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Sufficient Optimality Conditions for Semi-Infinite Multiobjective Fractional Programming under (Φ, ρ)-V-Invexity and Generalized (Φ, ρ)-V-Invexity

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Abstract. A new class of nonconvex smooth semi-infinite multiobjective fractional programming problems with both inequality and equality constraints is considered. We formulate and establish several parametric sufficient optimality conditions for efficient solutions in such nonconvex vector optimization problems under (Φ, ρ)-V-invexity and/or generalized (Φ, ρ)-V-invexity hypotheses. With the reference to the said functions, we extend some results of efficiency for a larger class of nonconvex smooth semi-infinite multiobjective programming problems in comparison to those ones previously established in the literature under other generalized convexity notions. Namely, we prove the sufficient optimality conditions for such nonconvex semi-infinite multiobjective fractional programming problems in which not all functions constituting them have the fundamental property of convexity, invexity and most generalized convexity notions.

1. Introduction

The term multiobjective programming, also known as vector programming, is used to denote a type of optimization problem in which two or more objectives are to be minimized subject to certain constraints. Multiobjective fractional programming refers to a vector optimization problem in which the objective functions are quotients. Nonlinear multiobjective fractional programming problems arise from many applied areas including portfolio selection, stock cutting, physics, engineering problems, optimal control, game theory and numerous decision problems in management science. Therefore, considerable attention has been given recently to obtaining new optimality results for various classes of nonlinear nonconvex multiobjective fractional programming problems (see, for example, [4], [10], [12], [26], [27], [29], [31] and the references therein).

Semi-infinite programming became in recent years a powerful tool for the mathematical modeling of many real-life problems. Semi-infinite programming problems, that is, optimization problems with a finite number of variables and infinitely many constraints, occur in a wide variety of fields, such as approximation theory, computer aided design, game theory, boundary values problems, robot trajectory planning

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and pollution control, defect minimization for operator equations, geometry, random graphs related to Newton flows, wavelet analysis, reliability testing, environmental protection planning, decision making under uncertainty, statistics, semidefinite programming, geometric programming, disjunctive programming, optimal control problems, robotics, and continuum mechanics, among others, (see, for instance, [15], [17], [20], [21], [22], and others).

Recently, many scholars have been making deeper research for optimality conditions for semi-infinite programming (see, for instance, [21], [22], [24], [30], [32], [34]). Although semi-infinite optimization became a very active research area in recent years, however, so far semi-infinite nonlinear multiobjective fractional programming problems have not received much attention in optimization literature. Some results for such vector optimization problems can be found mainly in papers by Zalmai and Zhang [35], [36], [37], [38].

In this paper, therefore, we consider a class of nonconvex semi-infinite multiobjective fractional programming problems with both inequality and equality constraints. Several parametric sufficient optimality conditions for efficiency are established for such a class of nonconvex smooth semi-infinite multiobjective fractional programming problems in which the functions involved are (Φ,ρ)-V-invex and/or generalized (Φ,ρ)-V-invex. Subsequently, we illustrate the results established in the paper by a suitable example of a nonconvex semi-infinite multiobjective fractional programming problem involving (Φ,ρ)-V-invex functions with respect to the same functional Φ and with respect to, not necessarily, the same ρ. It turns out that it is not possible to prove efficiency for the considered semi-infinite multiobjective fractional programming problem under a fairly large number of other generalized convexity notions existing in the literature. Thus, to the best of our knowledge, all the sufficient optimality conditions established in this paper for the considered class of nonconvex smooth semi-infinite multiobjective fractional programming problems are new in the area of semi-infinite programming.

2. (Φ,ρ)-V-Invexity and Generalized (Φ,ρ)-V-Invexity

In this section, we provide some definitions and some results that we shall use in the sequel.

The following convention for equalities and inequalities will be used in the paper.

For any $x=(x_1,x_2,...,x_n)^T,\ y=(y_1,y_2,...,y_n)^T$, we define:

(i) $x=y$ if and only if $x_i=y_i$ for all $i=1,2,...,n$;

(ii) $x>y$ if and only if $x_i>y_i$ for all $i=1,2,...,n$;

(iii) $x\geqq y$ if and only if $x_i\geqq y_i$ for all $i=1,2,...,n$;

(iv) $x\geq y$ if and only if $x\geqq y$ and $x\neq y$.

Following Jeyakumar and Mond [23] and Caristi et al. [11], we introduce a new class of nonconvex vector-valued functions. The class of so-called (Φ,ρ)-V-invex functions is a generalization and extension both the class of V-invex functions introduced by Jeyakumar and Mond [23] for differentiable multiobjective programming problems and the class of (Φ,ρ)-invex functions introduced by Caristi et al. [11] for smooth scalar optimization problems.

Let X be a nonempty open subset of R^n, $u\in X$ be given, and the function $f:(f_1,f_2,...,f_k):X\to R^k$ be differentiable at $u\in X$.

Definition 2.1. *If there exist a function $\Phi: X\times X\times R^{n+1}\to R$, where $\Phi(x,u,\cdot)$ is convex on R^{n+1}, $\Phi(x,u,(0,a))\geqq 0$ for all $x\in X$ and every $a\in R_+$ and $\rho=(\rho_1,...,\rho_k)\in R^k$, where ρ_i, $i=1,...,k$, are real numbers, and real-valued functions $\alpha_i: X\times X\to R_+\backslash\{0\}$, $i=1,...,k$, such that the following inequalities*

$$f_i(x)-f_i(u)\geqq\Phi\left(x,u,\alpha_i(x,u)\left(\nabla f_i(u),\rho_i\right)\right)\quad(>),\ i=1,...,k \tag{1}$$

hold for all $x\in X$, then f is said to be a (Φ,ρ)-V-invex (strictly (Φ,ρ)-V-invex) function at u on X.
If inequalities (1) are satisfied at each u, then f is said to be a (Φ,ρ)-V-invex (strictly (Φ,ρ)-V-invex) function on X.

Definition 2.2. *Each function f_i, $i=1,...,k$, satisfying inequality (1) is said to be (Φ,ρ_i)-α_i-invex (strictly (Φ,ρ_i)-α_i-invex) at u on X. If inequality (1) is satisfied at each u, then f_i is said to be a (Φ,ρ_i)-α_i-invex (strictly (Φ,ρ_i)-α_i-invex) function on X.*

Remark 2.3. *Note that the concept of (Φ, ρ)-V-invexity generalizes and extends many generalized convexity and generalized invexity notions, previously introduced in the literature. Indeed, there are the following special cases:*

i) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \nabla f_i(u)(x - u)$, $\alpha_i(x, u) \equiv 1$, $\rho_i = 0$, $i = 1, ..., k$, then we obtain the definition of a differentiable vector-valued convex function.

ii) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \alpha_i(x, u)\nabla f_i(u)(x - u)$ and $\rho_i = 0$, $i = 1, ..., k$, then we obtain the definition of a differentiable V-convex function.

iii) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \nabla f_i(u)\eta(x, u)$ for a certain mapping $\eta : X \times X \to R^n$ and, moreover, $\alpha_i(x, u) \equiv 1$, $\rho_i = 0$, $i = 1, ..., k$, then we obtain the definition of a differentiable invex function (in the scalar case, $k = 1$, see, Hanson [18]; in the vectorial case, see, Egudo and Hanson [14]).

iv) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \alpha_i(x, u)\nabla f_i(u)\eta(x, u)$ for a certain mapping $\eta : X \times X \to R^n$ and, moreover, $\rho_i = 0$, $i = 1, ..., k$, then we obtain the definition of a differentiable V-invex function (see, Jeyakumar and Mond [23]).

v) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \frac{1}{b_i(x,u)}\nabla f_i(u)\eta(x, u)$ for a certain mapping $\eta : X \times X \to R^n$, $b_i : X \times X \to R_+\backslash\{0\}$ and $\rho_i = 0$, $i = 1, ..., k$, then (Φ, ρ)-V-invexity reduces to the definition of a vector-valued (b, η)-invex function (see, Bector [7]).

vi) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \nabla f_i(u)(x - u) + \rho\|x - u\|^2$, then (Φ, ρ)-V-invexity reduces to the definition of a vector-valued ρ-convex function (see, in the scalar case, Vial [33]).

vii) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \nabla f_i(u)\eta(x, u) + \rho\|\theta(x, u)\|^2$ for a certain mapping $\eta : X \times X \to R^n$, where $\alpha_i(x, u) \equiv 1$ and $\theta : X \times X \to R^n$, $\theta(x, u) \neq 0$, whenever $x \neq u$, then (Φ, ρ)-V-invexity reduces to the definition of a vector-valued ρ-invex function (with respect to η and θ) (see, Craven [13] and also Ahmad [1]).

ix) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \alpha_i(x, u)\nabla f_i(u)\eta(x, u) + \rho\|\theta(x, u)\|^2$ for a certain mapping $\eta : X \times X \to R^n$, where $\theta : X \times X \to R^n$, $\theta(x, u) \neq 0$, whenever $x \neq u$, then (Φ, ρ)-V-invexity reduces to the definition of a V-ρ-invex function (with respect to η and θ) (see, Kuk et al. [25]).

x) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = F(x, u, \nabla f_i(u))$, where $F(x, u, \cdot)$ is a sublinear functional on R^n and $\alpha_i(x, u) \equiv 1$, $\rho_i = 0$, $i = 1, ..., k$, then (Φ, ρ)-V-invexity reduces to the definition of F-convexity introduced by Hanson and Mond [19], and considered by Gulati and Islam [16] in a vectorial case.

xi) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = F(x, u, \nabla f_i(u)) + \rho d^2(x, u)$, where $F(x, u, \cdot)$ is a sublinear functional on R^n, $\alpha_i(x, u) \equiv 1$, $i = 1, ..., k$, and $d : X \times X \to R$, then the concept of (Φ, ρ)-V-invexity reduces to the definition of (F, ρ)-convexity introduced by Preda [29].

xii) If $\Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) = \frac{1}{b_i(x,u)}\left(F(x, u, \nabla f_i(u)) + \rho_i d^2(x, u)\right)$, where $F(x, u, \cdot)$ is a sublinear functional on R^n, $b_i : X \times X \to R_+\backslash\{0\}$ and $d : X \times X \to R$, then (Φ, ρ)-V-invexity reduces to (b, F, ρ)-convexity introduced by Pandian [28].

xiii) If $\alpha_i(x, u) \equiv 1$, $\rho_i = 0$, $i = 1, ..., k$, then (Φ, ρ)-V-invexity reduces to the definition of differentiable (Φ, ρ)-invexity introduced by Caristi et al. [11].

Now, in the vectorial case, we introduce the definitions of generalized (Φ, ρ)-V-invex functions.

Definition 2.4. *If there exist a function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and every $a \in R_+$, $\rho = (\rho_1, ..., \rho_k) \in R^k$, where ρ_i, $i = 1, ..., k$, are real numbers, and real-valued functions $\alpha_i : X \times X \to R_+\backslash\{0\}$, $i = 1, ..., k$, such that the following relations*

$$f_i(x) < f_i(u) \Longrightarrow \Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) < 0, i = 1, ..., k \tag{2}$$

hold for all $x \in X$, then f is said to be a (Φ, ρ)-V-pseudo-invex function at u on X.
If relations (2) are satisfied at each u, then f is said to be a (Φ, ρ)-V-pseudo-invex function on X.

Each function f_i, $i = 1, ..., k$, satisfying the relation (2) is said to be (Φ, ρ_i)-α_i-pseudo-invex at u on X. If relation (2) is satisfied at each u, then f_i is said to be a (Φ, ρ_i)-α_i-pseudo-invex function on X.

Definition 2.5. *If there exist a function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and every $a \in R_+$, $\rho = (\rho_1, ..., \rho_k) \in R^k$, where ρ_i, $i = 1, ..., k$, are real numbers, and real-valued functions $\alpha_i : X \times X \to R_+\backslash\{0\}$, $i = 1, ..., k$, such that the following relations*

$$f_i(x) \leqq f_i(u) \Longrightarrow \Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) < 0, i = 1, ..., k \tag{3}$$

hold for all $x \in X$, then f is said to be a strictly (Φ, ρ)-V-pseudo-invex function at u on X.
If relations (3) are satisfied at each u, then f is said to be a strictly (Φ, ρ)-V-pseudo-invex function on X.

Each function f_i, $i = 1, ..., k$, satisfying the relation (3) is said to be strictly (Φ, ρ_i)-α_i-pseudo-invex at u on X. If relation (3) is satisfied at each u, then f_i is said to be a strictly (Φ, ρ_i)-α_i-pseudo-invex function on X.

Definition 2.6. *If there exist a function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and every $a \in R_+$, $\rho = (\rho_1, ..., \rho_k) \in R^k$, where ρ_i, $i = 1, ..., k$, are real numbers, and real-valued functions $\alpha_i : X \times X \to R_+ \backslash \{0\}$, $i = 1, ..., k$, such that the following relations*

$$f_i(x) \leqq f_i(u) \Longrightarrow \Phi(x, u, \alpha_i(x, u)(\nabla f_i(u), \rho_i)) \leqq 0, \; i = 1, ..., k \tag{4}$$

hold for all $x \in X$, then f is said to be a (Φ, ρ)-V-quasi-invex function at u on X.
If relations (4) are satisfied at each u, then f is said to be a (Φ, ρ)-V-quasi-invex function on X.

Each function f_i, $i = 1, ..., k$, satisfying the relation (4) is said to be (Φ, ρ_i)-α_i-quasi-invex at u on X. If relation (4) is satisfied at each u, then f_i is said to be a (Φ, ρ_i)-α_i-quasi-invex function on X.

3. Semi-Infinite Multiobjective Fractional Programming and Efficiency

In the paper, we consider the following semi-infinite multiobjective fractional programming problem:

$$\begin{aligned} &V\text{-minimize } \varphi(x) = \left(\varphi_1(x), ..., \varphi_p(x)\right) = \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right) \\ &\quad \text{s.t. } G_j(x, t) \leqq 0 \text{ for all } t \in T_j, \; j \in J = \{1, ..., q\}, \\ &\quad\quad H_k(x, s) = 0 \text{ for all } s \in S_k, \; k \in K = \{1, ..., r\}, \\ &\quad\quad\quad x \in X, \end{aligned} \tag{P}$$

where $f_i : X \to R, g_i : X \to R, i \in I = \{1, ..., p\}$, are real-valued functions defined on a nonempty open subset X of R^n such that, for each $i \in I$, $g_i(x) > 0$ for all $x \in X$, T_j, $j = 1, ..., q$, and $S_k, k = 1, ..., r$, are compact subsets of a complete metric space, $x \to G_j(x, t)$ is a function on X for all $t \in T_j$, for each $k \in K$, $x \to H_k(x, s)$, is a function on X for all $s \in S_k$, for each $j \in J$ and $k \in K$, $t \to G_j(x, t)$ and $s \to H_k(x, s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$ satisfying the constraints of problem (P).

Let

$$D := \left\{x \in X : G_j(x, t) \leqq 0 \text{ for all } t \in T_j, \; j = 1, ..., q, \; H_k(x, s) = 0 \text{ for all } s \in S_k, \; k = 1, ..., r\right\}$$

be the set of all feasible solutions of (P) and let $\widehat{T}_j(\overline{x})$ denote $\widehat{T}_j(\overline{x}) = \left\{t \in T_j : G_j(\overline{x}, t) = 0\right\}$.

It is well-known that a feasible solution $\overline{x}$ is efficient in problem (P) if and only if there exists no $x \in D$ such that

$$\varphi(x) \leq \varphi(\overline{x}).$$

In this section, for the considered semi-infinite multiobjective fractional programming problem (P), we establish a set of sufficient optimality conditions for efficient solutions under (Φ, ρ)-V-invexity and/or generalized (Φ, ρ)-V-invexity assumptions.

For the considered semi-infinite multiobjective fractional programming problem (P), we now give the parametric necessary optimality conditions established by Zalmai and Zhang [35].

Theorem 3.1. *Let $\overline{x} \in D$ be an efficient solution in the considered semi-infinite multiobjective fractional programming problem (P) with the corresponding optimal value equal to $\overline{v} = \varphi(\overline{x})$ and the generalized Guignard constraint qualification be satisfied at $\overline{x}$. Further, assume that, for each $i \in I$, f_i and g_i are continuously differentiable at $\overline{x}$, for each $j \in J$, the function $z \to G_j(z, t)$ is continuously differentiable at $\overline{x}$ for all $t \in T_j$, for each $k \in K$, the function $z \to H_k(z, s)$ is continuously differentiable at $\overline{x}$ for all $s \in S_k$. Then, there exist $\overline{\lambda} \in \Lambda = \left\{\lambda \in R^p : \lambda > 0, \; \sum_{i=1}^{p} \lambda_i = 1\right\}$*

and integers $\overline{w}_0$ *and* $\overline{w}$, *with* $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, *such that there exist* $\overline{w}_0$ *indices* j_m, *with* $1 \leqq j_m \leqq q$, *together with* $\overline{w}_0$ *points* $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, $\overline{w} - \overline{w}_0$ *indices* k_m, *with* $1 \leqq k_m \leqq r$, *together with* $\overline{w} - \overline{w}_0$ *points* $s^m \in S_{k_m}$, $m = 1, ..., \overline{w} - \overline{w}_0$ *and* $\overline{w}$ *real numbers* $\overline{\xi}_m$ *with* $\overline{\xi}_m > 0$, $m = 1, ..., \overline{w}_0$ *with the property that*

$$\sum_{i=1}^{p} \overline{\lambda}_i \left[\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})\right] + \sum_{m=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m) = 0. \tag{5}$$

For the considered semi-infinite multiobjective fractional programming problem (P), we define the vector-valued Lagrange function as follows:

$$L\left(\cdot, \lambda, \xi, v, w, w_0, \overline{t}, \overline{s}\right) := \left(L_1\left(\cdot, \lambda, \xi, v, w, w_0, \overline{t}, \overline{s}\right), ..., L_p\left(\cdot, \lambda, \xi, v, w, w_0, \overline{t}, \overline{s}\right)\right) : X \to R^p,$$

where each its component is defined by

$$L_i\left(z, \lambda, \xi, v, w, w_0, \overline{t}, \overline{s}\right) = \lambda_i \left[f_i(z) - v_i g_i(z)\right] + \tfrac{1}{p}\left[\sum_{m=1}^{w_0} \xi_m G_{j_m}(z, t^m) + \sum_{m=w_0+1}^{w} \xi_m H_{k_m}(z, s^m)\right]. \tag{6}$$

Theorem 3.2. *Let* $\overline{x} \in D$ *and* $\overline{v} = \varphi(\overline{x})$. *Also, let* f_i *and* g_i, $i \in I$, $z \to G_j(z,t), j \in J$, $z \to H_k(z,s)$, $k \in K$, *be differentiable at* $\overline{x}$ *for all* $t \in T_j$ *and for all* $s \in S_k$. *Further, assume that there exist* $\overline{\lambda} \in \Lambda$, *integers* $\overline{w}_0$ *and* $\overline{w}$, *with* $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, *such that there exist* $\overline{w}_0$ *indices* j_m, *with* $1 \leqq j_m \leqq q$, *together with* $\overline{w}_0$ *points* $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, $\overline{w} - \overline{w}_0$ *indices* k_m, *with* $1 \leqq k_m \leqq r$ *together with* $\overline{w} - \overline{w}_0$ *points* $s^m \in S_{k_m}$, $m = 1, ..., \overline{w} - \overline{w}_0$, *and* $\overline{w}$ *real numbers* $\overline{\xi}_m$ *with* $\overline{\xi}_m > 0$ *for* $m = 1, ..., \overline{w}_0$, *with the property that the relation (5) is fulfilled at* $\overline{x}$. *Assume, furthermore, that any one of the following eight sets of hypotheses is fulfilled:*

A) a) $f_i(\cdot) - \overline{v}_i g_i(\cdot)$, $i = 1, ..., p$, *is a* (Φ, ρ_i)-α_i*-invex function at* $\overline{x}$ *on D,*
b) $G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, *is a* $\left(\Phi, \rho_{G_{j_m}}\right)$-$\beta_{j_m}$*-invex function at* $\overline{x}$ *on D,*
c) $H_{k_m}(\cdot, s^m)$, $s^m \in S^+_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\xi}_m > 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, *is a* $\left(\Phi, \rho^+_{H_{k_m}}\right)$-$\gamma^+_{k_m}$*-invex function at* $\overline{x}$ *on D,*
d) $-H_{k_m}(\cdot, s^m)$, $s^m \in S^-_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\xi}_m < 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, *is a* $\left(\Phi, \rho^-_{H_{k_m}}\right)$-$\gamma^-_{k_m}$*-invex function at* $\overline{x}$ *on D,*
e) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \sum_{m=1}^{\overline{w}_0} \overline{\xi}_m \rho_{G_{j_m}} + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m \rho^+_{H_{k_m}} - \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m \rho^-_{H_{k_m}} \geqq 0,$

B) a) $f_i(\cdot)$, $i = 1, ..., p$, *is a* (Φ, ρ_i)-α_i*-invex function at* $\overline{x}$ *on D,*
b) $-\overline{v}_i g_i(\cdot)$, $i = 1, ..., p$, *is a* (Φ, ρ_i)-α_i*-invex function at* $\overline{x}$ *on D,*
c) $G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, *is a* $\left(\Phi, \rho_{G_{j_m}}\right)$-$\beta_{j_m}$*-invex function at* $\overline{x}$ *on D,*
d) $H_{k_m}(\cdot, s^m)$, $s^m \in S^+_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\xi}_m > 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, *is a* $\left(\Phi, \rho^+_{H_{k_m}}\right)$-$\gamma^+_{k_m}$*-invex function at* $\overline{x}$ *on D,*
e) $-H_{k_m}(\cdot, s^m)$, $s^m \in S^-_{k_m}(\overline{x}) \equiv \left\{s^m \in S_{k_m} : \overline{\xi}_m < 0\right\}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, *is a* $\left(\Phi, \rho^-_{H_{k_m}}\right)$-$\gamma^-_{k_m}$*-invex function at* $\overline{x}$ *on D,*
f) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \sum_{m=1}^{\overline{w}_0} \overline{\xi}_m \rho_{G_{j_m}} + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m \rho^+_{H_{k_m}} - \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m \rho^-_{H_{k_m}} \geqq 0,$

C) each component of the vector-valued Lagrange function $L\left(\cdot, \overline{\lambda}, \overline{\xi}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$, *that is, each function* $z \to L_i\left(z, \overline{\lambda}, \overline{\xi}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right)$, $i = 1, ..., p$, *is* (Φ, ρ_i)-α_i*-invex at* $\overline{x}$ *on D, where* $\overline{t} \equiv \left(t^1, ..., t^{\overline{w}_0}\right)$, $\overline{s} \equiv \left(s^{\overline{w}_0+1}, ..., s^{\overline{w}}\right)$ *and, moreover,* $\sum_{i=1}^{p} \rho_i \geqq 0$,

D) each function $\psi_i\left(\cdot, \overline{\xi}, \overline{v}, \overline{w}, \overline{w}_0, \overline{t}, \overline{s}\right) = f_i(\cdot) - \overline{v}_i g_i(\cdot) + \sum_{m=1}^{w_0} \overline{\xi}_m G_{j_m}(\cdot, t^m) + \sum_{m=w_0+1}^{w} \overline{\xi}_m H_{k_m}(\cdot, s^m)$, $i = 1, ..., p$, *is* (Φ, ρ_i)-α_i*-invex at* $\overline{x}$ *on D, where* $\overline{t} \equiv \left(t^1, ..., t^{\overline{w}_0}\right)$, $\overline{s} \equiv \left(s^{\overline{w}_0+1}, ..., s^{\overline{w}}\right)$ *and, moreover,* $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i \geqq 0$,

E) a) $f_i(\cdot) - \overline{v}_i g_i(\cdot)$, $i = 1, ..., p$, *is a* (Φ, ρ_i)-α_i*-invex function at* $\overline{x}$ *on D,*
b) $\overline{\xi}_m G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, *is a* $\left(\Phi, \rho_{G_{j_m}}\right)$-$\beta_{j_m}$*-invex function at* $\overline{x}$ *on D,*
c) $\overline{\xi}_m H_{k_m}(\cdot, s^m)$, $s^m \in S_{k_m}$, *is a* $\left(\Phi, \rho_{H_{k_m}}\right)$-$\gamma_{k_m}$*-invex function at* $\overline{x}$ *on D,*
e) $\sum_{i=1}^{p} \overline{\lambda}_i \rho_i + \sum_{m=1}^{\overline{w}_0} \rho_{G_{j_m}} + \sum_{m=\overline{w}_0+1}^{\overline{w}} \rho_{H_{k_m}} \geqq 0,$

F) a) $\overline{\lambda}_i\left[f_i(\cdot)-\overline{v}_i g_i(\cdot)\right]$, $i=1,...,p$, *is a* (Φ,ρ_i)*-*α_i*-invex function at* $\overline{x}$ *on* D,
b) $\overline{\xi}_m G_{j_m}(\cdot,t^m)$, $t^m\in\widehat{T}_{j_m}(\overline{x})$, $m=1,...,\overline{w}_0$, *is a* $\left(\Phi,\rho_{G_{j_m}}\right)$*-*$\beta_{j_m}$*-invex function at* $\overline{x}$ *on* D,
c) $\overline{\xi}_m H_{k_m}(\cdot,s^m)$, $s^m\in S_{k_m}$, *is a* $\left(\Phi,\rho_{H_{k_m}}\right)$*-*$\gamma_{k_m}$*-invex function at* $\overline{x}$ *on* D,
e) $\sum_{i=1}^{p}\rho_i+\sum_{m=1}^{\overline{w}_0}\rho_{G_{j_m}}+\sum_{m=\overline{w}_0+1}^{\overline{w}}\rho_{H_{k_m}}\geqq 0$,

G) a) $\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i(\cdot)-\overline{v}_i g_i(\cdot)\right]$ *is a* (Φ,ρ_α)*-*α*-invex function at* $\overline{x}$ *on* D,
b) $\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m G_{j_m}(\cdot,t^m)$, $t^m\in\widehat{T}_{j_m}(\overline{x})$, *is a* (Φ,ρ_G)*-*β*-invex function at* $\overline{x}$ *on* D,
c) $\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m H_{k_m}(\cdot,s^m)$, $s^m\in S_{k_m}$, *is a* (Φ,ρ_H)*-*γ*-invex function at* $\overline{x}$ *on* D,
d) $\rho_\alpha+\rho_G+\rho_H\geqq 0$,

H) a) $\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i(\cdot)-\overline{v}_i g_i(\cdot)\right]$ *is a* (Φ,ρ_α)*-*α*-invex function at* $\overline{x}$ *on* D,
b) $\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m G_{j_m}(\cdot,t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m H_{k_m}(\cdot,s^m)$, $t^m\in\widehat{T}_{j_m}(\overline{x})$, $s^m\in S_{k_m}$, *is a* $\left(\Phi,\rho_\beta\right)$*-*$\beta$*-invex function at* $\overline{x}$ *on* D,
c) $\rho_\alpha+\rho_\beta\geqq 0$.

Then $\overline{x}$ *is efficient in problem (P) with the corresponding optimal objective value equal to* $\overline{v}=\varphi(\overline{x})$.

Proof. By assumption, $\overline{x}\in D$, $\overline{v}=\varphi(\overline{x})$ and, moreover, there exist $\overline{\lambda}\in\Lambda$, integers $\overline{w}_0$ and $\overline{w}$, with $0\leqq\overline{w}_0\leqq\overline{w}\leqq n+1$, such that there exist $\overline{w}_0$ indices j_m, with $1\leqq j_m\leqq q$, together with $\overline{w}_0$ points $t^m\in\widehat{T}_{j_m}(\overline{x})$, $m=1,...,\overline{w}_0$, $\overline{w}-\overline{w}_0$ indices k_m, with $1\leqq k_m\leqq r$ together with $\overline{w}-\overline{w}_0$ points $s^m\in S_{k_m}$, $m=1,...,\overline{w}-\overline{w}_0$, and $\overline{w}$ real numbers $\overline{\xi}_m$ with $\overline{\xi}_m>0$ for $m=1,...,\overline{w}_0$, with the property that relation (5) is fulfilled at $\overline{x}$.

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not an efficient solution in problem (P). Hence, there exists $\widetilde{x}\in D$ such that

$$f_i(\widetilde{x})-\overline{v}_i g_i(\widetilde{x})\leqq f_i(\overline{x})-\overline{v}_i g_i(\overline{x})\ \text{ for } i\in I, \tag{7}$$

$$f_{i^*}(\widetilde{x})-\overline{v}_{i^*} g_{i^*}(\widetilde{x})< f_{i^*}(\overline{x})-\overline{v}_{i^*} g_{i^*}(\overline{x})\ \text{ for at least one } i^*\in I. \tag{8}$$

Proof of the theorem under hypothesis A).
In view of hypotheses a)-e), Definition 2.2 implies that the following inequalities

$$f_i(\widetilde{x})-\overline{v}_i g_i(\widetilde{x})-\left[f_i(\overline{x})-\overline{v}_i g_i(\overline{x})\right]\geqq\Phi\left(\widetilde{x},\overline{x},\alpha_i(\widetilde{x},\overline{x})\left(\nabla f_i(\overline{x})-\overline{v}_i\nabla g_i(\overline{x}),\rho_i\right)\right),\ i\in I, \tag{9}$$

$$G_{j_m}(\widetilde{x},t^m)-G_{j_m}(\overline{x},t^m)\geqq\Phi\left(\widetilde{x},\overline{x},\beta_{j_m}(\widetilde{x},\overline{x})\left(\nabla G_{j_m}(\overline{x},t^m),\rho_{G_{j_m}}\right)\right),t^m\in\widehat{T}_{j_m}(\overline{x}),\ m=1,...,\overline{w}_0, \tag{10}$$

$$H_{k_m}(\widetilde{x},s^m)-H_{k_m}(\overline{x},s^m)\geqq\Phi\left(\widetilde{x},\overline{x},\gamma^+_{k_m}(\widetilde{x},\overline{x})\left(\nabla H_{k_m}(\overline{x},s^m),\rho^+_{H_{k_m}}\right)\right),s^m\in S^+_{k_m}(\overline{x}),\ \ m\in\{\overline{w}_0+1,...,\overline{w}\}, \tag{11}$$

$$-H_{k_m}(\widetilde{x},s^m)+H_{k_m}(\overline{x},s^m)\geqq\Phi\left(\widetilde{x},\overline{x},\gamma^-_{k_m}(\widetilde{x},\overline{x})\left(-\nabla H_{k_m}(\overline{x},s^m),\rho^-_{H_{k_m}}\right)\right),s^m\in S^-_{k_m}(\overline{x}),\ \ m\in\{\overline{w}_0+1,...,\overline{w}\} \tag{12}$$

hold. Combining (7), (8) and (9), we get, respectively,

$$\Phi\left(\widetilde{x},\overline{x},\alpha_i(\widetilde{x},\overline{x})\left(\nabla f_i(\overline{x})-\overline{v}_i\nabla g_i(\overline{x}),\rho_i\right)\right)\leqq 0,\ i\in I, \tag{13}$$

$$\Phi\left(\widetilde{x},\overline{x},\alpha_{i^*}(\widetilde{x},\overline{x})\left(\nabla f_{i^*}(\overline{x})-\overline{v}_{i^*}\nabla g_{i^*}(\overline{x}),\rho_{i^*}\right)\right)<0 \text{ for at least one } i^*\in I. \tag{14}$$

By $\widetilde{x}\in D$ and $\overline{x}\in D$, it follows that $G_{j_m}(\widetilde{x},t^m)\leqq G_{j_m}(\overline{x},t^m)$, $t^m\in\widehat{T}_{j_m}(\overline{x})$, $m=1,...,\overline{w}_0$, $H_{k_m}(\widetilde{x},s^m)=H_{k_m}(\overline{x},s^m)$, $s^m\in S_{k_m}$, $m\in\{\overline{w}_0+1,...,\overline{w}\}$. Hence, (10)-(12) yield, respectively,

$$\Phi\left(\widetilde{x},\overline{x},\beta_{j_m}(\widetilde{x},\overline{x})\left(\nabla G_{j_m}(\overline{x},t^m),\rho_{G_{j_m}}\right)\right)\leqq 0,\quad t^m\in\widehat{T}_{j_m}(\overline{x}),\ m=1,...,\overline{w}_0, \tag{15}$$

$$\Phi\left(\widetilde{x},\overline{x},\gamma^+_{k_m}(\widetilde{x},\overline{x})\left(\nabla H_{k_m}(\overline{x},s^m),\rho^+_{H_{k_m}}\right)\right)\leqq 0,\quad s^m\in S^+_{k_m}(\overline{x}),m\in\{\overline{w}_0+1,...,\overline{w}\}, \tag{16}$$

$$\Phi\left(\widetilde{x},\overline{x},\gamma^-_{k_m}(\widetilde{x},\overline{x})\left(-\nabla H_{k_m}(\overline{x},s^m),\rho^-_{H_{k_m}}\right)\right)\leqq 0,\quad s^m\in S^-_{k_m}(\overline{x}),\ m\in\{\overline{w}_0+1,...,\overline{w}\}. \tag{17}$$

Multiplying each inequality (13) and (14) by $\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})} > 0$, $i \in I$, (15) by $\frac{\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})} > 0$, $m = 1, ..., \overline{w}_0$, (16) by $\frac{\overline{\xi}_m}{\gamma^+_{k_m}(\widetilde{x},\overline{x})} > 0$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, (17) by $\frac{-\overline{\xi}_m}{\gamma^-_{k_m}(\widetilde{x},\overline{x})} > 0$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\alpha_i(\widetilde{x},\overline{x})\left(\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x}),\rho_i\right)\right) < 0, \tag{18}$$

$$\sum_{m=1}^{\overline{w}_0} \frac{\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\beta_{j_m}(\widetilde{x},\overline{x})\left(\nabla G_{j_m}(\overline{x},t^m),\rho_{G_{j_m}}\right)\right) \leqq 0,\ t^m \in \widehat{T}_{j_m}(\overline{x}), \tag{19}$$

$$\sum_{m\in\{\overline{w}_0+1,...,\overline{w}\}} \frac{\overline{\xi}_m}{\gamma^+_{k_m}(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\gamma^+_{k_m}(\widetilde{x},\overline{x})\left(\nabla H_{k_m}(\overline{x},s^m),\rho^+_{H_{k_m}}\right)\right) \leqq 0, \quad s^m \in S^+_{k_m}(\overline{x}), \tag{20}$$

$$\sum_{m\in\{\overline{w}_0+1,...,\overline{w}\}} \frac{-\overline{\xi}_m}{\gamma^-_{k_m}(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\gamma^-_{k_m}(\widetilde{x},\overline{x})\left(-\nabla H_{k_m}(\overline{x},s^m),\rho^-_{H_{k_m}}\right)\right) \leqq 0, \quad s^m \in S^-_{k_m}(\overline{x}). \tag{21}$$

Hence, (18)-(21) yield

$$\begin{aligned}
&\sum_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\alpha_i(\widetilde{x},\overline{x})\left(\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x}),\rho_i\right)\right) + \sum_{m=1}^{\overline{w}_0} \frac{\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\beta_{j_m}(\widetilde{x},\overline{x})\left(\nabla G_{j_m}(\overline{x},t^m),\rho_{G_{j_m}}\right)\right) \\
&+ \sum_{m\in\{\overline{w}_0+1,...,\overline{w}\}} \frac{\overline{\xi}_m}{\gamma^+_{k_m}(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\gamma^+_{k_m}(\widetilde{x},\overline{x})\left(\nabla H_{k_m}(\overline{x},s^m),\rho^+_{H_{k_m}}\right)\right) \\
&+ \sum_{m\in\{\overline{w}_0+1,...,\overline{w}\}} \frac{-\overline{\xi}_m}{\gamma^-_{k_m}(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x},\overline{x},\gamma^-_{k_m}(\widetilde{x},\overline{x})\left(-\nabla H_{k_m}(\overline{x},s^m),\rho^-_{H_{k_m}}\right)\right) < 0.
\end{aligned} \tag{22}$$

Let us introduce the following notations:

$$A = \sum_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})} + \sum_{m=1}^{\overline{w}_0} \frac{\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})} + \sum_{m\in\{\overline{w}_0+1,...,\overline{w}\}} \frac{\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})} + \sum_{m\in\{\overline{w}_0+1,...,\overline{w}\}} \frac{-\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})}, \tag{23}$$

$$\widetilde{\lambda}_i = \frac{\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}{A},\ i \in I, \tag{24}$$

$$\widetilde{\xi}_m = \frac{\frac{\overline{\xi}_m}{\beta_{j_m}(\widetilde{x},\overline{x})}}{A},\ m = 1, ..., \overline{w}_0, \tag{25}$$

$$\widetilde{\xi}^+_m = \frac{\frac{\overline{\xi}_m}{\gamma^+_{k_m}(\widetilde{x},\overline{x})}}{A},\ m \in \{\overline{w}_0 + 1, ..., \overline{w}\}, \tag{26}$$

$$\widetilde{\xi}^-_m = \frac{\frac{-\overline{\xi}_m}{\gamma^-_{k_m}(\widetilde{x},\overline{x})}}{A},\ m \in \{\overline{w}_0 + 1, ..., \overline{w}\}. \tag{27}$$

By (23)-(27), it follows that $0 \leqq \widetilde{\lambda}_i \leqq 1$, but $\widetilde{\lambda}_i > 0$ for at least one $i \in I$, $0 \leqq \widetilde{\xi}_m \leqq 1$, $m = 1, ..., \overline{w}_0$, $0 \leqq \widetilde{\xi}_m^+ \leqq 1$, $s^m \in S_{k_m}^+(\overline{x})$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, $0 \leqq \widetilde{\xi}_m^- \leqq 1$, $s^m \in S_{k_m}^-(\overline{x})$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$ and, moreover,

$$\sum_{i=1}^{p} \widetilde{\lambda}_i + \sum_{m=1}^{\overline{w}_0} \widetilde{\xi}_m + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^+ + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^- = 1. \tag{28}$$

Combining (22) and (24)-(27), we get

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \alpha_i\left(\widetilde{x}, \overline{x}\right)\left(\nabla f_i\left(\overline{x}\right) - \overline{v}_i \nabla g_i\left(\overline{x}\right), \rho_i\right)\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\xi}_m \Phi\left(\widetilde{x}, \overline{x}, \beta_{j_m}\left(\widetilde{x}, \overline{x}\right)\left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right)$$
$$+ \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^+ \Phi\left(\widetilde{x}, \overline{x}, \gamma_{k_m}^+\left(\widetilde{x}, \overline{x}\right)\left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right)\right)$$
$$+ \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^- \Phi\left(\widetilde{x}, \overline{x}, \gamma_{k_m}^-\left(\widetilde{x}, \overline{x}\right)\left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right) < 0. \tag{29}$$

By Definition 2.1, it follows that $\Phi\left(\widetilde{x}, \overline{x}, \cdot\right)$ is a convex function on R^{n+1}. Since (28) holds, by the definition of a convex function, we have

$$\sum_{i=1}^{p} \widetilde{\lambda}_i \Phi\left(\widetilde{x}, \overline{x}, \alpha_i\left(\widetilde{x}, \overline{x}\right)\left(\nabla f_i\left(\overline{x}\right) - \overline{v}_i \nabla g_i\left(\overline{x}\right), \rho_i\right)\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\xi}_m \Phi\left(\widetilde{x}, \overline{x}, \beta_{j_m}\left(\widetilde{x}, \overline{x}\right)\left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right) \tag{30}$$
$$+ \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^+ \Phi\left(\widetilde{x}, \overline{x}, \gamma_{k_m}^+\left(\widetilde{x}, \overline{x}\right)\left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right)\right) + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^- \Phi\left(\widetilde{x}, \overline{x}, \gamma_{k_m}^-\left(\widetilde{x}, \overline{x}\right)\left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)$$
$$\geqq \Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^{p} \widetilde{\lambda}_i \alpha_i\left(\widetilde{x}, \overline{x}\right)\left(\nabla f_i\left(\overline{x}\right) - \overline{v}_i \nabla g_i\left(\overline{x}\right), \rho_i\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\xi}_m \beta_{j_m}\left(\widetilde{x}, \overline{x}\right)\left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right.\right.$$
$$\left.\left.+ \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^+ \gamma_{k_m}^+\left(\widetilde{x}, \overline{x}\right)\left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right) + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^- \gamma_{k_m}^-\left(\widetilde{x}, \overline{x}\right)\left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)\right).$$

By (29) and (30), it follows that

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\sum_{i=1}^{p} \widetilde{\lambda}_i \alpha_i\left(\widetilde{x}, \overline{x}\right)\left(\nabla f_i\left(\overline{x}\right) - \overline{v}_i \nabla g_i\left(\overline{x}\right), \rho_i\right) + \sum_{m=1}^{\overline{w}_0} \widetilde{\xi}_m \beta_{j_m}\left(\widetilde{x}, \overline{x}\right)\left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right)\right.\right. \tag{31}$$
$$\left.\left.+ \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^+ \gamma_{k_m}^+\left(\widetilde{x}, \overline{x}\right)\left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right) + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \widetilde{\xi}_m^- \gamma_{k_m}^-\left(\widetilde{x}, \overline{x}\right)\left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)\right) < 0.$$

Taking into account (24)-(27) in the inequality above, we obtain

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{A}\left(\sum_{i=1}^{p} \overline{\lambda}_i\left(\nabla f_i\left(\overline{x}\right) - \overline{v}_i \nabla g_i\left(\overline{x}\right), \rho_i\right) + \sum_{m=1}^{\overline{w}_0} \overline{\xi}_m\left(\nabla G_{j_m}(\overline{x}, t^m), \rho_{G_{j_m}}\right) + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m\left(\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^+\right)\right.\right.$$
$$\left.\left.+ \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \left(-\overline{\xi}_m\right)\left(-\nabla H_{k_m}(\overline{x}, s^m), \rho_{H_{k_m}}^-\right)\right)\right) < 0. \tag{32}$$

Thus, (32) gives

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{A}\left(\sum_{i=1}^{p}\overline{\lambda}_i\left(\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right)+\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right.,\right.$$

$$\left.\left.\sum_{i=1}^{p}\overline{\lambda}_i\rho_i+\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m\rho_{G_{j_m}}+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\rho^+_{H_{k_m}}-\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\rho^-_{H_{k_m}}\right)\right)<0.$$

Hence, the necessary optimality condition (5) implies

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{A}\left(0\ ,\ \sum_{i=1}^{p}\overline{\lambda}_i\rho_i+\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m\rho_{G_{j_m}}+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\rho^+_{H_{k_m}}-\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\rho^-_{H_{k_m}}\right)\right)<0. \tag{33}$$

By Definition 2.1, we have that $\Phi\left(\widetilde{x},\overline{x},(0,a)\right)\geqq 0$ for all $a\in R_+$. Therefore, by hypothesis d), it follows that the following inequality

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{A}\left(0\ ,\ \sum_{i=1}^{p}\overline{\lambda}_i\rho_i+\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m\rho_{G_{j_m}}+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\rho^+_{H_{k_m}}-\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\overline{\xi}_m\rho^-_{H_{k_m}}\right)\right)\geqq 0$$

holds, contradicting (33). Hence, $\overline{x}$ is efficient in problem (P) and this completes the proof of this theorem under hypothesis A).

Proof under hypothesis B) is similar to the proof under hypothesis A) and, therefore, it has been omitted.

Proof of the theorem under hypothesis C).

By assumption C), each function $z\to L_i\left(z,\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)$, $i=1,...,p$, is (Φ,ρ_i)-α_i-invex at $\overline{x}$ on D, where $\overline{t}\equiv\left(t^1,...,t^{\overline{w}_0}\right)$, $\overline{s}\equiv\left(s^{\overline{w}_0+1},...,s^{\overline{w}}\right)$. Hence, by Definition 2.2, it follows that the inequalities

$$L_i\left(x,\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)-L_i\left(\overline{x},\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)\geqq\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\nabla L_i\left(\overline{x},\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right),\rho_i\right)\right),\ i\in I \tag{34}$$

hold for all $x\in D$. Therefore, they are also satisfied for $x=\widetilde{x}\in D$. By definition of the Lagrange function (see (6)), we have

$$\overline{\lambda}_i\left[f_i\left(\widetilde{x}\right)-\overline{v}_ig_i\left(\widetilde{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)\right] \tag{35}$$

$$-\overline{\lambda}_i\left[f_i\left(\overline{x}\right)-\overline{v}_ig_i\left(\overline{x}\right)\right]-\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\overline{x},s^m)\right]$$

$$\geqq\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right]\ ,\ \rho_i\right)\right),\quad i\in I.$$

Multiplying (35) by $\frac{\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}{\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}$, $i\in I$, and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^{p}\frac{\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}{\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}\left(\overline{\lambda}_i\left[f_i\left(\widetilde{x}\right)-\overline{v}_ig_i\left(\widetilde{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)\right]\right)$$

$$-\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\left(\left[f_i\left(\overline{x}\right)-\overline{v}_i g_i\left(\overline{x}\right)\right]\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m H_{k_m}(\overline{x},s^m)\right]\right)\geqq$$

$$\frac{1}{\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right).$$

By $\widetilde{x}\in D$ and $\overline{x}\in D$, it follows that $\overline{\xi}_m G_{j_m}(\widetilde{x},t^m)\leqq\overline{\xi}_m G_{j_m}(\overline{x},t^m)$, $t^m\in\widehat{T}_{j_m}\left(\overline{x}\right)$, $m=1,...,\overline{w}_0$, $\overline{\xi}_m H_{k_m}(\widetilde{x},s^m)=\overline{\xi}_m H_{k_m}(\overline{x},s^m)$, $s^m\in S_{k_m}$, $m\in\{\overline{w}_0+1,...,\overline{w}\}$. Then, taking into account the above relations together with (7) and (8), we have

$$\frac{1}{\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]\right.\right.$$
$$\left.\left.+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right)<0. \tag{36}$$

Let us denote

$$\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)=\frac{\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}{\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}},i\in I. \tag{37}$$

By (36), it follows that $\sum_{i=1}^{p}\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)=1$. Hence, (36) and (37) yield

$$\sum_{i=1}^{p}\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right)<0. \tag{38}$$

By Definition 2.1, we have that $\Phi\left(\widetilde{x},\overline{x},\cdot\right)$ is a convex function on R^{n+1}. Since $\sum_{i=1}^{p}\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)=1$, using the definition of a convex function, we obtain

$$\sum_{i=1}^{p}\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right) \tag{39}$$
$$\geqq\Phi\left(\widetilde{x},\overline{x},\sum_{i=1}^{p}\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right).$$

Combining (38) and (39), we get

$$\Phi\left(\widetilde{x},\overline{x},\sum_{i=1}^{p}\widetilde{\alpha}_i\left(\widetilde{x},\overline{x}\right)\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\frac{1}{p}\left[\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right)<0.$$

Thus, by (37), it follows that

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{\sum_{i=1}^{p}\frac{1}{\alpha_i\left(\widetilde{x},\overline{x}\right)}}\left(\sum_{i=1}^{p}\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m),\sum_{i=1}^{p}\rho_i\right)\right)<0. \tag{40}$$

Hence, by the necessary optimality condition (5), the inequality (4) gives

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{\sum_{i=1}^{p}\frac{1}{\alpha_i(\widetilde{x},\overline{x})}}\left(0\ ,\ \sum_{i=1}^{p}\rho_i\right)\right)<0. \tag{41}$$

By Definition 2.1, it follows that $\Phi\left(\widetilde{x},\overline{x},(0,a)\right)\geqq 0$ for all $a\in R_+$. Since $\sum_{i=1}^{p}\rho_i\geqq 0$, the following inequality

$$\Phi\left(\widetilde{x},\overline{x},\frac{1}{\sum_{i=1}^{p}\frac{1}{\alpha_i(\widetilde{x},\overline{x})}}\left(0\ ,\ \sum_{i=1}^{p}\rho_i\right)\right)\geqq 0$$

holds, contradicting (41). Hence, $\overline{x}$ is efficient in problem (P) and this completes the proof of the theorem under hypothesis C).

Proof of theorem under hypothesis D).

By assumption D), each function $\psi_i\left(\cdot,\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)=f_i\left(\cdot\right)-v_ig_i\left(\cdot\right)+\sum_{m=1}^{w_0}\overline{\xi}_mG_{j_m}(\cdot,t^m)+\sum_{m=w_0+1}^{w}\overline{\xi}_mH_{k_m}(\cdot,s^m)$, $i=1,...,p$, is (Φ,ρ_i)-α_i-invex at $\overline{x}$ on D, where $\overline{t}\equiv\left(t^1,...,t^{\overline{w}_0}\right)$, $\overline{s}\equiv\left(s^{\overline{w}_0+1},...,s^{\overline{w}}\right)$. Hence, by Definition 2.1, it follows that the inequalities

$$\psi_i\left(x,\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)-\psi_i\left(\overline{x},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)\geqq\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\nabla\psi_i\left(\overline{x},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right),\rho_i\right)\right),\ i\in I \tag{42}$$

hold for all $x\in D$. Therefore, they are also satisfied for $x=\widetilde{x}\in D$. By the definition of ψ_i, $i=1,...,p$, we have

$$\begin{gathered}f_i\left(\widetilde{x}\right)-\overline{v}_ig_i\left(\widetilde{x}\right)+\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)-\\ \left[f_i\left(\overline{x}\right)-\overline{v}_ig_i\left(\overline{x}\right)\right]-\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\overline{x},t^m)-\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\overline{x},s^m)\geqq\\ \Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)+\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\ ,\ \rho_i\right)\right),\quad i\in I.\end{gathered} \tag{43}$$

Multiplying (43) by $\frac{\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}{\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}$, $i\in I$, and then adding both sides of the obtained inequalities, we get

$$\begin{gathered}\frac{1}{\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}\left[\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\left(\left[f_i\left(\widetilde{x}\right)-\overline{v}_ig_i\left(\widetilde{x}\right)\right]+\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)\right)\right.\\ \left.-\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\left(\left[f_i\left(\overline{x}\right)-\overline{v}_ig_i\left(\overline{x}\right)\right]+\sum_{i=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\overline{x},s^m)\right)\right]\\ \geqq\frac{1}{\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\ ,\ \rho_i\right)\right).\end{gathered} \tag{44}$$

By $\widetilde{x} \in D$ and $\overline{x} \in D$, it follows that $\overline{\xi}_m G_{j_m}(\widetilde{x}, t^m) \leqq \overline{\xi}_m G_{j_m}(\overline{x}, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, $\overline{\xi}_m H_{k_m}(\widetilde{x}, s^m) = \overline{\xi}_m H_{k_m}(\overline{x}, s^m)$, $s^m \in S_{k_m}$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$. Using the relations above together with (7) and (8), we get

$$\frac{1}{\sum\limits_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}} \sum_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})} \Phi\left(\widetilde{x}, \overline{x}, \alpha_i(\widetilde{x},\overline{x})\left([\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})] + \sum_{i=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m)\ ,\ \rho_i\right)\right) < 0. \tag{45}$$

Let us denote

$$\widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) = \frac{\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}{\sum_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}, i \in I. \tag{46}$$

By (46), it follows that $\sum_{i=1}^{p} \widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) = 1$. Thus, (45) and (46) yield

$$\sum_{i=1}^{p} \widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) \Phi\left(\widetilde{x}, \overline{x}, \alpha_i(\widetilde{x},\overline{x})\left(\overline{\lambda}_i [\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})] + \sum_{i=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m)\ ,\ \rho_i\right)\right) < 0. \tag{47}$$

By Definition 2.1, it follows that $\Phi(\widetilde{x}, \overline{x}, \cdot)$ is a convex function on R^{n+1}. Since $\sum_{i=1}^{p} \widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) = 1$, by the definition of a convex function, we have

$$\sum_{i=1}^{p} \widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) \Phi\left(\widetilde{x}, \overline{x}, \alpha_i(\widetilde{x},\overline{x})\left([\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})] + \sum_{i=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m)\ ,\ \rho_i\right)\right) \tag{48}$$

$$\geqq \Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) \alpha_i(\widetilde{x},\overline{x})\left([\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})] + \sum_{i=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m)\ ,\ \rho_i\right)\right).$$

Combining (47) and (48), we get

$$\Phi\left(\widetilde{x}, \overline{x}, \sum_{i=1}^{p} \widetilde{\alpha}_i^{\overline{\lambda}}(\widetilde{x},\overline{x}) \alpha_i(\widetilde{x},\overline{x})\left([\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})] + \sum_{i=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m)\ ,\ \rho_i\right)\right) < 0.$$

By (46) and $\overline{\lambda} \in \Lambda$, the inequality above yields

$$\Phi\left(\widetilde{x}, \overline{x}, \frac{1}{\sum_{i=1}^{p} \frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}}\left(\sum_{i=1}^{p} \overline{\lambda}_i [\nabla f_i(\overline{x}) - \overline{v}_i \nabla g_i(\overline{x})] + \sum_{i=1}^{\overline{w}_0} \overline{\xi}_m \nabla G_{j_m}(\overline{x}, t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}} \overline{\xi}_m \nabla H_{k_m}(\overline{x}, s^m), \sum_{i=1}^{p} \overline{\lambda}_i \rho_i\right)\right) < 0.$$

The rest of the proof is the same as the proof under hypothesis D).

Proofs of this theorem under hypotheses E)-H) are similar to the one of the above proofs and, therefore, they have been omitted in the paper. □

Now, we give an example of a nonconvex semi-infinite multiobjective fractional programming problem with (Φ, ρ)-V-invex functions. It turns out that, to prove efficiency of a feasible point for such a vector optimization problem, the concept of (Φ, ρ)-V-invexity may be applied.

Example 3.3. *Consider the following semi-infinite multiobjective fractional programming problem:*

$$V\text{-minimize } \varphi(x) = \frac{f(x)}{g(x)} = \left(\frac{x_1^2 + x_2^2 + 2}{2 + \arctan(x_1 x_2)}, \frac{x_1^2 + x_2^2 + 4}{4 + \arctan(x_1 x_2)}, ..., \frac{x_1^2 + x_2^2 + 2p}{2p + \arctan(x_1 x_2)} \right)$$

$$\text{s.t. } G_1(x,t) = -\frac{t}{1+t} - \frac{x_1 x_2}{x_1^2 + x_2^2 + 1} \leqq 0,\ t \in T_1 = [0,1], \qquad \text{(P1)}$$

where p is a finite positive integer. Note that the set of all feasible solutions

$$D = \left\{ (x_1, x_2) \in R^2 : -\frac{t}{1+t} - \frac{x_1 x_2}{x_1^2 + x_2^2 + 1} \leqq 0,\ t \in T_1 = [0,1] \right\}.$$

Further, it is not difficult to note that $\overline{x} = (0,0)$ is a feasible solution in problem (P1) at which the necessary optimality condition (5) is satisfied and $\overline{v} = (1, ..., 1) \in R^p$. By definition, it can be proved that the functions constituting the semi-infinite multiobjective fractional programming problem (P1) are (Φ, ρ)-V-invex functions at $\overline{x}$ on D, where

$$\Phi(x, \overline{x}, (\zeta, \rho)) = \tfrac{1}{2}(\zeta_1 + \zeta_2)\left(x_1^2 - \overline{x}_1^2 + x_2^2 - \overline{x}_2^2\right) + (2^\rho - 1)\left|(x_1 - \overline{x}_1)(x_2 - \overline{x}_2)\right|,$$

$$\alpha_i(x, \overline{x}) = 1,\ \rho_{f_i} = 1,\ \rho_{g_i} = 1,\ i = 1, ..., p,$$

$$\beta_1(x, \overline{x}) = \frac{1}{x_1^2 + x_2^2 + 1},\ \rho_{G_1} = -1$$

Since all hypotheses of Theorem 3.2 are fulfilled, $\overline{x} = (0,0)$ is efficient in problem (P1).

Remark 3.4. *Note that, to prove efficiency of $\overline{x}$ in the considered semi-infinite multiobjective fractional programming problem (P1) considered in Example 3.3, it is not possible to use the sufficient optimality conditions under invexity [7], [13], and also under many generalized convexity notions, previously defined in the literature (that is, r-invexity [3], F-convexity [16], (F, ρ)-convexity [29], [12], b-invexity [13], B-(p, r)-invexity [2], [5], V-invexity [23], G-invexity [6]). This is a consequence of the fact that a stationary point of each objective function g_i, $i = 1, ..., p$, and the stationary point of the constraint function $G_1(x,t)$ are not their global minimizers (see, Ben-Israel and Mond [9]). Then, each objective function g_i, $i = 1, ..., p$, and the constraint function $G_1(x,t)$ are neither invex [7], [13], nor generalized convex (for example, r-invex [3], V-invex [23], F-convex [16], B-(p, r)-invex [2], [5], G-invex [6]) with respect to any function $\eta : D \times D \to R^2$. As it follows even from this example of a nonconvex semi-infinite multiobjective fractional programming problem, the sufficient optimality conditions for efficiency established under (Φ, ρ)-V-invexity are useful for a larger class of such nonconvex vector optimization problems than the sufficient optimality conditions established under other generalized convexity, even those ones mentioned above.*

Now, for the considered semi-infinite multiobjective fractional programming problem (P), we prove seven sets of the sufficient optimality conditions for efficiency under various generalized (Φ, ρ)-V-invexity assumptions.

Theorem 3.5. *Let $\overline{x} \in D$ and $\overline{v} = \varphi(\overline{x})$. Also, let f_i and g_i, $i \in I$, $z \to G_j(z,t)$, $j \in J$, $z \to H_k(z,s)$, $k \in K$, be differentiable at $\overline{x}$ for all $t \in T_j$ and for all $s \in S_k$. Further, assume that there exist $\overline{\lambda} \in \Lambda$, integers $\overline{w}_0$ and $\overline{w}$, with $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, such that there exist $\overline{w}_0$ indices j_m, with $1 \leqq j_m \leqq q$, together with $\overline{w}_0$ points $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, $\overline{w} - \overline{w}_0$ indices k_m with $1 \leqq k_m \leqq r$ together with $\overline{w} - \overline{w}_0$ points $s^m \in S_{k_m}$, $m = 1, ..., \overline{w} - \overline{w}_0$, and $\overline{w}$ real numbers $\overline{\xi}_m$ with $\overline{\xi}_m > 0$ for $m = 1, ..., \overline{w}_0$, with the property that the relation (5) is fulfilled at $\overline{x}$. Assume, furthermore, that any one of the following seven sets of hypotheses is fulfilled:*

A) a) $f_i(\cdot) - \overline{v}_i g_i(\cdot)$, $i = 1, ..., p$, is a (Φ, ρ_i)-α_i-pseudo-invex function at $\overline{x}$ on D,

b) $G_{j_m}(\cdot, t^m)$, $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1, ..., \overline{w}_0$, is a $\left(\Phi, \rho_{G_{j_m}}\right)$-$\beta_{j_m}$-quasi-invex function at $\overline{x}$ on D,

c) $H_{k_m}(\cdot, s^m)$, $s^m \in S^+_{k_m}(\overline{x})$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^+_{H_{k_m}}\right)$-$\gamma^+_{k_m}$-quasi-invex function at $\overline{x}$ on D,

d) $-H_{k_m}(\cdot, s^m)$, $s^m \in S^-_{k_m}(\overline{x})$, $m \in \{\overline{w}_0 + 1, ..., \overline{w}\}$, is a $\left(\Phi, \rho^-_{H_{k_m}}\right)$-$\gamma^-_{k_m}$-quasi-invex function at $\overline{x}$ on D,

e) $\sum_{i=1}^p \overline{\lambda}_i \rho_i + \sum_{m=1}^{\overline{w}_0} \overline{\xi}_m \rho_{G_{j_m}} + \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m \rho^+_{H_{k_m}} - \sum_{m \in \{\overline{w}_0+1,...,\overline{w}\}} \overline{\xi}_m \rho^-_{H_{k_m}} \geqq 0$, *B) each component of the vector-valued Lagrange function* $L\left(\cdot,\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)$*, that is, each function* $z \to L_i\left(z,\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)$*,* $i = 1,...,p$*, is* (Φ,ρ_i)*-*α_i*-pseudo-invex at* $\overline{x}$ *on* D*, where* $\overline{t} \equiv \left(t^1,...,t^{\overline{w}_0}\right)$*,* $\overline{s} \equiv \left(s^{\overline{w}_0+1},...,s^{\overline{w}}\right)$ *and, moreover,* $\sum_{i=1}^{p}\rho_i \geqq 0$*,*

C) each function $\psi_i\left(\cdot,\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right) = f_i(\cdot) - \overline{v}_i g_i(\cdot) + \sum_{m=1}^{w_0}\overline{\xi}_m G_{j_m}(\cdot,t^m) + \sum_{m=w_0+1}^{w}\overline{\xi}_m H_{k_m}(\cdot,s^m)$*,* $i = 1,...,p$*, is* (Φ,ρ_i)*-*α_i*-pseudo-invex at* $\overline{x}$ *on* D*, where* $\overline{t} \equiv \left(t^1,...,t^{\overline{w}_0}\right)$*,* $\overline{s} \equiv \left(s^{\overline{w}_0+1},...,s^{\overline{w}}\right)$ *and, moreover,* $\sum_{i=1}^{p}\overline{\lambda}_i\rho_i \geqq 0$*,*

D) a) $f_i(\cdot) - \overline{v}_i g_i(\cdot)$*,* $i = 1,...,p$*, is a* (Φ,ρ_i)*-*α_i*-pseudo-invex function at* $\overline{x}$ *on* D*,*
b) $\overline{\xi}_m G_{j_m}(\cdot,t^m)$*,* $t^m \in \widehat{T}_{j_m}(\overline{x})$*,* $m = 1,...,\overline{w}_0$*, is a* $\left(\Phi,\rho_{G_{j_m}}\right)$*-*$\beta_{j_m}$*-quasi-invex function at* $\overline{x}$ *on* D*,*
c) $\overline{\xi}_m H_{k_m}(\cdot,s^m)$*,* $s^m \in S_{k_m}$*, is a* $\left(\Phi,\rho_{H_{k_m}}\right)$*-*$\gamma_{k_m}$*-quasi-invex function at* $\overline{x}$ *on* D*,*
e) $\sum_{i=1}^{p}\overline{\lambda}_i\rho_i + \sum_{m=1}^{\overline{w}_0}\rho_{G_{j_m}} + \sum_{m=\overline{w}_0+1}^{\overline{w}}\rho_{H_{k_m}} \geqq 0$*,*

E) a) $\overline{\lambda}_i\left[f_i(\cdot) - \overline{v}_i g_i(\cdot)\right]$*,* $i = 1,...,p$*, is a strictly* (Φ,ρ_i)*-*α_i*-pseudo-invex function at* $\overline{x}$ *on* D*,*
b) $\overline{\xi}_m G_{j_m}(\cdot,t^m)$*,* $t^m \in \widehat{T}_{j_m}(\overline{x})$*,* $m = 1,...,\overline{w}_0$*, is a* $\left(\Phi,\rho_{G_{j_m}}\right)$*-*$\beta_{j_m}$*-quasi-invex function at* $\overline{x}$ *on* D*,*
c) $\overline{\xi}_m H_{k_m}(\cdot,s^m)$*,* $s^m \in S_{k_m}$*, is a* $\left(\Phi,\rho_{H_{k_m}}\right)$*-*$\gamma_{k_m}$*-quasi-invex function at* $\overline{x}$ *on* D*,*
e) $\sum_{i=1}^{p}\rho_i + \sum_{m=1}^{\overline{w}_0}\rho_{G_{j_m}} + \sum_{m=\overline{w}_0+1}^{\overline{w}}\rho_{H_{k_m}} \geqq 0$*,*

F) a) $\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i(\cdot) - \overline{v}_i g_i(\cdot)\right]$*,* $i = 1,...,p$*, is a* (Φ,ρ_α)*-*α*-pseudo-invex function at* $\overline{x}$ *on* D*,*
b) $\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m G_{j_m}(\cdot,t^m)$*,* $t^m \in \widehat{T}_{j_m}(\overline{x})$*, is a* (Φ,ρ_G)*-*β*-quasi-invex function at* $\overline{x}$ *on* D*,*
c) $\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m H_{k_m}(\cdot,s^m)$*,* $s^m \in S_{k_m}$*, is a* (Φ,ρ_H)*-*γ*-quasi-invex function at* $\overline{x}$ *on* D*,*
d) $\rho_\alpha + \rho_G + \rho_H \geqq 0$*,*

G) a) $\sum_{i=1}^{p}\overline{\lambda}_i\left[f_i(\cdot) - \overline{v}_i g_i(\cdot)\right]$*,* $i = 1,...,p$*, is a* (Φ,ρ_α)*-*α*-pseudo-invex function at* $\overline{x}$ *on* D*,*
b) $\sum_{m=1}^{\overline{w}_0}\overline{\xi}_m G_{j_m}(\cdot,t^m) + \sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m H_{k_m}(\cdot,s^m)$*,* $t^m \in \widehat{T}_{j_m}(\overline{x})$*,* $s^m \in S_{k_m}$*, is a* $\left(\Phi,\rho_\beta\right)$*-*$\beta$*-quasi-invex function at* $\overline{x}$ *on* D*,*
c) $\rho_\alpha + \rho_\beta \geqq 0$*.*

Then $\overline{x}$ *is efficient in problem (P) with the corresponding optimal objective value equal to* $\overline{v} = \varphi(\overline{x})$*.*

Proof. By assumption, $\overline{x} \in D$, $\overline{v} = \varphi(\overline{x})$ and, moreover, there exist $\overline{\lambda} \in \Lambda$, integers $\overline{w}_0$ and $\overline{w}$, with $0 \leqq \overline{w}_0 \leqq \overline{w} \leqq n+1$, such that there exist $\overline{w}_0$ indices j_m, with $1 \leqq j_m \leqq q$, together with $\overline{w}_0$ points $t^m \in \widehat{T}_{j_m}(\overline{x})$, $m = 1,...,\overline{w}_0$, $\overline{w} - \overline{w}_0$ indices k_m with $1 \leqq k_m \leqq r$ together with $\overline{w} - \overline{w}_0$ points $s^m \in S_{k_m}$, $m = 1,...,\overline{w} - \overline{w}_0$, and $\overline{w}$ real numbers $\overline{\xi}_m$ with $\overline{\xi}_m > 0$ for $m = 1,...,\overline{w}_0$, with the property that the relation (5) is fulfilled at $\overline{x}$.

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not efficient in problem (P). Hence, there exists $\widetilde{x} \in D$ such that

$$f_i(\widetilde{x}) - \overline{v}_i g_i(\widetilde{x}) \leqq f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}) \text{ for } i \in I, \tag{49}$$

$$f_{i^*}(\widetilde{x}) - \overline{v}_{i^*} g_{i^*}(\widetilde{x}) < f_{i^*}(\overline{x}) - \overline{v}_{i^*} g_{i^*}(\overline{x}) \text{ for at least one } i^* \in I. \tag{50}$$

Proof of the theorem under hypothesis A).
By hypothesis a), Definition 2.4 implies that the following inequalities

$$\Phi\left(\widetilde{x},\overline{x},\alpha_i(\widetilde{x},\overline{x})\left(\nabla f_i(\overline{x}) - \overline{v}_i\nabla g_i(\overline{x}),\rho_i\right)\right) \leqq 0,\ i \in I, \tag{51}$$

$$\Phi\left(\widetilde{x},\overline{x},\alpha_{i^*}(\widetilde{x},\overline{x})\left(\nabla f_{i^*}(\overline{x}) - \overline{v}_{i^*}\nabla g_{i^*}(\overline{x}),\rho_{i^*}\right)\right) \leqq 0 \text{ for at least one } i^* \in I \tag{52}$$

hold. By $\widetilde{x} \in D$ and $\overline{x} \in D$, it follows that

$$\overline{\xi}_m G_{j_m}(\widetilde{x},t^m) \leqq \overline{\xi}_m G_{j_m}(\overline{x},t^m),\ t^m \in \widehat{T}_{j_m}(\overline{x}),\ m = 1,...,\overline{w}_0, \tag{53}$$

$$\overline{\xi}_m H_{k_m}(\widetilde{x},s^m) = \overline{\xi}_m H_{k_m}(\overline{x},s^m),\ s^m \in S_{k_m},\ m \in \{\overline{w}_0+1,...,\overline{w}\}. \tag{54}$$

Hence, by (53) and (54), hypotheses b)-d) yield, respectively,

$$\overline{\xi}_m\Phi\left(\widetilde{x},\overline{x},\beta_{j_m}(\widetilde{x},\overline{x})\left(\nabla G_{j_m}(\overline{x},t^m),\rho_{G_{j_m}}\right)\right) \leqq 0,\ t^m \in \widehat{T}_{j_m}(\overline{x}),\ m = 1,...,\overline{w}_0, \tag{55}$$

$$\overline{\xi}_m\Phi\left(\widetilde{x},\overline{x},\gamma^+_{k_m}\left(\widetilde{x},\overline{x}\right)\left(\nabla H_{k_m}(\overline{x},s^m),\rho^+_{H_{k_m}}\right)\right)\leqq 0,\quad s^m\in S^+_{k_m}\left(\overline{x}\right),\, m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}, \tag{56}$$

$$-\overline{\xi}_m\Phi\left(\widetilde{x},\overline{x},\gamma^-_{k_m}\left(\widetilde{x},\overline{x}\right)\left(-\nabla H_{k_m}(\overline{x},s^m),\rho^-_{H_{k_m}}\right)\right)\leqq 0,\quad s^m\in S^-_{k_m}\left(\overline{x}\right),\, m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}. \tag{57}$$

Multiplying the inequalities (51)-(52) by $\frac{\overline{\lambda}_i}{\alpha_i(\widetilde{x},\overline{x})}>0$, (55) by $\frac{1}{\beta_{j_m}(\widetilde{x},\overline{x})}>0$, (56) by $\frac{1}{\gamma^+_{k_m}(\widetilde{x},\overline{x})}>0$, (57) by $\frac{1}{\gamma^-_{k_m}(\widetilde{x},\overline{x})}>0$, and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^{p}\frac{\overline{\lambda}_i}{\alpha_i\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right),\rho_i\right)\right)+\sum_{m=1}^{\overline{w}_0}\frac{\overline{\xi}_m}{\beta_{j_m}\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\beta_{j_m}\left(\widetilde{x},\overline{x}\right)\left(\nabla G_{j_m}(\overline{x},t^m),\rho_{G_{j_m}}\right)\right)$$

$$+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\frac{\overline{\xi}_m}{\gamma^+_{k_m}\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\gamma^+_{k_m}\left(\widetilde{x},\overline{x}\right)\left(\nabla H_{k_m}(\overline{x},s^m),\rho^+_{H_{k_m}}\right)\right)$$

$$+\sum_{m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}}\frac{-\overline{\xi}_m}{\gamma^-_{k_m}\left(\widetilde{x},\overline{x}\right)}\Phi\left(\widetilde{x},\overline{x},\gamma^-_{k_m}\left(\widetilde{x},\overline{x}\right)\left(-\nabla H_{k_m}(\overline{x},s^m),\rho^-_{H_{k_m}}\right)\right)<0.$$

The last part of proof of this theorem is similar to the proof of Theorem 3.2 under hypothesis A).

Proof of the theorem under hypothesis B).

Using (49) and (50) together with $\overline{\lambda}\in\Lambda$, we get, respectively,

$$\overline{\lambda}_i\left[f_i\left(\widetilde{x}\right)-\overline{v}_ig_i\left(\widetilde{x}\right)\right]\leqq\overline{\lambda}_i\left[f_i\left(\overline{x}\right)-\overline{v}_ig_i\left(\overline{x}\right)\right]\ \text{ for } i\in I, \tag{58}$$

$$\overline{\lambda}_{i^*}\left[f_{i^*}\left(\widetilde{x}\right)-\overline{v}_{i^*}g_{i^*}\left(\widetilde{x}\right)\right]<\overline{\lambda}_{i^*}\left[f_{i^*}\left(\overline{x}\right)-\overline{v}_{i^*}g_{i^*}\left(\overline{x}\right)\right]\ \text{ for at least one } i^*\in I. \tag{59}$$

By $\widetilde{x}\in D$ and $\overline{x}\in D$, it follows that

$$\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)\leqq\overline{\xi}_mG_{j_m}(\overline{x},t^m),\ t^m\in\widehat{T}_{j_m}\left(\overline{x}\right),\, m=1,...,\overline{w}_0, \tag{60}$$

$$\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)=\overline{\xi}_mH_{k_m}(\overline{x},s^m),\ s^m\in S_{k_m},\, m\in\left\{\overline{w}_0+1,...,\overline{w}\right\}. \tag{61}$$

Adding both sides of (60) and (61), and then combining the inequalities obtained and (58) and (59), we get

$$\begin{aligned}&\overline{\lambda}_i\left[f_i\left(\widetilde{x}\right)-\overline{v}_ig_i\left(\widetilde{x}\right)\right]+\tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)\right]\\&\leqq\overline{\lambda}_i\left[f_i\left(\overline{x}\right)-\overline{v}_ig_i\left(\overline{x}\right)\right]+\tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\overline{x},s^m)\right],\ i\in I,\end{aligned} \tag{62}$$

$$\begin{aligned}&\overline{\lambda}_{i^*}\left[f_{i^*}\left(\widetilde{x}\right)-\overline{v}_{i^*}g_{i^*}\left(\widetilde{x}\right)\right]+\tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\widetilde{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\widetilde{x},s^m)\right]\\&<\overline{\lambda}_{i^*}\left[f_{i^*}\left(\overline{x}\right)-\overline{v}_{i^*}g_{i^*}\left(\overline{x}\right)\right]+\tfrac{1}{p}\left[\textstyle\sum_{m=1}^{\overline{w}_0}\overline{\xi}_mG_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_mH_{k_m}(\overline{x},s^m)\right]\ \text{for at least one } i^*\in I.\end{aligned} \tag{63}$$

By assumption B), each function $z\to L_i\left(z,\overline{\lambda},\overline{\xi},\overline{v},\overline{w},\overline{w}_0,\overline{t},\overline{s}\right)$, $i=1,...,p$, is (Φ,ρ_i)-α_i-pseudo-invex at $\overline{x}$ on D, where $\overline{t}\equiv\left(t^1,...,t^{\overline{w}_0}\right)$, $\overline{s}\equiv\left(s^{\overline{w}_0+1},...,s^{\overline{w}}\right)$. Hence, by Definition 2.4, (62) and (63) imply that, for every $i=1,...,p$,

$$\Phi\left(\widetilde{x},\overline{x},\alpha_i\left(\widetilde{x},\overline{x}\right)\left(\overline{\lambda}_i\left[\nabla f_i\left(\overline{x}\right)-\overline{v}_i\nabla g_i\left(\overline{x}\right)\right]+\tfrac{1}{p}\left[\textstyle\sum_{i=1}^{\overline{w}_0}\overline{\xi}_m\nabla G_{j_m}(\overline{x},t^m)+\sum_{m=\overline{w}_0+1}^{\overline{w}}\overline{\xi}_m\nabla H_{k_m}(\overline{x},s^m)\right],\rho_i\right)\right)<0.$$

The last part of proof is similar to the proof of Theorem 3.2 under hypothesis B).

Proofs of this theorem under hypotheses C)-G) are similar to the one of the proofs above and, therefore, they have been omitted in the paper. □

4. Concluding Remarks

In the paper, a fairly large number of sets of global parametric sufficient optimality conditions for efficiency has been established for a new class of nonconvex smooth semi-infinite multiobjective fractional programming problems under various (Φ, ρ)-V-invexity and/or generalized (Φ, ρ)-V-invexity assumptions. It appears that all these results are new for semi-infinite programming problems. Indeed, it turns out that the sufficient optimality conditions are applicable also for such semi-infinite multiobjective fractional programming problems in which not all functions constituting them have the fundamental property of convex, invex and the most concept of generalized convex functions - namely, that a stationary point of a function belonging to such a class of functions is also its global minimizer. This result was illustrated in the paper by the example of a nonconvex smooth semi-infinite multiobjective fractional programming problem in which the involved functions belong to the class of (Φ, ρ)-V-invex functions.

Further, it is also easy to see that all results obtained here can be modified and restarted in a straightforward manner for various types of optimization problems. Thus, the results established in this paper collectively provide a truly vast number of new optimality results for several classes of semi-infinite and classical (finite) programming problems.

However, some interesting topics for further research remain. It would be of interest to investigate whether these results are true also for various classes of nonsmooth semi-infinite programming problems. We shall investigate these questions in subsequent papers.

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