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New type of Lacunary Orlicz Difference Sequence Spaces Generated By Infinite Matrices

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Abstract. The main purpose of this paper is to introduce the spaces $\widehat{w}_{\theta}^{o}[A,M,\Delta,p]$, $\widehat{w}_{\theta}[A,M,\Delta,p]$ and $\widehat{w}_{\theta}^{\infty}[A,M,\Delta,p]$ generated by infinite matrices defined by Orlicz functions. Some properties of these spaces are discussed. Also we introduce the concept of $\widehat{S}_{\theta}[A,\Delta]-$statistical convergence and derive some results between the spaces $\widehat{S}_{\theta}[A,\Delta]$ and $\widehat{w}_{\theta}[A,\Delta]$. Further, we study some geometrical properties such as order continuous, the Fatou property and the Banach-Saks property of the new space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta,p]$. Finally, we introduce the notion of $\widehat{S}_{\theta}[A,\Delta]-$statistical convergence of order α of real number sequences and obtain some inclusion relations between the set of $\widehat{S}[A,\Delta]-$statistical convergence of order α.

1. Introduction

Let $p=(p_k)$ be a bounded sequence of positive real numbers. If $H=\sup_k p_k<\infty$, then for any complex numbers a_k and b_k

$$|a_k+b_k|^{p_k}\leq C\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{1}$$

where $C=\max(1,2^{H-1})$. Also, for any complex number α, (see [18])

$$|\alpha|^{p_k}\leq \max\left(1,|\alpha|^{H}\right). \tag{2}$$

We denote w,ℓ_∞,c and c_0, for the spaces of all, bounded, convergent, null sequences, respectively. Also, by ℓ_1 and ℓ_p, we denote the spaces of all absolutely summable and p-absolutely summable series, respectively. Recall that a sequence $(x(i))_{i=1}^{\infty}$ in a Banach space X is called *Schauder* (or *basis*) of X if for each $x\in X$ there exists a unique sequence $(a(i))_{i=1}^{\infty}$ of scalars such that $x=\sum_{i=1}^{\infty}a(i)x(i)$, i.e. $\lim_{n\to\infty}\sum_{i=1}^{n}a(i)x(i)=x$. A sequence space X with a linear topology is called a *K-space* if each of the projection maps $P_i:X\to\mathbb{C}$ defined by $P_i(x)=x(i)$ for $x=(x(i))_{i=1}^{\infty}\in X$ is continuous for each natural i. A *Fréchet space* is a complete metric linear space and the metric is generated by a *F-norm* and a Fréchet space which is a *K-space* is called an *FK-space* i.e. a *K-space* X is called an *FK-space* if X is a complete linear metric space. In other words, X is an *FK-space* if X is a Fréchet space with contnuous cordinatewise projections. All the sequence spaces mentioned above

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are *FK*-space except the space c_{00}. An *FK*-space X which contains the space c_{00} is said to have the *property AK* if for every sequence $(x(i))_{i=1}^{\infty} \in X, x = \sum_{i=1}^{\infty} x(i)e(i)$ where $e(i) = (0, 0, ...1^{i^{th} place}, 0, 0, ...)$.

A Banach space X is said to be a *Köthe sequence space* if X is a subspace of w such that

(a) if $x \in w, y \in X$ and $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, then $x \in X$ and $\|x\| \leq \|y\|$
(b) there exists an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbb{N}$.

We say that $x \in X$ is *order continuous* if for any sequence $(x_n) \in X$ such that $x_n(i) \leq |x(i)|$ for all $i \in \mathbb{N}$ and $x_n(i) \to 0$ as $n \to \infty$ we have $\|x_n\| \to 0$ as $n \to \infty$ holds.

A Köthe sequence space X is said to be *order continuous* if all sequences in X are order continuous. It is easy to see that $x \in X$ order continuous if and only if $\|(0, 0, ..., 0, x(n + 1), x(n + 2), ...)\| \to 0$ as $n \to \infty$.

A Köthe sequence space X is said to have the *Fatou property* if for any real sequence x and (x_n) in X such that $x_n \uparrow x$ coordinatewisely and $\sup_n \|x_n\| < \infty$, we have that $x \in X$ and $\|x_n\| \to \|x\|$ as $n \to \infty$.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $(t_k(z))$ is convergent in X with respect to the norm, where

$$t_k(z) = \frac{z_1 + z_2 + ... + z_k}{k} \text{ for all } k \in \mathbb{N}.$$

Some of works on geometric properties of sequence space can be found in [1, 2, 16, 19].

An *Orlicz function* M is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, convex, nondecreasing function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the *modulus function* and characterized by Nakano [20], followed by Ruckle [24]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1.1. *An Orlicz function satisfies the inequality* $M(\lambda x) \leq \lambda M(x)$ *for all* λ *with* $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{(x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0\right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1\right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage, different Orlicz sequence spaces were introduced and studied by Esi [3, 4, 6], Esi and Et [5], Güngör and Et [15], Parashar and Choudhary [22], Tripathy and Mahanta [26], Tripathy and Hazarika [27], and many others.

2. Classes of Lacunary Orlicz Difference Sequences

The strongly almost summable sequence spaces were introduced and studied by Maddox [18], Nanda [21], Güngör et al., [12], Esi [7], Gungor and Et [15] and many authors. For matrix maps on sequence spaces we refer to [23] and for difference sequence spaces we refer to [28–31] and references therein.

By lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of positive integers satisfying; $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. We denote the intervals, which θ determines, by $I_r = (k_{r-1}, k_r]$. Let A=$\left(a_{ij}\right)$ be an infinite matrix of non-negative real numbers with all rows are linearly independent for all $i, j = 1, 2, 3, ...$ and $B_{kn}(x) = \sum_{i=1}^{\infty} a_{ki} x_{n+i}$ if the series converges for each k and n. Now we define the following sequence spaces. Let M be an Orlicz function, $p = (p_k)$ be a sequence of positive real numbers and $\theta = (k_r)$ be a lacunary sequence, and for $\rho > 0$ then

$$\widehat{w}_{\theta}^{o}[A, M, \Delta, p] = \left\{ x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} = 0, \text{ uniformly on } n \right\},$$

$$\widehat{w}_{\theta}[A, M, \Delta, p] = \left\{ x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho} \right) \right]^{p_k} = 0, \text{ for some } L, \text{ uniformly on } n \right\}$$

and

$$\widehat{w}_{\theta}^{\infty}[A, M, \Delta, p] = \left\{ x \in w : \sup_{r} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho} \right) \right]^{p_k} < \infty, \text{ uniformly on } n \right\},$$

where $\Delta B_{kn}(x) = \sum_{i=1}^{\infty} (a_{ki} - a_{k+1,i}) x_{n+i}$.

Theorem 2.1. *For any Orlicz function M and a bounded sequence $p = (p_k)$ of positive real numbers, $\widehat{w}_{\theta}^{o}[A, M, \Delta, p]$, $\widehat{w}_{\theta}[A, M, \Delta, p]$ and $\widehat{w}_{\theta}^{\infty}[A, M, \Delta, p]$ are linear spaces over the set of complex field.*

Proof. We give the proof only for the space $\widehat{w}_{\theta}^{o}[A, M, \Delta, p]$ and for other spaces follow by applying similar method. Let $x = (x_k), y = (y_k) \in \widehat{w}_{\theta}^{o}[A, M, \Delta, p]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1} \right) \right]^{p_k} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{\left|\Delta B_{kn}(y)\right|}{\rho_2} \right) \right]^{p_k} = 0.$$

Define $\rho_3 = \max\left\{ 2|\alpha|\rho_1, 2\left|\beta\right|\rho_2 \right\}$. Since the operator ΔB_{kn} is linear and M is non-decreasing and convex, we have

$$\frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{\left|\Delta B_{kn}(\alpha x + \beta y)\right|}{\rho_3} \right) \right]^{p_k}$$

$$= \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{\left|\alpha \Delta B_{kn}(x) + \beta \Delta B_{kn}(y)\right|}{\rho_3} \right) \right]^{p_k}$$

$$\leq \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|\alpha \Delta B_{kn}(x)|}{\rho_3} \right) + M\left(\frac{\left|\beta \Delta B_{kn}(y)\right|}{\rho_3} \right) \right]^{p_k}$$

$$\leq \frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)+M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}$$

$$\leq \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)+M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}$$

$$\leq \frac{C}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho_1}\right)\right]^{p_k}+\frac{C}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(y)|}{\rho_2}\right)\right]^{p_k}\to 0 \text{ as } r\to\infty,$$

where $C=\max(1,2^{H-1})$, so $\alpha x+\beta y\in \widehat{w}_\theta^o[A,M,\Delta,p]$, hence it is a linear space. □

Theorem 2.2. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of positive real numbers, $\widehat{w}_\theta^o[A,M,\Delta,p]$ is a topological paranormed space, paranormed by*

$$g(x)=\inf\left\{\rho^{\frac{p_r}{H}}:\ \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{T}}\leq 1,\ r=1,2,3,...\right\}$$

where $T=\max(1,\sup_k p_k=H)$.

Proof. The subadditivity of g follows from the Theorem 2.1 by taking $\alpha=\beta=1$ and it is clear that $g(x)=g(-x)$. Since $M(0)=0$, we get $\inf\left\{\rho^{\frac{p_r}{H}}\right\}=0$ for $x=0$. Suppose that $x_k\neq 0$ for each $k\in\mathbb{N}$. This implies that $\Delta B_{kn}(x)\neq 0$ for each k and n. Let $\varepsilon\to 0$, then

$$\frac{|\Delta B_{kn}(x)|}{\varepsilon}\to\infty.$$

It follows that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\varepsilon}\right)\right]^{p_k}\right)^{\frac{1}{T}}\to\infty$$

which is a contradiction. Now we prove that scalar multiplication is continuous. Let λ be any complex number, by definition

$$g(\lambda x)=\inf\left\{\rho^{\frac{p_r}{H}}:\ \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(\lambda x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{T}}\leq 1,\ r=1,2,3,...\right\}$$

$$=\inf\left\{\rho^{\frac{p_r}{H}}:\ \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\lambda|\,|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{T}}\leq 1,\ r=1,2,3,...\right\}.$$

Suppose that $s=\frac{\rho}{\lambda}$, then $\rho=s|\lambda|$ and since $|\lambda|^{p_k}\leq\max\left(1,|\lambda|^H\right)$ we have

$$g(\lambda x)\leq|\lambda|^{p_k}\leq\max\left(1,|\lambda|^H\right)\inf\left\{s^{\frac{p_r}{H}}:\ \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{s}\right)\right]^{p_k}\right)^{\frac{1}{T}}\leq 1,\ r=1,2,3,...\right\}$$

which converges to zero as x converges to zero in $\widehat{w}_\theta^o[A,M,\Delta,p]$. Now suppose that $\lambda_i\to 0$ as $i\to\infty$ and x is fixed in $\widehat{w}_\theta^o[A,M,\Delta,p]$. For arbitrary $\varepsilon>0$ and let r_o be a positive integer such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\leq\left(\frac{\varepsilon}{2}\right)^T$$

for some $\rho > 0$ and $r > r_o$. This implies that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\lambda\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{T}} < \frac{\varepsilon}{2}$$

for some $\rho > 0$ and $r > r_o$. Let $0 < |\lambda| < 1$. Using the convexity of Orlicz function M, for $r > r_o$, we get

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k} \leq \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\lambda|\,|\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^T.$$

Since M is continuous everywhere in $[0,\infty)$, then we consider the function, for $r \leq r_o$

$$f(t) = \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}.$$

Then f is continous at zero. So there is a $\delta \in (0,1)$ such that $\left|f(t)\right| < \left(\frac{\varepsilon}{2}\right)^T$ for $0 < t < \delta$. Let A be such that $|\lambda_i| < \delta$ for $i > A$ and $r \leq r_o$

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\lambda_i\Delta B_{kn}(x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{T}} < \frac{\varepsilon}{2}$$

for $i > A$ and all r, so that $g(\lambda x) \to 0$ as $\lambda \to 0$. This completes the proof. □

Theorem 2.3. *Let the sequence $p = (p_k)$ be bounded. Then $\widehat{w}^o_\theta[A,M,\Delta,p] \subset \widehat{w}_\theta[A,M,\Delta,p] \subset \widehat{w}^\infty_\theta[A,M,\Delta,p]$.*

Proof. Let $x = (x_k) \in \widehat{w}^o_\theta[A,M,\Delta,p]$. Then we have

$$\begin{aligned}&\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|\Delta B_{kn}(x)|}{2\rho}\right)\right]^{p_k}\\ &\leq \frac{C}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k} + \frac{C}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|L|}{\rho}\right)\right]^{p_k}\\ &\leq \frac{C}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k} + C\max\left(1,\sup\left[M\left(\frac{|L|}{\rho}\right)\right]^H\right),\end{aligned}$$

where $H = \sup_k p_k < \infty$ and $C = \max\left(1, 2^{H-1}\right)$. Thus we have $x = (x_k) \in \widehat{w}_\theta[A,M,\Delta,p]$. The inclusion $\widehat{w}_\theta[A,M,\Delta,p] \subset \widehat{w}^\infty_\theta[A,M,\Delta,p]$ is obvious. □

Theorem 2.4. *If $0 < p_k < q_k$ and $\left(\frac{q_k}{p_k}\right)$ is bounded, then $\widehat{w}_\theta[A,M,\Delta,p] \subset \widehat{w}_\theta[A,M,\Delta,q]$.*

Proof. If we take $\left[M\left(\frac{|\Delta B_{kn}(x)-L|}{\rho}\right)\right]^{p_k} = w_k$ for all $k \in \mathbb{N}$, then using the same technique employed in the proof of Theorem 2.9 of Güngör et al., [12]. □

Corollary 2.5. *The following statements are valid.*

(i) If $0 < \inf_k p_k \leq 1$ for all $k \in \mathbb{N}$, then $\widehat{w}_\theta[A,M,\Delta,p] \subset \widehat{w}_\theta[A,M,\Delta]$.
(ii) If $1 \leq p_k \leq \sup_k p_k = H < \infty$ for all $k \in \mathbb{N}$, then $\widehat{w}_\theta[A,M,\Delta] \subset \widehat{w}_\theta[A,M,\Delta,p]$.

The proof of the following result is a routine work, so we omitted.

Proposition 2.6. *Let M be an Orlicz function satisfies Δ_2-condition. Then $\widehat{w}^o_\theta[A,\Delta,p] \subset \widehat{w}^o_\theta[A,M,\Delta,p]$, $\widehat{w}_\theta[A,\Delta,p] \subset \widehat{w}_\theta[A,M,\Delta,p]$ and $\widehat{w}^\infty_\theta[A,\Delta,p] \subset \widehat{w}^\infty_\theta[A,M,\Delta,p]$.*

3. New Sequence Space of Order α

In this section let $\alpha \in (0,1]$ be any real number, $\theta = (k_r)$ be a lacunary sequence, and p be a positive real number such that $1 < p < \infty$. Now we define the following sequence space.

$$\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) = \left\{ x \in w : \sup_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} |\Delta B_{kn}(x)|^p < \infty, \text{ uniformly on } n. \right\}$$

Special cases:

(a) For $p = 1$ we have $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) = \widehat{w}_{\theta\alpha}^{\infty}[A,\Delta]$.
(b) For $\alpha = 1$ and $p = 1$ we have $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) = \widehat{w}_{\theta}^{\infty}[A,\Delta]$.

Theorem 3.1. *Let $\alpha \in (0,1]$ and p be a positive real number such that $1 \le p < \infty$. Then the sequence space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ is a BK-space normed by*

$$\|x\|_{\alpha} = \sup_r \frac{1}{h_r^{\alpha}} \left(\sum_{k \in I_r} |\Delta B_{kn}(x)|^p \right)^{\frac{1}{p}}.$$

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.2. *Let $\alpha \in (0,1]$ and p be a positive real number such that $1 \le p < \infty$. Then $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta] \subset \widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$.*

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.3. *Let α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$ and p be a positive real number such that $1 \le p < \infty$. Then $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) \subset \widehat{w}_{\theta\beta}^{\infty}[A,\Delta](p)$.*

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.4. *Let α and β be fixed real numbers with $0 < \alpha \le \beta \le 1$ and p be a positive real number such that $1 \le p < \infty$. For any two lacubary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all r, then $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) \subset \widehat{w}_{\phi\alpha}^{\infty}[A,\Delta](p)$ if and only if $\sup_r \left(\frac{h_r^{\alpha}}{l_r^{\beta}}\right) < \infty$.*

Proof. Let $x = (x_k) \in \widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ and $\sup_{r \ge 1} \left(\frac{h_r^{\alpha}}{l_r^{\beta}}\right) < \infty$. Then

$$\sup_r \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} |\Delta B_{kn}(x)|^p < \infty$$

and there exists a positive number K such that $h_r^{\alpha} \le K l_r^{\beta}$ and so that $\frac{1}{l_r^{\beta}} \le \frac{K}{h_r^{\alpha}}$ for all r. Therefore, we have

$$\frac{1}{l_r^{\beta}} \sum_{k \in I_r} |\Delta B_{kn}(x)|^p \le \frac{K}{h_r^{\alpha}} \sum_{k \in I_r} |\Delta B_{kn}(x)|^p.$$

Now taking supremum over r, we get

$$\sup_r \frac{1}{l_r^{\beta}} \sum_{k \in I_r} |\Delta B_{kn}(x)|^p \le \sup_r \frac{K}{h_r^{\alpha}} \sum_{k \in I_r} |\Delta B_{kn}(x)|^p$$

and hence $x \in \widehat{w}_{\phi\alpha}^{\infty}[A,\Delta](p)$.

Next suppose that $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) \subset \widehat{w}_{\phi\alpha}^{\infty}[A,\Delta](p)$ and $\sup_r\left(\frac{h_r^\alpha}{l_r^\beta}\right) = \infty$. Then there exists an incresing sequence (r_i) of natural numbers such that $\lim_i\left(\frac{h_{r_i}^\alpha}{l_{r_i}^\beta}\right) = \infty$. Let L be a positive real number, then there exists $i_0 \in \mathbb{N}$ such that $\frac{h_{r_i}^\alpha}{l_{r_i}^\beta} > L$ for all $r_i \geq i_0$. Then $h_{r_i}^\alpha > Ll_{r_i}^\beta$ and so $\frac{1}{l_{r_i}^\beta} > \frac{L}{h_{r_i}^\alpha}$. Therefore we can write

$$\frac{1}{l_{r_i}^\beta}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p > \frac{L}{h_{r_i}^\alpha}\sum_{k\in I_r}|\Delta B_{kn}(x)|^p \text{ for all } r_i \geq i_0.$$

Now taking supremum over $r_i \geq i_0$ then we get

$$\sup_{r_i\geq i_0}\frac{1}{l_{r_i}^\beta}\sum_{k\in I_{r_i}}|\Delta B_{kn}(x)|^p > \sup_{r_i\geq i_0}\frac{L}{h_{r_i}^\alpha}\sum_{k\in I_{r_i}}|\Delta B_{kn}(x)|^p. \tag{3}$$

Since the relation (3) holds for all $L \in \mathbb{R}^+$ (we may take the number L sufficienlty large), we have

$$\sup_{r_i\geq i_0}\frac{1}{l_{r_i}^\beta}\sum_{k\in I_{r_i}}|\Delta B_{kn}(x)|^p = \infty$$

but $x = (x_k) \in \widehat{w}_{\theta\alpha}^{\infty}[A,\Delta,p]$ with

$$\sup_r\left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty.$$

Therefore $x \notin \widehat{w}_{\phi\alpha}^{\infty}[A,\Delta](p)$ which contradicts that $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) \subset \widehat{w}_{\phi\alpha}^{\infty}[A,\Delta](p)$. Hence $\sup_{r\geq 1}\left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty$. □

Corollary 3.5. *Let α and β be fixed real numbers with $0 < \alpha \leq \beta \leq 1$ and p be a positive real umber such that $1 \leq p < \infty$. For any two lacubary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all $r \geq 1$, then*

(a) $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) = \widehat{w}_{\phi\beta}^{\infty}[A,\Delta](p)$ *if and only if* $0 < \inf_r\left(\frac{h_r^\alpha}{l_r^\beta}\right) < \sup_r\left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty$.

(b) $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) = \widehat{w}_{\phi\alpha}^{\infty}[A,\Delta](p)$ *if and only if* $0 < \inf_r\left(\frac{h_r^\alpha}{l_r^\alpha}\right) < \sup_r\left(\frac{h_r^\alpha}{l_r^\alpha}\right) < \infty$.

(c) $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) = \widehat{w}_{\theta\beta}^{\infty}[A,\Delta](p)$ *if and only if* $0 < \inf_r\left(\frac{h_r^\alpha}{h_r^\beta}\right) < \sup_r\left(\frac{h_r^\alpha}{h_r^\beta}\right) < \infty$.

We state the following results without proof.

Theorem 3.6. $\ell_p[A,\Delta] \subset \widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) \subset \ell_\infty[A,\Delta]$.

Proof. The proof of the result is straightforward, so omitted. □

Theorem 3.7. *If $0 < p < q$, then $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p) \subset \widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](q)$.*

Proof. The proof of the result is straightforward, so omitted. □

4. Some Geometric Properties of the New Space

In this section we study some of the geometric properties like order continuous, the Fatou property and the Banach-Saks property of type p in this new sequence space.

Theorem 4.1. *The space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ is order continuous.*

Proof. To show that the space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ is an AK-space. It is easy to see that $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ contains c_{00} which is the space of real sequences which have only a finite number of non-zero coordinates. By using the definition of AK-properties, we have that $x=(x(i))\in\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ has a unique representation $x=\sum_{i=1}^{\infty}x(i)e(i)$ i.e. $\|x-x^{[j]}\|_{\alpha}=\|(0,0,..,x(j),x(j+1),...)\|_{\alpha}\to 0$ as $j\to\infty$, which means that $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ has AK. Therefore BK-space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ contains c_{00} has AK-property, hence the space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ is order continuous. □

Theorem 4.2. *The space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ has the Fatou property.*

Proof. Let x be a real sequence and (x_j) be any nondecreasing sequence of non-negative elements form $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ such that $x_j(i)\to x(i)$ as $j\to\infty$ coordinatewisely and $\sup_j\|x_j\|_{\alpha}<\infty$.

Let us denote $T=\sup_j\|x_j\|_{\alpha}$. Since the supremum is homogeneous, then we have

$$\frac{1}{T}\sup_r\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}|\Delta B_{kn}\left(x_j(i)\right)|^p\right)^{\frac{1}{p}}\leq\sup_r\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}\left|\frac{\Delta B_{kn}\left(x_j(i)\right)}{\|x_n\|_{\alpha}}\right|^p\right)^{\frac{1}{p}}=\frac{1}{\|x_n\|_{\alpha}}\|x_n\|_{\alpha}=1.$$

Also by the assumptions that (x_j) is non-dreceasing and convergent to x coordinatewisely and by the Beppo-Levi theorem, we have

$$\frac{1}{T}\lim_{j\to\infty}\sup_r\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}|\Delta B_{kn}\left(x_j(i)\right)|^p\right)^{\frac{1}{p}}=\sup_r\frac{1}{h_r^{\alpha}}\left(\sum_{k\in I_r}\left|\frac{\Delta B_{kn}(x(i))}{T}\right|^p\right)^{\frac{1}{p}}\leq 1,$$

whence

$$\|x\|_{\alpha}\leq T=\sup_j\|x_j\|_{\alpha}=\lim_{j\to\infty}\|x_j\|_{\alpha}<\infty.$$

Therefore $x\in\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$. On the other hand, since $0\leq x_j$ for any natural number j and the sequence (x_j) is non-decreasing, we obtain that the sequence $\left(\|x_j\|_{\alpha}\right)$ is bounded form above by $\|x\|_{\alpha}$. Therefore $\lim_{j\to\infty}\|x_j\|_{\alpha}\leq\|x\|_{\alpha}$ which contadicts the above inequality proved already, yields that $\|x\|_{\alpha}=\lim_{j\to\infty}\|x_j\|_{\alpha}$. □

Theorem 4.3. *The space $\widehat{w}_{\theta\alpha}^{\infty}[A,\Delta](p)$ has the Banach-Saks property.*

Proof. The proof of the result follows from the standard technique. □

5. Lacunary Statistical Convergence

The notion of statistical convergence was introduced by Fast [8] and studied various authors (see [7, 9, 25]). The notion of lacunary statistical convergence was introduced by Fridy and Orhan [10] and has been investgated for the real case in [11]. For more details on lacunary statistical convergence we refer to [13, 14] and many others. In this section, we define the concept of $\widehat{S_\theta}[A,\Delta]$-statistical convergence and establish the relationship of $\widehat{S_\theta}[A,\Delta]$ with $\widehat{w}_\theta[A,\Delta]$. Also we introduce the notion of $\widehat{S_\theta}[A,\Delta]-$statistical convergence of order α of real number sequences and obtain some inclusion relations between the set of $\widehat{S}[A,\Delta]-$statistical convergence of order α.

Definition 5.1. *[8] A sequence $x=(x_k)$ is said to be statistically convergent to L, if for every $\varepsilon>0$*

$$\lim_n\frac{1}{n}|\{k\leq n:\ |x_k-L|\geq\varepsilon\}|=0.$$

In this case we write $S-\lim x=L$ or $x_k\to L(S)$.

Definition 5.2. *[10] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary statistically convergent or S_θ-convergent to L, if for every $\varepsilon > 0$*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \ |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim x = L$ or $x_k \to L(S_\theta)$ and $S_\theta = \{x \in w : S_\theta - \lim x = L \text{ for some } L\}$.

Definition 5.3. *Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $\widehat{S}_\theta [A, \Delta]$ −convergent to L, if for every $\varepsilon > 0$*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \ |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = 0.$$

In this case we write $\widehat{S}_\theta [A, \Delta] - \lim x = L$ or $x_k \to L(\widehat{S}_\theta [A, \Delta])$.

Theorem 5.4. *Let $\theta = (k_r)$ be a lacunary sequence.*

(a) If $x_k \to L(\widehat{w}_\theta [A, \Delta])$ then $x_k \to L(\widehat{S}_\theta [A, \Delta])$,
(b) If $x \in l_\infty [A, \Delta]$ and $x_k \to L(\widehat{S}_\theta [A, \Delta])$, then $x_k \to L(\widehat{w}_\theta [A, \Delta])$,
(c) $\widehat{w}_\theta [A, \Delta] \cap l_\infty [A, \Delta] = \widehat{S}_\theta [A, \Delta] \cap l_\infty [A, \Delta]$, where

$$l_\infty [A, \Delta] = \left\{ x \in w : \ \sup_{k,n} |\Delta B_{kn}(x)| < \infty \right\}.$$

Proof. (a) Suppose that $\varepsilon > 0$ and $x_k \to L(\widehat{w}_\theta [A, \Delta])$, then we have

$$\sum_{k \in I_r} |\Delta B_{kn}(x) - L| \geq \sum_{\substack{k \in I_r \\ |\Delta B_{kn}(x) - L| \geq \varepsilon}} |\Delta B_{kn}(x) - L| \geq \varepsilon |\{k \in I_r : \ |\Delta B_{kn}(x) - L| \geq \varepsilon\}|.$$

Therefore $x_k \to L(\widehat{S}_\theta [A, \Delta])$.

(b) Suppose that $x \in l_\infty [A, \Delta]$ and $x_k \to L(\widehat{S}_\theta [A, \Delta])$, i.e., for some $K > 0$, $|\Delta B_{kn}(x) - L| \leq K$ for all k and n. Given $\varepsilon > 0$, we get

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta B_{kn}(x) - L| = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |\Delta B_{kn}(x) - L| \geq \varepsilon}} |\Delta B_{kn}(x) - L| + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |\Delta B_{kn}(x) - L| < \varepsilon}} |\Delta B_{kn}(x) - L|$$

$$\leq \frac{K}{h_r} |\{k \in I_r : \ |\Delta B_{kn}(x) - L| \geq \varepsilon\}| + \varepsilon,$$

as $r \to \infty$, the right side goes to zero, which implies that $x_k \to L(\widehat{w}_\theta [A, \Delta])$.

(c) Follows from (a) and (b). □

Definition 5.5. *Let $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be almost statistically $[A, \Delta]-$ convergent of oder α or $\widehat{S^\alpha} [A, \Delta]$-convergent of oder α if there is a real number L such that for every $\varepsilon > 0$*

$$\lim_{n \to \infty} \frac{1}{n^\alpha} |\{k \leq n : \ |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = 0.$$

In this case we write $\widehat{S^\alpha} [A, \Delta] - \lim x = L$ or $x_k \to L(\widehat{S^\alpha} [A, \Delta])$.

Definition 5.6. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be $\widehat{S^{\alpha}_{\theta}}[A, \Delta]$ −convergent of oder α if there is a real number L such that for every $\varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} |\{k \in I_r : \ |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = 0. \tag{4}$$

In this case we write $\widehat{S^{\alpha}_{\theta}}[A, \Delta] - \lim x = L$ or $x_k \to L(\widehat{S^{\alpha}_{\theta}}[A, \Delta])$.

Theorem 5.7. *Let $0 < \alpha \leq 1$ and $x = (x_k)$ and $(y = (y_k))$ be sequences of real numbers.*

(a) If $\widehat{S^{\alpha}}[A, \Delta] - \lim_k x_k = x_0$ and $c \in \mathbb{C}$, then $\widehat{S^{\alpha}}[A, \Delta] - \lim_k (cx_k) = cx_0$;

(b) If $\widehat{S^{\alpha}}[A, \Delta] - \lim_k x_k = x_0$ and $\widehat{S^{\alpha}}[A, \Delta] - \lim_k y_k = y_0$, then $\widehat{S^{\alpha}}[A, \Delta] - \lim_k (x_k + y_k) = x_0 + y_0$.

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\frac{1}{n^{\alpha}} |\{k \leq n : |\Delta B_{kn}(cx) - cx_0| \geq \varepsilon\}| = \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

(b) For every $\varepsilon > 0$. The result follows from the from the following inequality.

$$\begin{aligned} &\frac{1}{n^{\alpha}} |\{k \leq n : |\Delta B_{kn}(x + y) - (x_0 + y_0)| \geq \varepsilon\}| \\ &\leq \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{n^{\alpha}} \left| \left\{ k \leq n : |\Delta B_{kn}(y) - y_0| \geq \frac{\varepsilon}{2} \right\} \right| \end{aligned}$$

□

Theorem 5.8. *Let $0 < \alpha \leq 1$ and $x = (x_k)$ and $(y = (y_k))$ be sequences of real numbers.*

(a) If $\widehat{S^{\alpha}_{\theta}}[A, \Delta] - \lim_k x_k = x_0$ and $c \in \mathbb{C}$, then $\widehat{S^{\alpha}_{\theta}}[A, \Delta] - \lim_k (cx_k) = cx_0$;

(b) If $\widehat{S^{\alpha}_{\theta}}[A, \Delta] - \lim_k x_k = x_0$ and $\widehat{S^{\alpha}_{\theta}}[A, \Delta] - \lim_k y_k = y_0$, then $\widehat{S^{\alpha}_{\theta}}[A, \Delta] - \lim_k (x_k + y_k) = x_0 + y_0$.

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows form the following inequality

$$\frac{1}{h_r^{\alpha}} |\{k \in I_r : |\Delta B_{kn}(cx) - cx_0| \geq \varepsilon\}| = \frac{1}{h_r^{\alpha}} \left| \left\{ k \in I_r : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

(b) For every $\varepsilon > 0$. The result follows from the from the following inequality.

$$\begin{aligned} &\frac{1}{h_r^{\alpha}} |\{k \in I_r : |\Delta B_{kn}(x + y) - (x_0 + y_0)| \geq \varepsilon\}| \\ &\leq \frac{1}{h_r^{\alpha}} \left| \left\{ k \in I_r : |\Delta B_{kn}(x) - x_0| \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{h_r^{\alpha}} \left| \left\{ k \in I_r : |\Delta B_{kn}(y) - y_0| \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

□

Theorem 5.9. *If $0 < \alpha < \beta \leq 1$, then $\widehat{S^{\alpha}_{\theta}}[A, \Delta] \subset \widehat{S^{\beta}_{\theta}}[A, \Delta]$ and the inclusion is strict.*

Proof. The proof of the result follows form the following inequality.

$$\frac{1}{h_r^{\beta}} |\{k \in I_r : |\Delta B_{kn}(x) - L| \geq \varepsilon\}| = \frac{1}{h_r^{\alpha}} \left| \left\{ k \in I_r : |\Delta B_{kn}(x) - L| \geq \frac{\varepsilon}{|c|} \right\} \right|.$$

To prove the inclusion is strict, let θ be given and we consider a sequence $x = (x_k)$ be defined by

$$\Delta B_{kn}(x_k) = \begin{cases} [\sqrt{h_r}], & \text{if } k = 1, 2, 3, ..., [\sqrt{h_r}]; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $x \in \widehat{S^{\beta}_{\theta}}[A, \Delta]$ for $\frac{1}{2} < \beta \leq 1$ but $x \notin \widehat{S^{\alpha}_{\theta}}[A, \Delta]$ for $0 < \alpha \leq \frac{1}{2}$. □

Corollary 5.10. *If a sequence is* $\widehat{S_\theta^\alpha}[A,\Delta]$*-convergent to L then it is* $\widehat{S_\theta}[A,\Delta]$*-convergent to L.*

Theorem 5.11. *Let* $0<\alpha\le 1$ *and* $\theta=(k_r)$ *be a lacunary sequence. If* $\liminf_r q_r>1$*, then* $\widehat{S^\alpha}[A,\Delta]\subset\widehat{S_\theta^\alpha}[A,\Delta]$.

Proof. Suppose that $\liminf_r q_r>1$, then there exists a $\delta>0$ such that $q_r\ge 1+\delta$ for sufficently large r which implies that

$$\frac{h_r}{k_r}\ge\frac{\delta}{1+\delta}\Rightarrow\left(\frac{h_r}{k_r}\right)^\alpha\ge\left(\frac{\delta}{1+\delta}\right)^\alpha\Rightarrow\frac{1}{k_r^\alpha}\ge\frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}.$$

If $x_k\to L\left(\widehat{S^\alpha}[A,\Delta]\right)$ then for every $\varepsilon>0$ and for sufficiently large r we have

$$\frac{1}{k_r^\alpha}\left|\{k\le k_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|\ge\frac{1}{k_r^\alpha}\left|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|$$

$$\ge\frac{\delta^\alpha}{(1+\delta)^\alpha}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|.$$

This complete the proof of the theorem. □

Theorem 5.12. *Let* $0<\alpha\le 1$ *and* $\theta=(k_r)$ *be a lacunary sequence. If* $\limsup_r q_r<\infty$*, then* $\widehat{S^\alpha}[A,\Delta]\subset\widehat{S}[A,\Delta]$.

Proof. If $\limsup_r q_r<\infty$, then there exists an $K>0$ such that $q_r<K$ for all r. Suppose that $x_k\to L\left(\widehat{S^\alpha}[A,\Delta]\right)$ and let $M_r=\left|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|$. Then form relation (4) for given $\varepsilon>0$ there is an $r_0\in\mathbb{N}$ such that for $0<\alpha\le 1$

$$\frac{M_r}{h_r^\alpha}<\varepsilon\Rightarrow\frac{M_r}{h_r}<\varepsilon\text{ for all }r>r_0.$$

The rest of the proof of the theorem follows by using the similar technique of Lemma 3 [10]. □

Theorem 5.13. *If*

$$\lim_{r\to\infty}\inf\frac{h_r^\alpha}{k_r}, \tag{5}$$

then $\widehat{S}[A,\Delta]\subset\widehat{S^\alpha}[A,\Delta]$.

Proof. For a given $\varepsilon>0$ we have

$$\{k\le k_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\supset\{k\le I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}.$$

Then we have

$$\frac{1}{k_r^\alpha}\left|\{k\le k_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|\ge\frac{1}{k_r^\alpha}\left|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|=\frac{h_r^\alpha}{k_r}\frac{1}{h_r^\alpha}\left|\{k\in I_r:|\Delta B_{kn}(x)-L|\ge\varepsilon\}\right|.$$

By taking limit as $r\to\infty$ and from relation (5) we have

$$x_k\to L\left(\widehat{S}[A,\Delta]\right)\Rightarrow x_k\to L\left(\widehat{S^\alpha}[A,\Delta]\right).$$

□

Definition 5.14. *Let M be an Orlicz function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\alpha \in (0,1]$, $\theta = (k_r)$ be a lacunary sequence, and for $\rho > 0$, now we define*

$$\widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p] = \left\{x \in w : \lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} = 0, \text{ for some } L, \text{ uniformly on } n\right\}.$$

If $M(x) = x$ and $p_k = p$ for all $k \in \mathbb{N}$ then we shall write $\widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p] = \widehat{w}_{\theta}^{\alpha}[A,\Delta](p)$ and if $M(x) = x$ then we shall write $\widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p] = \widehat{w}_{\theta}^{\alpha}[A,\Delta,p]$.

Theorem 5.15. *Let (p_k) be a bounded and $0 < \inf_k p_k \le p_k \le sup_k p_k = H < \infty$. Let $\alpha,\beta \in (0,1]$ be real numbers such that $\alpha \le \beta$, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence, then $\widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p] \subset \widehat{S}_{\theta}^{\beta}[A,\Delta]$.*

Proof. Let $x = (x_k) \in \widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p]$. Let $\varepsilon > 0$ be given. As $h_r^{\alpha} \le h_r^{\beta}$ for each r we can write

$$\frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} = \frac{1}{h_r^{\alpha}} \left[\sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|\ge\varepsilon}} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} + \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|<\varepsilon}} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} \right]$$

$$\ge \frac{1}{h_r^{\beta}} \left[\sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|\ge\varepsilon}} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} + \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|<\varepsilon}} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} \right]$$

$$\ge \frac{1}{h_r^{\beta}} \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|\ge\varepsilon}} \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} \ge \frac{1}{h_r^{\beta}} \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|\ge\varepsilon}} \min\left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right), \quad \varepsilon_1 = \frac{\varepsilon}{\rho}$$

$$\ge \frac{1}{h_r^{\beta}} |\{k \in I_r : |\Delta B_{kn}(x) - L| \ge \varepsilon\}| \min\left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right).$$

From the above inequality we have $(x_k) \in \widehat{S}_{\theta}^{\beta}[A,\Delta]$. □

Corollary 5.16. *Let $0 < \alpha \le 1$, M be an Orlicz function and $\theta = (k_r)$ be a lacunary sequence, then $\widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p] \subset \widehat{S}_{\theta}^{\alpha}[A,\Delta]$.*

Theorem 5.17. *Let M be an Orlicz function, $x = (x_k)$ be a sequence in $l_\infty[A,\Delta]$, and $\theta = (k_r)$ be a lacunary sequence. If $\lim_{r\to\infty} \frac{h_r}{h_r^{\alpha}} = 1$, then $\widehat{S}_{\theta}^{\alpha}[A,\Delta] \subset \widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p]$.*

Proof. Suppose that $x = (x_k)$ is a in $l_\infty[A,\Delta]$ and $\widehat{S}^{\alpha}[A,\Delta] - \lim_k x_k = L$. As $x = (x_k) \in l_\infty[A,\Delta]$ there exists $K > 0$ such that $|\Delta B_{kn}(x)| \le K$ for all k and n. For given $\varepsilon > 0$ we have

$$\frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} = \frac{1}{h_r^{\alpha}} \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|\ge\varepsilon}} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k} + \frac{1}{h_r^{\alpha}} \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|<\varepsilon}} \left[M\left(\frac{|\Delta B_{kn}(x) - L|}{\rho}\right)\right]^{p_k}$$

$$\le \frac{1}{h_r^{\alpha}} \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|\ge\varepsilon}} \max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^h, \left[M\left(\frac{K}{\rho}\right)\right]^H\right\} + \frac{1}{h_r^{\alpha}} \sum_{\substack{k\in I_r \\ |\Delta B_{kn}(x)-L|<\varepsilon}} \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k}$$

$$\le \max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^h, \left[M\left(\frac{K}{\rho}\right)\right]^H\right\} \frac{1}{h_r^{\alpha}} |\{k \in I_r : |\Delta B_{kn}(x) - L| \ge \varepsilon\}| + \frac{h_r}{h_r^{\alpha}} \max\left\{\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\rho}\right)\right]^H\right\}.$$

Therefore we have $(x_k) \in \widehat{w}_{\theta}^{\alpha}[A,M,\Delta,p]$. □

Theorem 5.18. *Let M be an Orlicz function and if* $\inf_k p_k > 0$, *then limit of any sequence* $x = (x_k)$ *in* $\widehat{w}_{\theta}^{\alpha}[A, M, \Delta, p]$ *is unique.*

Proof. Let $\lim_k p_k = s > 0$. Suppose that $(x_k) \to l_1 \left(\widehat{w}_{\theta}^{\alpha}[A, M, \Delta, p]\right)$ and $(x_k) \to l_2 \left(\widehat{w}_{\theta}^{\alpha}[A, M, \Delta, p]\right)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - l_1|}{\rho}\right)\right]^{p_k} = 0, \text{ uniformly on } n$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - l_2|}{\rho}\right)\right]^{p_k} = 0, \text{ uniformly on } n.$$

Let $\rho = \max\{2\rho_1, 2\rho_2\}$. As M is nondecreasing and convex, we have

$$\frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|l_1 - l_2|}{\rho}\right)\right]^{p_k} \leq \frac{D}{h_r^{\alpha}} \sum_{k\in I_r} \frac{1}{2^{p_k}} \left(\left[M\left(\frac{|\Delta B_{kn}(x) - l_1|}{\rho}\right)\right]^{p_k} + \left[M\left(\frac{|\Delta B_{kn}(x) - l_2|}{\rho}\right)\right]^{p_k}\right)$$

$$\frac{D}{h_r^{\alpha}} \sum_{k\in I_r} \left(\left[M\left(\frac{|\Delta B_{kn}(x) - l_1|}{\rho}\right)\right]^{p_k} + \frac{D}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|\Delta B_{kn}(x) - l_2|}{\rho}\right)\right]^{p_k}\right) \to 0 \text{ as } r \to \infty,$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Therefore we get

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} \left[M\left(\frac{|l_1 - l_2|}{\rho}\right)\right]^{p_k} = 0.$$

As $\lim_k p_k = s$, we have

$$\lim_{k\to\infty} \left[M\left(\frac{|l_1 - l_2|}{\rho}\right)\right]^{p_k} = \left[M\left(\frac{|l_1 - l_2|}{\rho}\right)\right]^{s}$$

and so $l_1 = l_2$. Hence the limit is unique. □

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