ & b\end{pmatrix}=\begin{pmatrix} p_n(a)+h_n(b) & \sum_{i+j=n,i\neq n}\big(p_i(a)+h_i(b)\big)m_j-m_j\big(q_i(b)+h'_i(a)\big)+S_n(m) \\ & q_n(b)+h'_n(a)\end{pmatrix}$$

is proper if and only if there exist two sequences $\{l_n\}_{n\in\mathbb{N}}:\mathscr{A}\longrightarrow Z(\mathscr{A})$ and $\{l'_n\}_{n\in\mathbb{N}}:\mathscr{B}\longrightarrow Z(\mathscr{B})$ of linear maps such that

(1) *$\{p_n-l_n\}_{n\in\mathbb{N}}$ and $\{q_n-l'_n\}_{n\in\mathbb{N}}$ are higher derivations on $\mathscr{A}$ and $\mathscr{B}$, respectively,*

(2) *$l_n([a,a'])=0$ and $l'_n([b,b'])=0$ for all $n\in\mathbb{N}, a,a'\in\mathscr{A}, b,b'\in\mathscr{B}$, and*

(3) *$l_n(a)m=mh'_n(a)$ and $h_n(b)m=ml'_n(b)$, for all $n\in\mathbb{N}, a\in\mathscr{A}, b\in\mathscr{B}, m\in\mathscr{M}$.*

Proof. To prove "if" part, if we set

$$\mathscr{D}_n\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_n(a) - l_n(a) & \sum_{i+j=n, i\neq n}\Big((p_i(a) - l_i(a))m_j - m_j(q_i(b) - l'_i(b))\Big) + S_n(m) \\ & q_n(b) - l'_n(b) \end{pmatrix},$$

and

$$f_n\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} l_n(a) + h_n(b) & 0 \\ & l'_n(b) + h'_n(a) \end{pmatrix}.$$

Theorem 2.2 confirms that $\mathscr{D} = \{\mathscr{D}_n\}_{n\in\mathbb{N}}$ is a higher derivation and a direct verification reveals that $\mathscr{F} = \{f_n\}_{n\in\mathbb{N}}$ satisfies the required properties. Clearly $\mathcal{L} = \mathscr{D} + \mathscr{F}$, so $\mathcal{L}$ is proper.

For the converse, suppose that $\mathcal{L}$ is proper i.e. $\mathcal{L} = \mathscr{D} + \mathscr{F}$ for some sequence of center valued mapping $\mathscr{F} = \{f_n\}_{n\in\mathbb{N}} : \mathfrak{A} \longrightarrow Z(\mathfrak{A})$ that vanishing at commutators, and a higher derivation $\mathscr{D} : \mathfrak{A} \longrightarrow \mathfrak{A}$. According to (2.1), we may assume that

$$f_n\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} l_n(a) + k_n(b) & 0 \\ 0 & k'_n(a) + l'_n(b) \end{pmatrix}.$$

By Theorem 2.2, $\mathscr{D}$ can be written as

$$\mathscr{D}_n\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p'_n(a) & \sum_{i+j=n, i\neq n}\Big(p'_i(a)m_j - m_j q'_i(b)\Big) + S_n(m) \\ & q'_n(b) \end{pmatrix}.$$

Now, from Theorem 2.1 we have $p_n(a) + h_n(b) = p'_n(a) + l_n(a) + k_n(b)$, for all $a \in \mathscr{A}$, $b \in \mathscr{B}$ and $n \in \mathbb{N}$. Set $a = 0$ in the last equation hence $h_n(b) = k_n(b)$, again set $b = 0$, thus $p_n(a) - l_n(a) = p'_n(a)$, i.e $\{p_n - l_n\}_{n\in\mathbb{N}}$ is a higher derivation on $\mathscr{A}$. That $l_n([a, a']) = 0$, $l_n(a)m = mh'_n(a)$ follows from the properties of $Z(\mathfrak{A})$. The other statements can be derived similarly. □

With the notation as in Theorem 2.1, suppose that $\{p_n = p'_n + l_n\}_{n\in\mathbb{N}}$, where $\{p'_n\}_{n\in\mathbb{N}}$ is a higher derivation on $\mathscr{A}$ and $\{l_n\}_{n\in\mathbb{N}} : \mathscr{A} \longrightarrow Z(\mathscr{A})$ annihilates the commutators. Mimic the methods of Cheung [3], we define a powerful subset $\mathcal{V}_{\mathscr{A}}$ of $\mathscr{A}$ by

$$\mathcal{V}_{\mathscr{A}} = \bigcap_{n\in\mathbb{N}}\{a \in \mathscr{A} : \ l_n(a)m = mh'_n(a) \ for\ all\ m \in \mathscr{M}\}.$$

With some modifications in the proof of [3, Proposition 10], using an induction process, it can be shown that $\mathcal{V}_{\mathscr{A}}$ is a subalgebra of $\mathscr{A}$ containing all commutators and idempotents. For more details on the properties of $\mathcal{V}_{\mathscr{A}}$ and its relation to what was introduced by Cheung [3] see [7].

It can also be readily verified that

$$\mathcal{V}_{\mathscr{A}} \subseteq \bigcap_{n\in\mathbb{N}}\{a \in \mathscr{A} : h'_n(a) \in \pi_{\mathscr{B}}(Z(\mathfrak{A}))\}. \tag{2.2}$$

We denote by $\mathcal{W}_{\mathscr{A}}$ the smallest subalgebra of $\mathscr{A}$ contains all commutators and idempotents. Trivially $\mathcal{W}_{\mathscr{A}} \subseteq \mathcal{V}_{\mathscr{A}}$ and if $\mathcal{W}_{\mathscr{A}} = \mathscr{A}$ then for each $n \in \mathbb{N}$, $h'_n(\mathscr{A}) \subseteq \pi_{\mathscr{B}}(Z(\mathfrak{A}))$, consequently $\pi_{\mathscr{B}}(\mathcal{L}(\mathscr{A})) \subseteq \pi_{\mathscr{B}}(Z(\mathfrak{A}))$.

As an immediate consequence of the above theorem, we have the next corollary providing some necessary conditions for the properness of Lie higher derivations on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$.

Corollary 2.4. *Let $\mathcal{L} = \{\mathcal{L}_n\}_{n\in\mathbb{N}}$ be a Lie higher derivation on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ and $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ as given in Theorem 2.1. If $\mathcal{L}$ is proper then*

(i) *The Lie higher derivations $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ are proper on $\mathscr{A}$, $\mathscr{B}$, respectively.*

(ii) *$\mathcal{V}_{\mathscr{A}} = \mathscr{A}$ and $\mathcal{V}_{\mathscr{B}} = \mathscr{B}$.*

As another consequence of Theorem 2.1 we have the next result providing some sufficient conditions for a Lie higher derivation on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ to be proper.

Corollary 2.5. *Every Lie higher derivation on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ is proper if the following conditions are satisfied:*

(i) *Every Lie higher derivation on $\mathscr{A}$ and $\mathscr{B}$ are proper.*

(ii) $\mathcal{W}_{\mathscr{A}} = \mathscr{A}$ *and* $\mathcal{W}_{\mathscr{B}} = \mathscr{B}$.

In [3, Example 8] Cheung presented an improper Lie derivation L on certain triangular algebra $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ such that $\mathscr{A}$ and $\mathscr{B}$ do not have improper Lie derivations. It induces the ordinary Lie higher derivation $\mathcal{L} = \{\frac{L^n}{n!}\}_{n\in\mathbb{N}}$ on $\mathfrak{A}$ which is clearly improper. By the above corollary we should have either $\mathcal{W}_{\mathscr{A}} \neq \mathscr{A}$ or $\mathcal{W}_{\mathscr{B}} \neq \mathscr{B}$.

3. The Case Where $\mathscr{M}$ is Faithful

In this section we focus on the case when $\mathscr{M}$ is a faithful $(\mathscr{A}, \mathscr{B})-$module, i.e. $\mathscr{M}$ is both a faithful left $\mathscr{A}-$module and a faithful right $\mathscr{B}-$module. Similar to what was explored by Cheung, [2, 3], for (Lie) derivations on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$, we show that the faithfulness of $\mathscr{M}$ also provides some simplifications for the structures of (Lie) higher derivations. The proposed simplifications in the construction of (Lie) higher derivations are basically come from a unique algebra isomorphism $\tau : \pi_{\mathscr{B}}(Z(\mathfrak{A})) \longrightarrow \pi_{\mathscr{A}}(Z(\mathfrak{A}))$ satisfying $am = m\tau^{-1}(a)$, for all $m \in \mathscr{M}, a \in \mathscr{A}$, whose existence guaranteed by Cheung [3, Proposition 3].

In this case, for example, the center $Z(\mathfrak{A})$ of $\mathfrak{A}$ (see (2.1)) can be simplified as

$$Z(\mathfrak{A}) = \{a \oplus b |\, am = mb \ for\, all\ m \in \mathscr{M}\}.$$

It also follows that if $\mathscr{M}$ is faithful then by setting $l_n = \tau \circ h'_n, n \in \mathbb{N}$, as h'_n annihilates the commutators, l_n behaves the same and these imply that the equality holds in (2.2), i.e.

$$\mathcal{V}_{\mathscr{A}} = \bigcap_{n\in\mathbb{N}} \{a \in \mathscr{A} : h'_n(a) \in \pi_{\mathscr{B}}(Z(\mathfrak{A}))\}. \tag{3.1}$$

In particular, if $\mathscr{M}$ is faithful then $\mathcal{W}_{\mathscr{A}} = \mathscr{A}$ if and only if $\pi_{\mathscr{B}}(\mathcal{L}(\mathscr{A})) \subseteq \pi_{\mathscr{B}}(Z(\mathfrak{A}))$.

The faithfulness of $\mathscr{M}$ also brings some simplifications in Theorems 2.1, 2.2, 2.3. It can be verified that in this case, the condition (2) in Theorem 2.1 is superfluous, as it can be derived from (1), (3). Similarly, (1) in Theorem 2.2 can be derived from (2).

In Theorem 2.3 the conditions (1) and (2) can also be dropped and, by the above discussions, (3) is equivalent to saying that $\pi_{\mathscr{A}}(\mathcal{L}(\mathscr{B})) \subseteq \pi_{\mathscr{A}}(Z(\mathfrak{A}))$ and $\pi_{\mathscr{B}}(\mathcal{L}(\mathscr{A})) \subseteq \pi_{\mathscr{B}}(Z(\mathfrak{A}))$. We thus restate the faithful version of Theorem 2.3 as follows (see also Corollary 2.4).

Corollary 3.1. *Let $\mathcal{L} = \{\mathcal{L}_n\}_{n\in\mathbb{N}}$ be a Lie higher derivation on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$. If $\mathscr{M}$ is faithful then the following assertions are equivalent:*

(a) *$\mathcal{L}$ is proper.*

(b) $\pi_{\mathscr{B}}(\mathcal{L}(\mathscr{A})) \subseteq \pi_{\mathscr{B}}(Z(\mathfrak{A}))$ *and* $\pi_{\mathscr{A}}(\mathcal{L}(\mathscr{B})) \subseteq \pi_{\mathscr{A}}(Z(\mathfrak{A}))$.

(c) $\mathcal{W}_{\mathscr{A}} = \mathscr{A}$ *and* $\mathcal{W}_{\mathscr{B}} = \mathscr{B}$.

As a rapid consequence, we derive the next result from Li and Shen [5] which was proved by a different method. It also is an extension of [14, Theorem 2.1] and [10, Theorem 2.1].

Corollary 3.2 ([5, Theorem 3.1])**.** *Let $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ be a triangular algebra with faithful $\mathscr{M}$ such that $Z(\mathscr{A}) = \pi_{\mathscr{A}}(Z(\mathfrak{A}))$ and $Z(\mathscr{B}) = \pi_{\mathscr{B}}(Z(\mathfrak{A}))$. Then every Lie higher derivation on $\mathfrak{A}$ is proper.*

Proof. It is obvious that for every Lie higher derivation $\mathcal{L}$ on $\mathfrak{A}$, $\pi_{\mathscr{B}}(\mathcal{L}(\mathscr{A})) \subseteq Z(\mathscr{B})$ and $\pi_{\mathscr{A}}(\mathcal{L}(\mathscr{B})) \subseteq Z(\mathscr{A})$. As $Z(\mathscr{A}) = \pi_{\mathscr{A}}(Z(\mathfrak{A}))$ and $Z(\mathscr{B}) = \pi_{\mathscr{B}}(Z(\mathfrak{A}))$, the conclusion follows from Corollary 3.1. □

One may collect the assertions of the Corollaries 2.5, 3.1 and 3.2 to arrive the next result which is the "higher" version of a nice result of Cheung, [3, Theorem 11].

Theorem 3.3. *A Lie higher derivation on $\mathfrak{A} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ is proper if the following two conditions hold:*

(a) $Z(\mathscr{B}) = \pi_{\mathscr{B}}(Z(\mathfrak{A}))$ *and* $\mathscr{M}$ *is faithful left* $\mathscr{A}$*-module, or* $\mathscr{A} = \mathcal{W}_{\mathscr{A}}$ *and* $\mathscr{M}$ *is faithful left* $\mathscr{A}$*-module, or every Lie higher derivation of* $\mathscr{A}$ *is proper and* $\mathscr{A} = \mathcal{W}_{\mathscr{A}}$.

(b) $Z(\mathscr{A}) = \pi_{\mathscr{A}}(Z(\mathfrak{A}))$ *and* $\mathscr{M}$ *is faithful right* $\mathscr{B}$*-module, or* $\mathscr{B} = \mathcal{W}_{\mathscr{B}}$ *and* $\mathscr{M}$ *is faithful right* $\mathscr{B}$*-module, or every Lie higher derivation of* $\mathscr{B}$ *is proper and* $\mathscr{B} = \mathcal{W}_{\mathscr{B}}$.

In [3] Cheung has applied his results for some classic triangular algebras, such as upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. Based on the Corollaries 2.5, 3.1, 3.2, in the most cases the "higher" versions of Cheung's results on some classical triangular algebras are hold, however we omit the similar results here.

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