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Multivalued F-Contractions and Related Fixed Point Theorems with an Application

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Abstract. In this paper, we introduce the notions of α-F-contractions, by combining the notions of α-ψ-contraction and F-contraction. Using our new notions we obtain some fixed point theorems for multivalued mappings. As an application we establish an existence theorem for integral equations. An example is also constructed to show an importance of our results.

1. Introduction

Recently, Wardowski [1] introduced a new family of mappings so called F or $\mathfrak{F}$ family. Using the mappings from $\mathfrak{F}$ family he introduced a new contraction condition called F-contraction. This F-contraction nicely generalize the most famous contraction condition, that is, Banach contraction condition. Several researcher working in the metric fixed point theory tried or trying to introduce a contraction condition which generalize Banach contraction condition, see for example [2–40]. Semat *et al.* [2] succeeded to generalized Banach contraction condition by introducing α-ψ-contraction. Many authors appreciate these two condition conditions which can be seen in [4–19]. In this paper, we combine these two ideas to introduce some new contraction conditions for multivalued mappings and corresponding fixed point theorem. We also show that many new results in different setting can be obtained from our results. As an application of our result we establish an existence theorem for integral equations. For completeness we recollect some basic results and definitions.

Let (X, d) be metric space. We denote by $CB(X)$ the class of all nonempty bounded and closed subsets of X. The Hausdorff-Pompeu metric that is,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, \text{ where } A, B \in CB(X).$$

For subsets A and B of a partially ordered metric space X, we say that $A \prec_r B$, if for each $a \in A$ and $b \in B$, we have $a \leq b$. Wardowski [1] introduced following definition.

Definition 1.1. *Let $\mathfrak{F}$ be the class of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following three assumptions:*

(F_1) *F is strictly increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$.*

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(F_2) *For each sequence* $\{\mathfrak{d}_n\}$ *of positive real numbers we have* $\lim_{n\to\infty} \mathfrak{d}_n = 0$ *if and only if* $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

(F_3) *There exists* $k \in (0, 1)$ *such that* $\lim_{\mathfrak{d}\to 0^+} \mathfrak{d}^k F(\mathfrak{d}) = 0$.

Following are some examples of such functions.

- $F_a = \ln x$ for each $x \in (0, \infty)$.
- $F_b = x + \ln x$ for each $x \in (0, \infty)$.
- $F_c = -\frac{1}{\sqrt{x}}$ for each $x \in (0, \infty)$.

Secelean [3] showed that condition (F_2) can be replaced by one of following condition which are equivalent to (F_2) but easy to handle.

(F_{2a}) $\inf F = -\infty$
or

(F_{2b}) there exists a sequence $\{\mathfrak{d}_n\}$ of positive numbers such that $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

Secelean concluded it on the bases of following lemma.

Lemma 1.2. *[3] Let* $F : (0, \infty) \to \mathbb{R}$ *be an increasing mapping and* $\{\mathfrak{d}_n\}$ *be a sequence of positive real numbers. Then the following condition holds.*

(i) if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$, *then* $\lim_{n\to\infty} \mathfrak{d}_n = 0$.

(ii) if $\inf F = -\infty$ *and* $\lim_{n\to\infty} \mathfrak{d}_n = 0$, *then* $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

Wardowski [1] introduced F-contraction and corresponding fixed point theorem as.

Definition 1.3. *[1] Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is F-contraction if there exist* $F \in \mathfrak{F}$ *and* $\tau > 0$ *such that for each* $x, y \in X$ *with* $d(Tx, Ty) > 0$, *we have*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Remark 1.4. *[1] Note that if T is* F_a*-contraction, then it is also Banach contraction. But it is not a case with* F_b*-contraction.*

Theorem 1.5. *[1] Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be an F-contraction. Then T has a unique fixed point.*

Minak *et al*. [5] introduced following result.

Theorem 1.6. *[5] Let* (X, d) *be a complete metric space and let* $T : X \to X$. *Assume that there exists* $F \in \mathfrak{F}$ *and* $\tau > 0$ *such that*

$$\tau + F(d(Tx, Ty)) \leq F\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\right),$$

for each $x, y \in X$ *with* $d(Tx, Ty) > 0$. *If T or F is continuous, then T has a unique fixed point.*

Sgroi and Vetro [6] introduced following theorem.

Theorem 1.7. *[6] Let* (X, d) *be a complete metric space and let* $T : X \to CB(X)$. *Assume that there exists* $F \in \mathfrak{F}$ *and* $\tau > 0$ *such that*

$$2\tau + F(H(Tx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx)), \tag{1}$$

for each $x, y \in X$ *with* $Tx \neq Ty$, *where* $a_1, a_2, a_3, a_4, L \geq 0$ *satisfying* $a_1 + a_2 + a_3 + 2a_4 = 1$ *and* $a_3 \neq 1$. *Then T has a fixed point.*

2. Main Results

We start this section by slightly modifying the definitions given in [11] and [12].

Definition 2.1. *Let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CB(X)$ is strictly α-admissible if for each $x \in X$ and $y \in Tx$ such that $\alpha(x, y) > 1$, we have $\alpha(y, z) > 1$ for each $z \in Ty$.*

Definition 2.2. *Let $\alpha : X\times X \to [0, \infty)$ be a function. A mapping $T : X \to CB(X)$ is strictly α_*-admissible mapping if for each $x, y \in X$ with $\alpha(x, y) > 1$, we have $\alpha_*(Tx, Ty) > 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(u, v) : u \in Tx \text{ and } v \in Ty\}$.*

Remark 2.3. *Note that if a mapping $T : X \to CB(X)$ is strictly α_*-admissible, then it is strictly α-admissible. Converse is not true in general.*

Example 2.4. *Let $X = [-1, 1]$. Define $T : X \to CB(X)$ by*

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = -1 \\ \{1\} & \text{if } x = 0 \\ \{-x\} & \text{if } x \notin \{-1, 0\} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y. \end{cases}$$

Following the details of [13, Example 1], it is straight forward to see that T is strictly α-admissible but not α_*-admissible.

Definition 2.5. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CB(X)$ is α-F-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(N(x, y)), \tag{2}$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + Ld(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be an α-F-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) T is strictly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (ii), there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. As $\alpha(x_0, x_1) > 1$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1). \tag{3}$$

Since F is strictly increasing, we have

$$F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1)H(Tx_0, Tx_1)). \tag{4}$$

From (2), we have

$$\begin{aligned}\tau + F(d(x_1, x_2)) &\leq \tau + F(\alpha(x_0, x_1)H(Tx_0, Tx_1)) \\ &\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + Ld(x_1, Tx_0)\Big) \\ &\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + L.0\Big) \\ &\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 (d(x_0, x_1) + d(x_1, x_2))\Big) \\ &= F\Big((a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2)\Big). \end{aligned} \quad (5)$$

Since F is strictly increasing, we get from above that

$$d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1) + (a_3 + a_4)d(x_1, x_2).$$

That is,

$$(1 - a_3 - a_4)d(x_1, x_2) < (a_1 + a_2 + a_4)d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_1, x_2) < d(x_0, x_1).$$

Now, from (5), we have

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)).$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Let $x_2 \notin Tx_2$. Since, T is strictly α-admissible, we have $\alpha(x_1, x_2) > 1$. There exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \alpha(x_1, x_2)H(Tx_1, Tx_2). \quad (6)$$

Since, F is strictly increasing, we have

$$F(d(x_2, x_3)) \leq F(\alpha(x_1, x_2)H(Tx_1, Tx_2)). \quad (7)$$

From (2), we have

$$\begin{aligned}\tau + F(d(x_2, x_3)) &\leq \tau + F(\alpha(x_1, x_2)H(Tx_1, Tx_2)) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, Tx_1) + a_3 d(x_2, Tx_2) + a_4 d(x_1, Tx_2) + Ld(x_2, Tx_1)\Big) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + a_4 d(x_1, x_3) + L.0\Big) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + a_4 (d(x_1, x_2) + d(x_2, x_3))\Big) \\ &= F\Big((a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3)\Big). \end{aligned} \quad (8)$$

Since F is strictly increasing, we get from above that

$$d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4)d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_2, x_3) < d(x_1, x_2).$$

Now from (8), we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

So we have

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau \leq F(d(x_0, x_1)) - 2\tau.$$

Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1},\ x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) > 1 \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{9}$$

Letting $n \to \infty$ in (9), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (9) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{10}$$

Letting $n \to \infty$ in (10), we get

$$\lim_{n\to\infty} n d_n^k = 0. \tag{11}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $nd_n^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{12}$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (12), we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) = \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iii), we have $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, Tx^*) = 0$. On contrary suppose that $d(x^*, Tx^*) > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, Tx^*) > 0$ for each $n \geq n_0$. For each $n \geq n_0$, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &< d(x^*, x_{n+1}) + \alpha(x_n, x^*) H(Tx_n, Tx^*) \\ &< d(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) + a_3 d(x^*, Tx^*) + a_4 d(x_n, Tx^*) + L d(x^*, x_{n+1}). \end{aligned} \tag{13}$$

Letting $n \to \infty$ in (13), we have

$$d(x^*, Tx^*) \leq (a_3 + a_4) d(x^*, Tx^*) < d(x^*, Tx^*).$$

Which is a contradiction. Thus $d(x^*, Tx^*) = 0$. □

Example 2.7. *Let $X = \mathbb{N} \cup \{0\}$ be endowed with the usual metric $d(x, y) = |x - y|$ for each $x, y \in X$. Define $T : X \to CB(X)$ by*

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = 0, 1 \\ \{x - 1, x\} & \text{if } x > 1 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in \{0, 1\} \\ \frac{1}{2} & \text{if } x, y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(x) = x + \ln x$ for each $x \in (0, \infty)$. Under this F, condition (2) *reduces to*

$$\frac{\alpha(x, y)H(Tx, Ty)}{N(x, y)} e^{\alpha(x,y)H(Tx,Ty)-N(x,y)} \leq e^{-\tau} \tag{14}$$

for each $x, y \in X$ with $\min\{\alpha(x, y)H(Tx, Ty), N(x, y)\} > 0$. Assume that $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $\tau = \frac{1}{2}$. Clearly, $\min\{\alpha(x, y)H(Tx, Ty), d(x, y)\} > 0$ for each $x, y > 1$ with $x \neq y$. From (14) *for each $x, y > 1$ with $x \neq y$, we have*

$$\frac{1}{2} e^{-\frac{1}{2}|x-y|} < e^{-\frac{1}{2}}.$$

Thus, T is α-F-contraction of Hardy-Rogers-type with $F(x) = x + \ln x$. For $x_0 = 1$, we have $x_1 = 0 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$. Moreover, it is easy to see that T is strictly α-admissible mapping and for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 2.6, T has a fixed point in X.

Remark 2.8. *Note that [6, Theorem 3.4] is not applicable on above example with $F(x) = x + \ln x$. Since for $x = 3$ and $y = 2$, from* (1), *we have $\frac{1}{a_1+a_4} e^{1-a_1-a_4} \leq e^{-2\tau}$, which is impossible.*

Definition 2.9. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CB(X)$ is α_*-F-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha_*(Tx, Ty)H(Tx, Ty)) \leq F(N(x, y)), \tag{15}$$

for each $x, y \in X$, whenever $\min\{\alpha_*(Tx, Ty)H(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 2.10. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be an α_*-F-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) T is strictly α_*-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. The proof of this theorem runs along the same lines as the proof of Theorem 2.6 is done. □

Remark 2.11. *We may replace the condition (iii) of Theorem 2.6 and Theorem 2.10 by continuity of T.*

Definition 2.12. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CB(X)$ is α-F-contraction, if there exist continuous F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \le F(M(x, y)), \tag{16}$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

with $L \ge 0$.

Theorem 2.13. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be an α-F-contraction satisfying the following conditions:*

(i) T is strictly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (ii), there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. From (16), we have

$$\begin{aligned}\tau + F(\alpha(x_0, x_1)H(Tx_0, Tx_1)) &\le F\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_1, Tx_0) + d(x_0, Tx_1)}{2}\right\} + Ld(x_1, Tx_0)\right)\\ &= F\Big(\max\{d(x_0, x_1), d(x_1, Tx_1)\}\Big). \end{aligned}\tag{17}$$

As $\alpha(x_0, x_1) > 1$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \le \alpha(x_0, x_1)H(Tx_0, Tx_1). \tag{18}$$

Since, F is strictly increasing, we have

$$F(d(x_1, x_2)) \le F(\alpha(x_0, x_1)H(Tx_0, Tx_1)). \tag{19}$$

From (17) and (19), we have

$$\tau + F(d(x_1, x_2)) \le F\Big(\max\{d(x_0, x_1), d(x_1, Tx_1)\}\Big). \tag{20}$$

If we assume that $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then we have a contradiction to (20). Thus, $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$. From (20), we have

$$\tau + F(d(x_1, x_2)) \le F(d(x_0, x_1)). \tag{21}$$

Since T is strictly α-admissible, therefore $\alpha(x_0, x_1) > 1$ implies $\alpha(x_1, x_2) > 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Let $x_2 \notin Tx_2$. From (16), we have

$$\begin{aligned}\tau + F(\alpha(x_1, x_2)H(Tx_1, Tx_2)) &\le F\Big(\max\left\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_2, Tx_1) + d(x_1, Tx_2)}{2}\right\} + Ld(x_2, Tx_1)\Big)\\ &= F\Big(\max\{d(x_1, x_2), d(x_2, Tx_2)\}\Big). \end{aligned}\tag{22}$$

As $\alpha(x_1, x_2) > 1$, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \le \alpha(x_1, x_2)H(Tx_1, Tx_2). \tag{23}$$

Since F is strictly increasing, we have

$$F(d(x_2, x_3)) \le F(\alpha(x_1, x_2)H(Tx_1, Tx_2)). \tag{24}$$

From (22) and (24), we have

$$\tau + F(d(x_2, x_3)) \le F\Big(\max\{d(x_1, x_2), d(x_2, Tx_2)\}\Big). \tag{25}$$

If we assume that $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$, then we have a contradiction to (25). Thus, $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2)$. From (25), we have

$$\tau + F(d(x_2, x_3)) \le F(d(x_1, x_2)). \tag{26}$$

From (21) and (26), we have

$$F(d(x_2, x_3)) \le F(d(x_0, x_1)) - 2\tau. \tag{27}$$

Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1},\ x_{n-1} \ne x_n \text{ and } \alpha(x_{n-1}, x_n) > 1 \text{ for each } n \in \mathbb{N}.$$

Moreover,

$$F(d(x_n, x_{n+1})) \le F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{28}$$

Letting $n \to \infty$ in (28), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (28) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \le -d_n^k n\tau \le 0 \text{ for each } n \in \mathbb{N}. \tag{29}$$

Letting $n \to \infty$ in (29), we get

$$\lim_{n\to\infty} n d_n^k = 0. \tag{30}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \le 1$ for each $n \ge n_1$. Thus, we have

$$d_n \le \frac{1}{n^{1/k}}, \quad \text{for each } n \ge n_1. \tag{31}$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (31), we have

$$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) = \sum_{i=n}^{m-1} d_i \le \sum_{i=n}^{\infty} d_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iii), we have $\alpha(x_n, x^*) > 1$

for each $n \in \mathbb{N}$. We claim that $d(x^*, Tx^*) = 0$. On contrary suppose that $d(x^*, Tx^*) > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, Tx^*) > 0$ for each $n \geq n_0$. From (16), for each $n \geq n_0$, we have

$$\begin{aligned}\tau + F(d(x_{n+1}, Tx^*)) &\leq \tau + F(\alpha(x_n, x^*)H(Tx_n, Tx^*)) \\ &\leq F\Big(\max\Big\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x^*, Tx_n) + d(x_n, Tx^*)}{2}\Big\} + Ld(x^*, Tx_n)\Big)\end{aligned}$$

Letting $n \to \infty$ in above inequality and by continuity of F, we get

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)).$$

This implies $\tau \leq 0$. Which is a contradiction. Thus $d(x^*, Tx^*) = 0$. □

Example 2.14. *Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for each $x, y \in X$. Define $T : X \to CB(X)$ by*

$$Tx = \begin{cases} [0, \frac{x}{4}] & \text{if } x \in [0, 2) \\ \{2\} & \text{if } x = 2 \\ \{x + 1, (x + 1)^2\} & \text{otherwise,} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 2] \\ 0 & \text{otherwise.} \end{cases}$$

Take $\tau = \ln 2$, $L = 6$ and $F(x) = \ln x$ for each $x > 0$. Then it is easy to check that T is α-F-contraction and all other condition of Theorem 2.13 hold. Therefore, T has a fixed point.

Definition 2.15. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CB(X)$ is α_*-F-contraction, if there exist continuous F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha_*(Tx, Ty)H(Tx, Ty)) \leq F(M(x, y)), \tag{32}$$

for each $x, y \in X$, whenever $\min\{\alpha_*(Tx, Ty)H(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\Big\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 2.16. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be an α_*-F-contraction satisfying the following conditions:*

(i) T is strictly α_*-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. The proof of this theorem is similar to proof of Theorem 2.13. □

Remark 2.17. *If we assume that T is continuous then we can leave condition (iii) and continuity of F from Theorem 2.13 and Theorem 2.16.*

3. Consequences

In this section, we obtain some fixed point theorems as consequences of our results. It is worth mentioning that these results are also new, as for as our knowledge.

3.1. Metric space endowed with partial ordering

Here we prove some results for fixed points of multivalued mappings from a partially ordered metric spaces into the space of nonempty closed and bounded subsets of the metric space. We begin this subsection by introducing the following definition.

Definition 3.1. *Let $(X, d, \preceq)$ be an ordered metric space. A mapping $T : X \to CB(X)$ is F_q-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$, $\tau > 0$ and $q > 1$ such that*

$$\tau + F(qH(Tx, Ty)) \leq F(N(x, y)), \tag{33}$$

for each $x, y \in X$ with $x \preceq y$, whenever $\min\{qH(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + Ld(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.2. *Let $(X, d, \preceq)$ be a complete ordered metric space and let $T : X \to CB(X)$ be an F_q-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) for each $x \in X$ and $y \in Tx$ such that $x \preceq y$, this implies $y \preceq z$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $x_0 \preceq x_1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, we have $x_n \preceq x$ for each $n \in \mathbb{N}$.

Then the mapping T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 2.6 hold. Thus, T has a fixed point. □

Definition 3.3. *Let $(X, d, \preceq)$ be an ordered metric space. A mapping $T : X \to CB(X)$ is F_q-contraction, if there exist continuous F in $\mathfrak{F}$, $\tau > 0$ and $q > 1$ such that*

$$\tau + F(qH(Tx, Ty)) \leq F(M(x, y)), \tag{34}$$

for each $x, y \in X$ with $x \preceq y$, whenever $\min\{qH(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 3.4. *Let $(X, d, \preceq)$ be a complete ordered metric space and let $T : X \to CB(X)$ be an F_q-contraction satisfying the following conditions:*

(i) for each $x \in X$ and $y \in Tx$ such that $x \preceq y$, this implies $y \preceq z$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $x_0 \preceq x_1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, we have $x_n \preceq x$ for each $n \in \mathbb{N}$.

Then the mapping T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 2.13 hold. Thus, T has a fixed point. □

Remark 3.5. *If we replace assumption (i) of above results by*
(i′) If $x \preceq y$, then we have $Tx \prec_r Ty$. Then Theorem 3.2 and Theorem 3.4 follow from Theorem 2.10 and Theorem 2.16, respectively.

3.2. Metric space endowed with graph

In this subsection, we drive some fixed point theorems for multivalued mappings from a metric spaces X, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Throughout this subsection, we assume that G is a directed graph such that the set of its vertices $V(G)$ coincides with X (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $E(G) \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Let us also assume that G has no parallel edges. We can identify G with the pair $(V(G), E(G))$.

Definition 3.6. *Let (X, d) be a metric space endowed with a graph G. A mapping $T : X \to CB(X)$ is graphic F_q-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$, $\tau > 0$ and $q > 1$ such that*

$$\tau + F(qH(Tx, Ty)) \leq F(N(x, y)), \tag{35}$$

for each $x, y \in X$ with $(x, y) \in E(G)$, whenever $\min\{qH(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + Ld(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.7. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to CB(X)$ be a graphic F_q-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) for each $x \in X$ and $y \in Tx$ such that $(x, y) \in E(G)$, this implies $(y, z) \in E(G)$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E(G)$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}$.

Then the mapping T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } (x, y) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 2.6 hold. Thus, the mapping T has a fixed point. □

Definition 3.8. *Let (X, d) be a metric space endowed with a graph G. A mapping $T : X \to CB(X)$ is graphic F_q-contraction, if there exist continuous F in $\mathfrak{F}$, $\tau > 0$ and $q > 1$ such that*

$$\tau + F(qH(Tx, Ty)) \leq F(M(x, y)), \tag{36}$$

for each $x, y \in X$ with $(x, y) \in E(G)$, whenever $\min\{qH(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 3.9. *Let (X, d) be a complete metric space endowed with a graph G and let $T : X \to CB(X)$ be a graphic F_q-contraction satisfying the following conditions:*

(i) for each $x \in X$ and $y \in Tx$ such that $(x, y) \in E(G)$, this implies $(y, z) \in E(G)$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E(G)$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}$.

Then the mapping T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } (x, y) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 2.13 hold. Thus, mapping T has a fixed point. □

Remark 3.10. *If we replace assumption (i) of above result by*
(i′) If $(x, y) \in E(G)$, then we have $(a, b) \in E(G)$ for each $a \in Tx$ and $b \in Ty$. Then Theorem 3.7 and Theorem 3.9 follow from Theorem 2.10 and Theorem 2.16, respectively.

4. Application

In this section, as a consequence of our result, we establish an existence theorem for an integral equation. Let $X = (C[a, b], \mathbb{R})$ be the space of all realvalued continuous functions defined on $[a, b]$. Note that X is complete [30] with respect to the metric $d_\tau(x, y) = \sup_{t\in[a,b]}\{|x(t) - y(t)|e^{-|\tau t|}\}$. Consider an integral equation of the form

$$x(t) = f(t) + \int_{g(t)}^{h(t)} K(t, s, x(s))ds, \tag{37}$$

for $t, s \in [a, b]$. Where $K : [a, b]\times[a, b]\times\mathbb{R} \to \mathbb{R}$ and $f, g, h : [a, b] \to \mathbb{R}$ are continuous functions and $g(t) \leq h(t)$ for each $t \in [a, b]$.

Theorem 4.1. *Let $X = (C[a, b], \mathbb{R})$ and let $T : X \to X$ be the operator defined as*

$$Tx(t) = f(t) + \int_{g(t)}^{h(t)} K(t, s, x(s))ds, \tag{38}$$

for $t, s \in [a, b]$. Where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f, g, h : [a, b] \to \mathbb{R}$ are continuous functions and $g(t) \leq h(t)$ for each $t \in [a, b]$. Assume that there exist $\beta : X \to (0, \infty)$ and $\alpha : X \times X \to (0, \infty)$ such that the following conditions hold:

(i) there exist $\tau > 0$ such that

$$|K(t,s,x) - K(t,s,y)| \leq \frac{e^{-\tau}}{\beta(x+y)}|x-y|$$

for each $t, s \in [a,b]$ and $x, y \in X$, moreover,

$$\Big|\int_{g(t)}^{h(t)} \frac{e^{|\tau s|}}{\beta(x(s)+y(s))}ds\Big| \leq \frac{e^{|\tau t|}}{\alpha(x,y)}$$

for each $t \in [a,b]$;

(ii) for $x, y \in X$, $\alpha(x,y) > 1$ implies $\alpha(Tx,Ty) > 1$;

(iii) there exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) > 1$;

(iv) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then the integral equation (37) *has a solution in X.*

Proof. First we show that T is an α-F-contraction Hardy-Rogers-type.

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_{g(t)}^{h(t)} |K(t,s,x(s)) - K(t,s,y(s))|ds \\
&\leq \int_{g(t)}^{h(t)} \frac{e^{-\tau}}{\beta(x(s)+y(s))}|x(s)-y(s)|ds \\
&= \int_{g(t)}^{h(t)} \frac{e^{-\tau}e^{|\tau s|}}{\beta(x(s)+y(s))}|x(s)-y(s)|e^{-|\tau s|}ds \\
&\leq e^{-\tau}d_\tau(x,y)\int_{g(t)}^{h(t)} \frac{e^{|\tau s|}}{\beta(x(s)+y(s))}ds \\
&\leq \frac{e^{|\tau t|}}{\alpha(x,y)}e^{-\tau}d_\tau(x,y).
\end{aligned}$$

Thus, we have

$$\alpha(x,y)|Tx(t) - Ty(t)|e^{-|\tau t|} \leq e^{-\tau}d_\tau(x,y).$$

Equivalently

$$\alpha(x,y)d_\tau(Tx,Ty) \leq e^{-\tau}d_\tau(x,y).$$

Clearly natural logarithm belongs to $\mathfrak{F}$. Applying it on above inequality, we get

$$\ln(\alpha(x,y)d_\tau(Tx,Ty)) \leq \ln(e^{-\tau}d_\tau(x,y)),$$

after some simplification, we get

$$\tau + \ln(\alpha(x,y)d_\tau(Tx,Ty)) \leq \ln(d_\tau(x,y)).$$

Thus, T is an α-F-contraction of Hardy-Rogers-type with $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $F(x) = \ln x$. All other conditions of Theorem 2.6 are immediately hold. Therefore, the operator (38) has a fixed point, that is, the integral equation (37) has a solution in X. □

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