Filomat 30:14 (2016), 3771–3778
DOI 10.2298/FIL1614771D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Extremal Solution for a Nonlinear Boundary Value Problems of Fractional p-Laplacian Differential Equation

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Abstract. This paper is concerned with the existence and uniqueness of extremal solution for a nonlinear boundary value problems of fractional differential equation involving Riemann-Liouville derivative and p-Laplacian operator. By applying monotone iterative technique and lower and upper solutions method, we obtain sufficient conditions for the existence and uniqueness of extremal solution and construct the sequences of iteration to approximate it. The paper extends the applications of lower and upper solutions method and obtains some new results.

1. Introduction

This paper investigates the following nonlinear boundary value problem of a fractional differential equation with p-Laplacian operator:

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u(t))) = f(t,u(t),D^{\alpha}_{0^+}u(t)),\ t\in(0,1] \\ t^{\frac{1-\beta}{p-1}}D^{\alpha}_{0^+}u(t)|_{t=0} = \delta, g(\widetilde{u}(0),\widetilde{u}(1)) = 0, \end{cases} \tag{1}$$

where $0 < \alpha,\beta \le 1, 1 < \alpha+\beta \le 2, \delta \ge 0$, $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α, $\phi_p(t) = |t|^{p-2}t, p > 1$ is the p -Laplacian operator and $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p}+\frac{1}{q} = 1$. The nonlinear term $f \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\widetilde{u}(0) = t^{1-\alpha}u(t)|_{t=0}$ and $\widetilde{u}(1) = t^{1-\alpha}u(t)|_{t=1}$.

The existence of solutions for fractional boundary value problems with p-Laplacian have been considered by some authors via classic fixed-point theorems and coincidence degree theory [1–4]. The monotone iterative technique, combined with the method of lower and upper solutions, is also a powerful tool to prove the existence of solutions for boundary value problems of nonlinear differential equations. Recently, some authors used the methods to investigate some nonlinear boundary value problems of nonlinear fractional equations [7–11]. Others also applied the methods to show the existence of solutions for some integer-order p-Laplacian boundary value problems, see [12–15]. However, to the best of our knowledge,

2010 *Mathematics Subject Classification*. Primary 34B15; Secondary 26A33,34B99
Keywords. Fractional differential equation, nonlinear boundary value conditions, p-Laplacian operator, lower and upper solutions method, extremal solution
Received: 25 October 2014; Accepted: 14 May 2015
Communicated by Jelena Manojlović
Research supported by Natural Science Foundation of Shandong Provincial(ZR2012AQ007, ZR2013AM009) and the Doctoral Fund of Shandong Jianzhu University.
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there are few results on fractional order p-Laplacian boundary value problems by the way of the upper and lower method and the monotone iteration, only see [16].

It is worth mentioning that the p-Laplacian operator brings some difficulty in constructing monotone sequences. All known results require the nonlinear terms satisfying monotonicity on unknown function u or its derivatives, which make it easy to iterate. The unnecessary condition is removed in this paper. Furthermore, the nonlinear boundary value condition is considered in the paper which means that our method and main results here are different from those in [2, 3, 13–16].

2. Linear Problems and Comparison Principles

We firstly introduce some spaces. Let the Banach space $C[0,1] = \{u : [0,1] \to \mathbb{R} |\, u(t)$ is continuous on $[0,1]\}$ with the norm $\|u\|_C = \max_{t\in[0,1]} |u(t)|$. Denote $C_{1-\alpha}[0,1]$ by

$$C_{1-\alpha}[0,1] = \{u \in C(0,1] : t^{1-\alpha}u \in C[0,1]\},\ \alpha \in (0,1].$$

Then $C_{1-\alpha}[0,1]$ is a Banach space with the norm $\|u\|_{C_{1-\alpha}} = \|t^{1-\alpha}u(t)\|_C$. It is clear that $C[0,1] := C_0[0,1] \subset C_{1-\alpha}[0,1] \subset C_{1-\beta}[0,1]$ with $\|u\|_{C_{1-\beta}} \le \|u\|_{C_{1-\alpha}} \le \|u\|_C$ for $1 \ge \alpha \ge \beta > 0$.

Denote the space $\mathcal{X}$ by

$$\mathcal{X} = \{u(t) \in C_{1-\alpha}[0,1] : (D^\alpha u)(t) \in C_r[0,1] \text{ and } t^r D^\alpha_{0^+}u(t)|_{t=0} = \delta\},$$

where $r = \frac{1-\beta}{p-1}, 0 < \alpha, \beta \le 1, p > 1$. It is easy to know that the space $\mathcal{X}$ is a Banach space with the norm $\|u\| = \|u\|_{C_{1-\alpha}} + \|D^\alpha u(t)\|_{C_r}$.

Definition 2.1. *[6] The Riemann-Liouville fractional integral $I^\alpha_{0^+}$ and fractional derivative $D^\alpha_{0^+}$ are defined by*

$$I^\alpha_{0^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$

and

$$D^\alpha_{0^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}f(s)ds = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{0^+}f)(t),$$

where $n-1 < \alpha \le n,\ n \in \mathbb{N}$, provided the integrals exist.

Lemma 2.2. *[6] Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^\alpha_{0+}D^\alpha_{0+}u(t) = u(t) - ct^{\alpha-1} \text{ for some } c \in \mathbb{R}.$$

We first show the existence results for the following fractional equation with initial conditions

$$\begin{cases} D^\beta_{0^+}(\phi_p(D^\alpha_{0^+}u(t))) = f(t, u(t), D^\alpha_{0^+}u(t)),\ t \in (0,1], \\ t^{\frac{1-\beta}{p-1}} D^\alpha_{0^+}u(t)|_{t=0} = \phi_p(\delta), \widetilde{u}(0) = k. \end{cases} \tag{2}$$

Let $v(t) := \phi_p(D^\alpha_{0^+}u(t))$. Since $\widetilde{u}(0) = k$, by Lemma 2.2, we have

$$u(t) = kt^{\alpha-1} + I^\alpha_{0^+}\phi_q(v(t)) = kt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(v(s))ds =: (Tv)(t), \tag{3}$$

and $\phi_p(t^{\frac{1-\beta}{p-1}} D^\alpha_{0^+}u(t)) = t^{1-\beta}\phi_p(D^\alpha_{0^+}u(t)) = t^{1-\beta}v(t)$ for $0 < t \le 1$.

Substituting the above $u(t)$ and $v(t)$ into the nonlinear term $f(t, u(t), D^{\alpha}_{0^+}u(t))$ of problem (2), we get that

$$\begin{cases} D^{\beta}_{0^+}v(t) = f(t,(Tv)(t), v(t)),\ t \in (0,1], \\ t^{1-\beta}v(t)|_{t=0} = \phi_p(\delta). \end{cases} \tag{4}$$

If the problem (4) has a solution $v(t)$, then substituting it into (3), we can get a solution u of the problem (2). So, we shall show that the problem (4) has at least one solutions under a proper condition.

Lemma 2.3. *Assume that* $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and there exists nonnegative constant* M *such that*

$$|f(t, u_1(t), u_2(t)) - f(t, v_1(t), v_2(t))| \le M|v_2(t) - u_2(t)|,\ t \in (0,1], \tag{5}$$

then the problem (4) *has a unique solution* $x(t) \in C_{1-\beta}[0,1]$.

Proof. By Lemma 2.2, in accordance with $t^{1-\beta}v(t)|_{t=0} = \phi_p(\delta)$, the problem (4) is equivalent to the integral equation

$$v(t) = \phi_p(\delta)t^{\beta-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,(Tv)(s), v(s))ds. \tag{6}$$

It can be written in the form $v = \mathbf{B}v$, where the operator $\mathbf{B}$ is defined by

$$\mathbf{B}v(t) := \phi_p(\delta)t^{\beta-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,(Tv)(s), v(s))ds.$$

It is clear that $\phi_p(\delta)t^{\beta-1} \in C_{1-\beta}[0,1]$. Since $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and

$$\|I^{\beta}_{0^+} f(s,(Tv)(s), v(s))\|_{C_{1-\beta}} \le \frac{\Gamma(\beta)}{\Gamma(2\beta)}\|f\|_{C_{1-\beta}},$$

we get that the operator $\mathbf{B} : C_{1-\beta}[0,1] \to C_{1-\beta}[0,1]$.

To prove $\mathbf{B}$ is a contraction operator, we use the way that is derived from [7, Theorme 1]. Let us choose constants m, n such that $1 < m < \frac{1}{1-\beta}$ and $\frac{1}{m} + \frac{1}{n} = 1$. We use the following norm

$$\|v\|_* = \max_{t\in[0,1]} t^{1-\beta}e^{\kappa t}|v(t)|,$$

with a positive κ such that

$$(\kappa n)^{\frac{1}{n}} > \frac{M}{\Gamma(\beta)} e^{\kappa}[\frac{\Gamma^2(m(\beta-1)+1)}{\Gamma(2m(\beta-1)+2)}]^{\frac{1}{m}} \equiv \varrho.$$

Note that

$$\int_0^t ((t-s)^{\beta-1}s^{\beta-1})^m \mathrm{d}s = \frac{\Gamma^2(m(\beta-1)+1)}{\Gamma(2m(\beta-1)+2)} t^{2m(\beta-1)+1}.$$

For any $u, v \in C_{1-\beta}[0,1]$, using the Hölder inequality for integrals

$$\int_0^t |a(t)||b(t)|\mathrm{d}t \le \left(\int_0^t |a(t)|^m \mathrm{d}t\right)^{\frac{1}{m}} \left(\int_0^t |b(t)|^n \mathrm{d}t\right)^{\frac{1}{n}},$$

and the inequality (5), we obtain

$$\begin{aligned}
\|\mathbf{B}u(t)-\mathbf{B}v(t)\|_* &\le \max_{t\in[0,1]} t^{1-\beta}e^{\kappa t}\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|f(s,(Tu)(s),u(s))-f(s,(Tv)(s),v(s))|\mathrm{d}s\\
&\le \max_{t\in[0,1]} \frac{t^{1-\beta}e^{\kappa t}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|Mu(s)-Mv(s)|\mathrm{d}s\\
&= \max_{t\in[0,1]} \frac{t^{1-\beta}e^{\kappa t}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\beta-1}e^{-\kappa s}s^{1-\beta}e^{\kappa s}|Mu(s)-Mv(s)|\mathrm{d}s\\
&\le \frac{M}{\Gamma(\beta)}\|u-v\|_* \max_{t\in[0,1]} t^{1-\beta}e^{\kappa t}\int_0^t (t-s)^{\beta-1}s^{\beta-1}e^{-\kappa s}\mathrm{d}s\\
&\le \frac{M}{\Gamma(\beta)}\|u-v\|_* \max_{t\in[0,1]} t^{1-\beta}e^{\kappa t}\left(\int_0^t |(t-s)^{\beta-1}s^{\beta-1}|^m\mathrm{d}s\right)^{\frac{1}{m}}\left(\int_0^t |e^{-\kappa s}|^n\mathrm{d}s\right)^{\frac{1}{n}}\\
&\le \frac{M}{\Gamma(\beta)}\|u-v\|_* \max_{t\in[0,1]} t^{\beta-1+\frac{1}{m}}e^{\kappa t}[\frac{\Gamma^2(m(\beta-1)+1)}{\Gamma(2m(\beta-1)+2)}]^{\frac{1}{m}}(\frac{1}{\kappa n})^{\frac{1}{n}}\\
&\le \frac{M}{\Gamma(\beta)}\|u-v\|_* e^{\kappa}[\frac{\Gamma^2(m(\beta-1)+1)}{\Gamma(2m(\beta-1)+2)}]^{\frac{1}{m}}(\frac{1}{\kappa n})^{\frac{1}{n}}\\
&\le \frac{\varrho}{(\kappa n)^{\frac{1}{n}}}\|u-v\|_* < \|u-v\|_*.
\end{aligned}$$

So the operator **B** has a unique fixed point by the Banach fixed point theorem, and then the problem (4) has a unique solution. □

Lemma 2.4. *Assume that* $0<\alpha,\beta\le 1, M(t)\in C[0,1]$ *and* $\eta(t)\in C_{1-\beta}[0,1]$. *Then the linear fractional initial problem*

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u(t)))+M(t)\phi_p(D^{\alpha}_{0^+}u(t))=\eta(t),\ t\in(0,1]\\ t^{\frac{1-\beta}{p-1}}D^{\alpha}_{0^+}u(t)|_{t=0}=\delta,\ \widetilde{u}(0)=k, \end{cases} \tag{7}$$

has a unique solution $u\in\mathcal{X}$.

Proof. Let $v(t):=\phi_p(D^{\alpha}_{0^+}u(t))$. The problem (7) is transformed into the following fractional initial value problems

$$\begin{cases} D^{\alpha}_{0^+}u(t)=\phi_q(v(t)), t\in(0,1],\\ \widetilde{u}(0)=k, \end{cases} \tag{8}$$

and

$$\begin{cases} D^{\beta}_{0^+}v(t)+M(t)v(t)=\eta(t),\ t\in(0,1],\\ t^{1-\beta}v(t)|_{t=0}=\phi_p(\delta). \end{cases} \tag{9}$$

Let $f(t,Tv,v):=\eta-M(t)v$. Since $M(t)\in C[0,1]$, $M(t)$ is a bounded function with $|M(t)|\le\|M\|_C<\infty$. For $v_1,v_2\in C_{1-\beta}[0,1]$, we get

$$|f(t,Tv_1,v_1)-f(t,Tv_2,v_2)|=|M(t)||v_2-v_1|\le\|M\|_C|v_2-v_1|.$$

Therefore, the above problem (9) has a unique solution $v(t)\in C_{1-\beta}[0,1]$ by Lemma 2.3. So $D^{\alpha}_{0^+}u(t)\in C_{\frac{1-\beta}{p-1}}[0,1]$. In addition, the problem (8) has a solution $u(t)\in C_{1-\alpha}[0,1]$ in the form as (3) by Lemma 2.2. Substituting the solution $v(t)$ of (9) into the solution $u(t)$ of (8), we get a unique solution $u(t)\in\mathcal{X}$ of the problem (7). □

Corollary 2.5. *Assume that* $0 < \alpha, \beta \le 1, \lambda \in \mathbb{R}$ *and* $\eta(t) \in C_{1-\beta}[0,1]$. *Then the linear fractional initial problem*

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) + \lambda\phi_p(D_{0^+}^{\alpha}u(t)) = \eta(t), \ t \in (0,1] \\ t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha}u(t)|_{t=0} = \delta, \ \widetilde{u}(0) = k, \end{cases} \tag{10}$$

has a unique solution $x \in \mathcal{X}$.

The next lemma gives two comparison results which play very important roles in our main results.

Lemma 2.6. *(Comparison results)*
(i) If $x(t) \in C_{1-\beta}[0,1]$ *satisfies*

$$\begin{cases} D_{0^+}^{\beta}x(t) + M(t)x(t) \ge 0, \ t \in (0,1], \\ t^{1-\beta}x(t)|_{t=0} \ge 0, \end{cases}$$

where $M(t) \in C([0,1],[0,\infty))$. *Then* $x(t) \ge 0$ *for* $t \in (0,1]$.
(ii) If $y(t) \in C_{1-\alpha}[0,1]$ *satisfies*

$$\begin{cases} D_{0^+}^{\alpha}y(t) \ge 0, t \in (0,1], \\ \widetilde{y}(0) = k \ge 0. \end{cases}$$

Then $y(t) \ge 0$ *for* $t \in (0,1]$.

Proof. Assume that (i) is not true. It means that there exist points $a, b \in (0,1]$ such $x(a) = 0, x(b) < 0$ and $x(t) \ge 0, t \in (0,a], x(t) < 0, t \in (a,b]$. Let c be the first minimal point of $x(t)$ on $[a,b]$, then $x(t) < 0, t \in (a,c]$. Then, since $M(t) \ge 0$, $D_{0^+}^{\beta}x(t) \ge 0, x \in [a,c]$. So $\int_a^c D_{0^+}^{\beta}x(t)dt \ge 0$, that is,

$$D := \int_a^c \frac{d}{dt} I^{1-\beta}x(t)dt = I^{1-\beta}x(c) - I^{1-\beta}x(a) \ge 0. \tag{11}$$

On the other hand, we have

$$\begin{aligned} D &= \frac{1}{\Gamma(1-\beta)} \left[\int_0^c (c-s)^{-\beta}x(s)ds - \int_0^a (a-s)^{-\beta}x(s)ds \right] \\ &= \frac{1}{\Gamma(1-\beta)} \left[\int_0^a ((c-s)^{-\beta} - (a-s)^{-\beta})x(s)ds + \int_a^c (c-s)^{-\beta}x(s)ds \right] \\ &< 0. \end{aligned}$$

It contradicts (11), so the assertion holds. This completes the proof of result (1).

The result (ii) is obvious by (3). □

3. Main Results

In this section, we show the existence and uniqueness of extremal solution of the problem (1) by monotone iterative technique and the method of upper and lower solutions. First of all, we give the definitions of a couple of lower and upper solutions.

Definition 3.1. *A function* $u(t) \in \mathcal{X}$ *is called a lower solution of the problem* (1) *if it satisfies*

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) \le f(t, u(t), D_{0^+}^{\alpha}u(t)), \ t \in (0,1] \\ t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha}u(t)|_{t=0} = \delta, g(\widetilde{u}(0), \widetilde{u}(1)) \ge 0. \end{cases} \tag{12}$$

Likewise, a function $v(t) \in \mathcal{X}$ *is called an upper solution of the problem* (1) *if it satisfies*

$$\begin{cases} D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}v(t))) \ge f(t, v(t), D_{0^+}^{\alpha}v(t)), \ t \in (0,1] \\ t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha}v(t)|_{t=0} = \delta, g(\widetilde{v}(0), \widetilde{v}(1)) \le 0. \end{cases} \tag{13}$$

We need the following assumptions in our main results.
(H1)Assume that $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.
(H2) There exists a function $M(t) \geq 0, t \in [0,1]$ such that

$$f(t, u(t), D_{0^+}^{\alpha} u(t)) - f(t, v(t), D_{0^+}^{\alpha} v(t)) \leq M(t)[\phi_p(D_{0^+}^{\alpha} v(t)) - \phi_p(D_{0^+}^{\alpha} u(t))]$$

for $u_0(t) \leq u(t) \leq v(t) \leq v_0(t),\ t \in (0,1]$.
(H3) There exist constants $\lambda > 0, \mu \geq 0$ such that

$$g(x_1, y_1) - g(x_2, y_2) \leq \lambda(x_2 - x_1) - \mu(y_2 - y_1)$$

for $\widetilde{u}_0(0) \leq x_1 \leq x_2 \leq \widetilde{v}_0(0)$ and $\widetilde{u}_0(1) \leq y_1 \leq y_2 \leq \widetilde{v}_0(1)$.

Now we give main results

Theorem 3.2. *Assume that $u_0,\ v_0 \in \mathcal{X}$ are lower and upper solutions of the problem* (1)*, respectively and $u_0(t) \leq v_0(t),\ t \in (0,1]$. In addition, assume that (H1), (H2), (H3) hold. Then there exist sequences $\{u_n(t)\}, \{v_n(t)\} \subset \mathcal{X}$ such that the problem* (1) *has extremal solutions in the sector*

$$[u_0,\ v_0] = \{u \in \mathcal{X} : u_0(t) \leq u(t) \leq v_0(t),\ t \in (0,1],\ \widetilde{u}_0(0) \leq \widetilde{u}(0) \leq \widetilde{v}_0(0)\}.$$

Proof. Let $F(u(t)) := f(t, u(t), D_{0^+}^{\alpha} u(t))$. For $n = 1, 2, ...,$ we define

$$D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha} u_n(t))) + M(t)\phi_p(D_{0^+}^{\alpha} u_n(t)) = F(u_{n-1}(t)) + M(t)\phi_p(D_{0^+}^{\alpha} u_{n-1}(t)),\ t \in (0,1],$$
$$t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha} u_n(t)|_{t=0} = \delta,\ \widetilde{u}_n(0) = \widetilde{u}_{n-1}(0) + \tfrac{1}{\lambda} g(\widetilde{u}_{n-1}(0), \widetilde{u}_{n-1}(1)),$$

and

$$D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha} v_n(t))) + M(t)\phi_p(D_{0^+}^{\alpha} v_n(t)) = F(v_{n-1}(t)) + M(t)\phi_p(D_{0^+}^{\alpha} v_{n-1}(t)),\ t \in (0,1],$$
$$t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha} v_n(t)|_{t=0} = \delta, \widetilde{v}_n(0) = \widetilde{v}_{n-1}(0) + \tfrac{1}{\lambda} g(\widetilde{v}_{n-1}(0), \widetilde{v}_{n-1}(1)).$$

In view of Lemma 2.4, functions u_1, v_1 are well defined in the space $\mathcal{X}$.

First, we show that $u_0(t) \leq u_1(t) \leq v_1(t) \leq v_0(t),\ t \in (0,1]$ and $\widetilde{u}_0(0) \leq \widetilde{u}_1(0) \leq \widetilde{v}_1(0) \leq \widetilde{v}_0(0)$.

Let $\delta(t) := \phi_p(D_{0^+}^{\alpha} u_1(t)) - \phi_p(D_{0^+}^{\alpha} u_0(t))$. By the definition of u_1 and the assumption that u_0 is a lower solution, we obtain that

$$D_{0^+}^{\beta} \delta(t) + M(t)\delta(t) = F(u_0(t)) - D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha} u_0(t)) \geq 0,$$

and $t^{1-\beta}\delta(t)|_{t=0} = \phi_p(t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha} u_1(t))|_{t=0} - \phi_p(t^{\frac{1-\beta}{p-1}} D_{0^+}^{\alpha} u_0(t))|_{t=0} = 0$.

Then, we get $\phi_p(D_{0^+}^{\alpha} u_0(t)) \leq \phi_p(D_{0^+}^{\alpha} u_1(t))$ for $t \in (0,1]$ by (i) of Lemma 2.6. Since $\phi_p(x)$ is monotonous, $D_{0^+}^{\alpha} u_0(t) \leq D_{0^+}^{\alpha} u_1(t)$, that is, $D_{0^+}^{\alpha}(u_1(t) - u_0(t)) \geq 0$. In view of $\widetilde{u}_1(0) = \widetilde{u}_0(0) + \frac{1}{\lambda} g(\widetilde{u}_0(0), \widetilde{u}_0(1)) \geq 0$, we have $u_1(t) \geq u_0(t),\ t \in (0,1]$ by (ii) of Lemma 2.6 and $\widetilde{u}_0(0) \leq \widetilde{u}_1(0)$.

By a similar way, we can show that $v_1(t) \leq v_0(t),\ t \in (0,1]$ and $\widetilde{v}_1(0) \leq \widetilde{v}_0(0)$.

Now, we put $\xi(t) = \phi_p(D_{0^+}^{\alpha} v_1(t)) - \phi_p(D_{0^+}^{\alpha} u_1(t))$. By the definitions of u_1, v_1 and (H2), we have

$$D_{0^+}^{\beta} \xi(t) + M(t)\xi(t) = F(v_0(t)) - F(u_0(t)) + M(t)[\phi_p(D_{0^+}^{\alpha} v_0) - \phi_p(D_{0^+}^{\alpha} u_0)] \geq 0,$$

and $t^{1-\beta}\xi(t)|_{t=0} = 0$.

Hence, $\xi(t) \geq 0$ by Lemma 2.6, that is, $\phi_p(D_{0^+}^{\alpha} v_1(t)) \geq \phi_p(D_{0^+}^{\alpha} u_1(t))$, and then $D_{0^+}^{\alpha} v_1(t) \geq D_{0^+}^{\alpha} u_1(t)$. In addition, we have, by (H3) and (H1),

$$\begin{aligned}
\widetilde{v}_1(0) - \widetilde{u}_1(0) &= \widetilde{v}_0(0) + \frac{1}{\lambda} g(\widetilde{v}_0(0), \widetilde{v}_0(1)) - [\widetilde{u}_0(0) + \frac{1}{\lambda} g(\widetilde{u}_0(0), \widetilde{u}_0(1))] \\
&= \frac{1}{\lambda}[\lambda(\widetilde{v}_0(0) - \widetilde{u}_0(0)) - (g(\widetilde{u}_0(0), \widetilde{u}_0(1)) - g(\widetilde{v}_0(0), \widetilde{v}_0(1)))] \\
&\geq \frac{1}{\lambda}[\lambda(\widetilde{v}_0(0) - \widetilde{u}_0(0)) - \lambda(\widetilde{v}_0(0) - \widetilde{u}_0(0)) + \mu(\widetilde{v}_0(1) - \widetilde{u}_0(1))] \\
&= \frac{\mu}{\lambda}(\widetilde{v}_0(1) - \widetilde{u}_0(1)) \geq 0.
\end{aligned} \tag{14}$$

The inequality (14) and Lemma 2.6 imply that $v_1(t) \geq u_1(t),\ t \in (0,1]$ and $\widetilde{u}_1(0) \leq \widetilde{v}_1(0)$.

In the following, we show that u_1, v_1 are lower and upper solutions of the problem (1), respectively.

$$\begin{aligned} D^{\beta}_{0^+}(\phi_p(D^{\alpha}_{0^+}u_1(t))) &= F(u_0(t)) - F(u_1(t)) + F(u_1(t)) - M(t)[\phi_p(D^{\alpha}_{0^+}u_1(t)) - \phi_p(D^{\alpha}_{0^+}u_0(t))] \\ &\leq M[\phi_p(D^{\alpha}_{0^+}u_1(t)) - \phi_p(D^{\alpha}_{0^+}u_0(t))] - M[\phi_p(D^{\alpha}_{0^+}u_1(t)) - \phi_p(D^{\alpha}_{0^+}u_0(t))] + F(u_1(t)) \\ &= F(u_1(t)), \end{aligned}$$

and

$$\begin{aligned} 0 &= g(\widetilde{u}_0(0), \widetilde{u}_0(1)) - g(\widetilde{u}_1(0), \widetilde{u}_1(1)) + g(\widetilde{u}_1(0), \widetilde{u}_1(1)) - \lambda[\widetilde{u}_1(0) - \widetilde{u}_0(0)] \\ &\leq g(\widetilde{u}_1(0), \widetilde{u}_1(1)) - \mu(\widetilde{u}_1(1) - \widetilde{u}_0(1)) \end{aligned}$$

by assumptions (H2) and (H3). Since $\widetilde{u}_1(1) \geq \widetilde{u}_0(1)$, $g(\widetilde{u}_1(0), \widetilde{u}_1(1)) \geq 0$. Thus we prove that u_1 is a lower solution of the problem (1). Similarly, we can prove that v_1 is an upper solution of the problem (1).

Using the mathematical induction, we can obtain that

$$u_0(t) \leq u_1(t) \leq ... \leq u_n(t) \leq u_{n+1}(t) \leq v_{n+1}(t) \leq v_n(t) \leq ... \leq v_1(t) \leq v_0(t), \tag{15}$$

$D^{\alpha}_{0^+}u_0 \leq D^{\alpha}_{0^+}u_1 \leq ... \leq D^{\alpha}_{0^+}u_n \leq D^{\alpha}_{0^+}u_{n+1} \leq D^{\alpha}_{0^+}v_{n+1} \leq D^{\alpha}_{0^+}v_n \leq ... \leq D^{\alpha}_{0^+}v_1 \leq D^{\alpha}_{0^+}v_0$

and

$$\widetilde{u}_0(0) \leq \widetilde{u}_1(0) \leq ... \leq \widetilde{u}_n(0) \leq \widetilde{u}_{n+1}(0) \leq \widetilde{v}_{n+1}(0) \leq \widetilde{v}_n(0) \leq ... \leq \widetilde{v}_1(0) \leq \widetilde{v}_0(0), \tag{16}$$

for $t \in (0,1]$ and $n = 1, 2, 3, ...$

Similar to [18], we know that the sequences $\{t^{1-\alpha}u_n\}$ and $\{t^{1-\alpha}v_n\}$ are uniformly bounded and equi-continuous. So the Arzela-Ascoli theorem educes that they are relatively compact sets of the space $\mathcal{X}$. Therefore, $\{t^{1-\alpha}u_n\}$ and $\{t^{1-\alpha}v_n\}$ converge to $t^{1-\alpha}x(t)$ and $t^{1-\alpha}y(t)$ uniformly on $[0,1]$, respectively. That is

$$\lim_{n\to\infty} u_n(t) = x(t), \ \ \lim_{n\to\infty} v_n(t) = y(t), \ \ t \in (0,1].$$

$$\lim_{n\to\infty} D^{\alpha}_{0^+}u_n(t) = D^{\alpha}_{0^+}x(t), \ \ \lim_{n\to\infty} D^{\alpha}_{0^+}v_n(t) = D^{\alpha}_{0^+}y(t), \ \ t \in (0,1].$$

Moreover, $x(t)$ and $y(t)$ are the solutions of the problem (1) and $u_0(t) \leq x(t) \leq y(t) \leq v_0(t)$ on $(0,1]$.

To prove that $x(t), y(t)$ are extremal solutions of (1), let $u \in [u_0, v_0]$ be any solution of the problem (1). We suppose that $u_n(t) \leq u(t) \leq v_n(t),\ t \in (0,1]$ for some n. Let

$$q(t) = \phi_p(D^{\alpha}_{0^+}u(t)) - \phi_p(D^{\alpha}_{0^+}u_{n+1}(t)), \ p(t) = \phi_p(D^{\alpha}_{0^+}v_{n+1}(t)) - \phi_p(D^{\alpha}_{0^+}u(t)).$$

Then, by assumptions (H2), we can prove that

$D^{\beta}_{0^+}q(t) + M(t)q(t)(t) = F(u(t)) - F(u_n(t)) + M(t)[\phi_p(D^{\alpha}_{0^+}u(t)) - \phi_p(D^{\alpha}_{0^+}u_n(t))] \geq 0,$

$t^{1-\beta}q(t)|_{t=0} = 0,$

and

$D^{\beta}_{0^+}p(t) + M(t)p(t) = F(v_n(t)) - F(u(t)) + M(t)[\phi_p(D^{\alpha}_{0^+}v_n(t)) - \phi_p(D^{\alpha}_{0^+}v(t))] \geq 0,$

$t^{1-\beta}p(t)|_{t=0} = 0.$

Hence, $q(t) \geq 0,\ p(t) \geq 0$ by Lemma 2.6, that is, $\phi_p(D^{\alpha}_{0^+}u(t)) \geq \phi_p(D^{\alpha}_{0^+}u_{n+1}(t)),\ \phi_p(D^{\alpha}_{0^+}(u)) \leq \phi_p(D^{\alpha}_{0^+}v_{n+1}(t))$. Thus, $D^{\alpha}_{0^+}(u(t) - u_{n+1}(t)) \geq 0,\ D^{\alpha}_{0^+}(v_{n+1}(t) - u(t)) \geq 0$. Besides, by (H3), we have

$$\begin{aligned} \widetilde{u}(0) - \widetilde{u}_{n+1}(0) &= \widetilde{u}(0) + \frac{1}{\lambda}g(\widetilde{u}(0), \widetilde{u}(1)) - [\widetilde{u}_n(0) + \frac{1}{\lambda}g(\widetilde{u}_n(0), \widetilde{u}_n(1))] \\ &= \frac{1}{\lambda}[\lambda\widetilde{u}(0) + g(\widetilde{u}(0), \widetilde{u}(1)) - (\lambda\widetilde{u}_n(0) + g(\widetilde{u}_n(0), \widetilde{u}_n(1))] \\ &\geq \frac{\mu}{\lambda}(\widetilde{u}(1) - \widetilde{u}_n(1)) \geq 0. \end{aligned}$$

$$\widetilde{v}_{n+1}(0)-\widetilde{u}(0)=\widetilde{v}_n(0)+\frac{1}{\lambda}g(\widetilde{v}_n(0),\widetilde{v}_n(1))-[\widetilde{u}(0)+\frac{1}{\lambda}g(\widetilde{u}(0),\widetilde{u}(1))]$$
$$=\frac{1}{\lambda}[\lambda\widetilde{v}_n+g(\widetilde{u}(0),\widetilde{u}(1))-(\lambda\widetilde{u}+g(\widetilde{u}_n(0),\widetilde{u}_{n+1}(1))]$$
$$\geq\frac{\mu}{\lambda}(\widetilde{v}_{n+1}(1)-\widetilde{u}(1))\geq 0.$$

These and Lemma 2.6 derive that $u_{n+1}(t)\leq u(t)\leq v_{n+1}(t),\ t\in(0,1]$. So, by induction, $x(t)\leq u(t)\leq y(t)$ on $(0,1]$ by taking $n\to\infty$. The proof is completed. □

Theorem 3.3. *The assumptions of Theorem 3.2 hold and there exists a function* $N(t)\geq 0, t\in[0,1]$ *such that*

$$N(t)[\phi_p(D^\alpha_{0^+}v(t))-\phi_p(D^\alpha_{0^+}u(t))]\leq f(t,u(t),D^\alpha_{0^+}u(t))-f(t,v(t),D^\alpha_{0^+}v(t)) \tag{17}$$

for $u_0(t)\leq u(t)\leq v(t)\leq v_0(t),\ t\in(0,1]$ *and* $\widetilde{u}_0(0)=\widetilde{v}_0(0)$. *Then the problem* (1) *has one unique solution in the order interval* $[u_0,\ v_0]$.

Proof. From the Theorem 3.2, we know $x(t)$ and $y(t)$ are extremal solutions and $x(t)\leq y(t), t\in(0,1]$. It is sufficient to prove $x(t)\geq y(t), t\in(0,1]$. In fact, let $w(t)=\phi_p(D^\alpha_{0^+}x(t))-\phi_p(D^\alpha_{0^+}y(t)), t\in(0,1]$, we have, by (17),

$$\begin{cases}D^\beta_{0^+}w(t)=F(x(t))-F(y(t))\geq N(t)[\phi_p(D^\alpha_{0^+}y(t))-\phi_p(D^\alpha_{0^+}x(t))]=-N(t)w(t),\\ t^{1-\beta}w(t)|_{t=0}=\phi_p(t^{\frac{1-\beta}{p-1}}D^\alpha_{0^+}x(t))|_{t=0}-\phi_p(t^{\frac{1-\beta}{p-1}}D^\alpha_{0^+}y(t))|_{t=0}=0.\end{cases}$$

Then, $w(t)\geq 0, t\in(0,1]$ by (i) of Lemma 2.6 Thus, $D^\alpha_{0^+}x(t)\geq D^\alpha_{0^+}y(t), t\in(0,1]$ since $\phi_p(x)$ is monotonic function. In addition, by (16) and $\widetilde{u}_0(0)=\widetilde{v}_0(0)$, we have $\widetilde{x}(0)=\widetilde{y}(0)$. So $x(t)\geq y(t), t\in(0,1]$ by (ii) of Lemma 2.6. Therefore, we get $x=y$ is a unique solution of the problem (1). The proof is completed. □

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