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On Some Characterizations of Sub-b-s-Convex Functions

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Abstract. A new class of generalized convex functions called sub-b-s-convex functions is defined by modulating the definitions of s-convex functions and sub-b-convex functions. Similarly, a new class sub-b-s-convex sets, which are generalizations of s-convex sets and sub-b-convex sets, is introduced. Furthermore, some basic properties of sub-b-s-convex functions in both general case and differentiable case are presented. In addition the sufficient conditions of optimality for both unconstrained and inequality constrained programming are established and proved under the sub-b-s-convexity.

1. Introduction

Owing to the importance of the convexity and generalized convexity in the study of optimality to solve mathematical programming, researchers worked a lot on the generalized convex functions. For example, in earlier papers, C.R. Bector and R. Singh(1991)[4] introduced a class of b-vex functions. H. Hudzik and L. Maligranda(1994)[10] discussed two kinds of s-convexity ($0 < s < 1$) and proved that s-convexity in the second sense is essentially stronger than the s-convexity in the first sense whenever ($0 < s < 1$). E.A. Youness(1999) [20] introduced a class of sets and a class of functions called E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions. X.M. Yang(2001)[19] gave some examples for E.A. Youness's paper[20] and perfected it. For more results on generalized E-convex functions, place refer to [1, 8, 9] and closely related references therein.

Recently, these classes of generalized convex functions caused a lot of research interests. Especially for the research of b-invex function. Such as, X.J. Long and J.W. Peng(2006)[13] discussed a class of functions called semi-b-preinvex functions, which is a generalization of the semi preinvex functions and the b-vex functions. Yu-Ru Syau *et al.*(2009)[17] introduced a class of functions, called E-b-vex functions, which is defined as a generalization of b-vex functions and E-vex functions. T. Emam(2011)[18] researched a new class of functions called roughly b-invex functions, discussed some their properties, and obtained sufficient optimality criteria for nonlinear programming involving these functions. M.T. Chao *et al.*(2012)[6] studied a new generalized sub-b-convex functions and a class of sub-b-convex sets, and presented the sufficient

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conditions of optimality for both unconstrained and inequality constrained sub-b-convex programming. For more information on generalized convex functions, see [5, 7, 15]. These scholars's researches promoted the development of the generalized convex functions like b-invex function. Meanwhile, we now find a class of generalized convex function, which are not sub-b-convex functions, also has some similar properties of sub-b-convex function and even s-convex function, and more generalized than these two types of generalized convex functions. Therefore, these extensions of convexity such as sub-b-convexity and s-convexity sparking our research interest, so we turn our attention to this new research.

Inspired by the research works[2, 6, 10–12, 14, 16], the purpose of this paper is to present a new class of generalized convex functions which is called sub-b-s-convex functions and discuss some properties of the class of functions satisfying the sub-b-convexity. We also give the sufficient conditions of optimality for both unconstrained and inequality constrained programming, which are obtained under the sub-b-s-convexity. Therefore, under the sub-b-s-convexity, we can solve the sub-b-convex and s-convex optimization programs which were solved separately in different frames.

The remainder of this paper is organized as follows. In Sect. 2, a new class of functions, called sub-b-s-convex function, which further extends to the concept of sub-b-convexity is introduced. Correspondingly, a new class of sets called sub-b-s-convex sets is introduced, and some properties of sub-b-s-convex function and sub-b-s-convex sets are developed. In Sect. 3, we introduce a new sub-b-s-convex programming and establish the sufficient conditions of optimality under the sub-b-s-convexity. Sect. 4 is devoted of drawing the conclusions.

2. Basic Results

In this section, we first recalled the definitions of sub-b-convexity and s-convexity of function. The class of sub-b-convex functions is defined by M.T. Chao *et al.*[6] as follows. Through out the paper, let S be a nonempty convex set in $\mathbb{R}^n$.

Definition 2.1. *The function* $f\colon \mathrm{S} \to \mathbb{R}$ *is said to be a sub-b-convex function on* S *with respect to map b:* $\mathrm{S} \times \mathrm{S} \times [0,1] \to \mathbb{R}$, *if*

$$f\Big(\lambda x + (1-\lambda)y\Big) \le \lambda f(x) + (1-\lambda) f(y) + b(x,y,\lambda)$$

holds for all $x, y \in \mathrm{S}$ *and* $\lambda \in [0,1]$.

Among others, H. Hudzik *et al.*[10] considered the class of functions which is s-convex in the second sense defined in the following way:

Definition 2.2. *The function* $f\colon \mathrm{S} \to \mathbb{R}$ *is said to be s-convex in the second sense(i.s.s. in short of in the second sense) if*

$$f\Big(\lambda x + (1-\lambda)y\Big) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in \mathrm{S}$, $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$. *The class of s-convex in the second sense is usually denoted by* K_s^2.

In the following, by combining Definition 2.1 and Definition 2.2, we introduce the concepts of sub-b-s-convex function and sub-b-s-convex set *i.s.s.*. Then we study some of their basic properties.

Definition 2.3. *The function* $f\colon \mathrm{S} \to \mathbb{R}$ *is said to be sub-b-s-convex function i.s.s. on* S *with respect to map b:* $\mathrm{S} \times \mathrm{S} \times [0,1] \to \mathbb{R}$, *if*

$$f\Big(\lambda x + (1-\lambda)y\Big) \le \lambda^s f(x) + (1-\lambda)^s f(y) + b(x,y,\lambda) \tag{2.1}$$

holds for all $x, y \in S$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. On the other hand, If

$$f\Big(\lambda x + (1-\lambda)y\Big) \geq \lambda^s f(x) + (1-\lambda)^s f(y) + b(x,y,\lambda) \tag{2.2}$$

holds for all $x, y \in S$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$, then the function f is said to be sub-b-s-concave function i.s.s.. If the inequality signs in the previous two inequalities are strict, then f is called strictly sub-b-s-convex and sub-b-s-concave function i.s.s., respectively.

Remark 2.4. *When $s = 1$, the sub-b-s-convex function i.s.s. is reduced to be the sub-b-convex function. Moreover, when $s = 1$ and $b(x,y,\lambda) \leq 0$, the sub-b-s-convex function is reduced to be the convex function.*

Remark 2.5. *Sub-b-s-convex function can be concave. It is easy to see that when $b(x,y,\lambda) = f(\lambda x + (1-\lambda)y)$, if $f(x) \geq 0$, then f is a sub-b-s-convex function i.s.s.. In this case, if f is a concave function, then f is both sub-b-s-convex and concave function.*

Example 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as*

$$f(x) = -(x-2)^2 + 4, x \in [0,4],$$

and let $b(x,y,\lambda) \equiv 4$, then f is both sub-b-s-convex and concave function.

In what following, we are going to find out, whether or not, the sub-*b*-*s*-convex function *i.s.s.* shares some similar properties with the sub-*b*-convex function. The first observation is given as follows.

Theorem 2.7. *If $f, g : S \to \mathbb{R}$ are sub-b-s-convex functions i.s.s. with respect to the same map b, then $f + g$ and αf, $(\alpha \geq 0)$ are sub-b-s-convex with respect to the same map b.*

Corollary 2.8. *If f_i: $S \to \mathbb{R}$,$(i = 1, 2, \cdots, m)$ are sub-b-s-convex functions i.s.s. with respect to maps b_i: $S \times S \times [0,1] \to \mathbb{R}$,$(i = 1, 2, \cdots, m)$, respectively, then the function*

$$f = \sum_{i=1}^{m} a_i f_i, a_i \geq 0, (i = 1, 2, \cdots, m) \tag{2.3}$$

is sub-b-s-convex with respect to $b = \sum_{i=1}^{m} a_i b_i$.

Proposition 2.9. *If f_i: $S \to \mathbb{R}$,$(i = 1, 2, \cdots, m)$ are sub-b-s-convex functions i.s.s. with respect to maps b_i: $S \times S \times [0,1] \to \mathbb{R}$, $(i = 1, 2, \cdots, m)$, respectively, then the function $f = \max\{f_i, i = 1, 2, \cdots, m\}$ is a sub-b-s-convex function i.s.s. with respect to $b = \max\{b_i, i = 1, 2, \cdots, m\}$.*

Theorem 2.10. *Assume f: $S \to \mathbb{R}$ is a sub-b-s-convex functions i.s.s. with respect to b: $S \times S \times [0,1] \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ is an increasing function. If g satisfies the following conditions*

$$(i)\ g(\alpha x) = \alpha g(x), \forall x \in \mathbb{R}, \alpha \geq 0, \tag{2.4}$$

$$(ii)\ g(x+y) = g(x) + g(y), \forall x, y \in \mathbb{R}, \tag{2.5}$$

then $f' = g \circ f$ is a sub-b-s-convex function i.s.s. with respect to $b' = g \circ b$.

Proof Since f is a sub-*b*-*s*-convex functions *i.s.s.* with respect to b and g is an increasing function, it follows that

$$\begin{aligned}(g \circ f)\Big(\lambda x + (1-\lambda)y\Big) &= g\Big(f\Big(\lambda x + (1-\lambda)y\Big)\Big) \\ &\leq g\Big(\lambda^s f(x) + (1-\lambda)^s f(y) + b(x,y,\lambda)\Big).\end{aligned}$$

Since $\lambda \in [0,1]$, by combining the two conditions of (2.4) and (2.5), it yields that

$$\begin{aligned}(g\circ f)\Big(\lambda x+(1-\lambda)y\Big) &\le \lambda^s g\Big(f(x)\Big)+(1-\lambda)^s g\Big(f(y)\Big)+g\Big(b(x,y,\lambda)\Big)\\ &= \lambda^s (g\circ f)(x)+(1-\lambda)^s (g\circ f)(y)+(g\circ b)(x,y,\lambda).\end{aligned}$$

That is, $f' = g\circ f$ is a sub-b-s-convex function *i.s.s.* with respect to $b' = g\circ b$ and the proof is completed.

Remark 2.11. *In Theorem 2.7, Corollary 2.8, Proposition 2.9 and Theorem 2.10, if the sub-b-s-convex function f and g are replaced with the strict sub-b-s-convex function i.s.s., then we can obtain the similar conclusions. Obviously, Theorem 2.7, Corollary 2.8 and Proposition 2.9 satisfy the conditions of Theorem 2.10, so Theorem 2.7, Corollary 2.8 and Proposition 2.9 can be regarded as the special cases of Theorem 2.10.*

In what following, we introduce a new concept of sub-b-s-convex set *i.s.s.*.

Definition 2.12. *Let $X \subseteq \mathbb{R}^{n+1}$ be a nonempty set. X is said to be a sub-b-s-convex set i.s.s. with respect to b: $\mathbb{R}^n\times\mathbb{R}^n\times[0,1]\to\mathbb{R}$, if*

$$\Big(\lambda x+(1-\lambda)y, \lambda^s\alpha+(1-\lambda)^s\beta+b(x,y,\lambda)\Big)\in X \tag{2.6}$$

holds for all $(x,\alpha),(y,\beta)\in X$, $x,y\in\mathbb{R}^n$, $\lambda\in[0,1]$, and some fixed $s\in(0,1]$.

Here, we give a characterization of sub-b-s-convex function $f\colon S\to\mathbb{R}$ *i.s.s.* in terms of their epigraph $E(f)$, which is given by

$$E(f)=\Big\{(x,\alpha)|x\in S,\alpha\in\mathbb{R}, f(x)\le\alpha\Big\}. \tag{2.7}$$

Theorem 2.13. *A function $f\colon S\to\mathbb{R}$ is a sub-b-s-convex function i.s.s. with respect to b: $S\times S\times[0,1]\to\mathbb{R}$, if and only if $E(f)$ is a sub-b-s-convex set i.s.s. with respect to b.*

Proof Suppose that f is a sub-b-s-convex function *i.s.s.* with respect to b. Let (x_1,α_1), $(x_2,\alpha_2)\in E(f)$. Then, $f(x_1)\le\alpha_1$, $f(x_2)\le\alpha_2$. So we have

$$\begin{aligned}f\Big(\lambda x_1+(1-\lambda)x_2\Big) &\le \lambda^s f(x_1)+(1-\lambda)^s f(x_2)+b(x_1,x_2,\lambda)\\ &\le \lambda^s\alpha_1+(1-\lambda)^s\alpha_2+b(x_1,x_2,\lambda)\end{aligned}$$

holds for all $x_1,x_2\in S$, $\lambda\in[0,1]$ and some fixed $s\in(0,1]$. Hence, it is easy to see that

$$\Big(\lambda x_1+(1-\lambda)x_2, \lambda^s\alpha+(1-\lambda)^s\beta+b(x_1,x_2,\lambda)\Big)\in E(f).$$

Thus, by Definition 2.12, $E(f)$ is a sub-b-s-convex set *i.s.s.* with respect to b.

Conversely, let's assume that $E(f)$ is a sub-b-s-convex set *i.s.s.* with respect to b. Let $x_1,x_2\in S$, then $\Big(x_1,f(x_1)\Big),\Big(x_2,f(x_2)\Big)\in E(f)$. Thus, for $\lambda\in[0,1]$ and some fixed $s\in(0,1]$, we have that

$$\Big(\lambda x_1+(1-\lambda)x_2, \lambda^s f(x_1)+(1-\lambda)^s f(x_2)+b(x_1,x_2,\lambda)\Big)\in E(f).$$

This implies that

$$f\Big(\lambda x_1+(1-\lambda)x_2\Big)\le\lambda^s f(x_1)+(1-\lambda)^s f(x_2)+b(x_1,x_2,\lambda).$$

That is, f is a sub-b-s-convex function *i.s.s.* with respect to b and the proof of Theorem 2.13 is completed.

Proposition 2.14. *If X_i is a family of sub-b-s-convex sets i.s.s. with respect to the same map $b(x,y,\lambda)$, then $\bigcap_{i\in I} X_i$ is a sub-b-s-convex set i.s.s. with respect to $b(x,y,\lambda)$.*

Proof Let $(x,\alpha),(y,\beta) \in \bigcap_{i\in I} X_i$, then, for each $i \in I$, $(x,\alpha),(y,\beta) \in X_i$.
Since X_i is a sub-*b*-*s*-convex set *i.s.s.* with respect to b, for all $\lambda \in [0,1]$ and some fixed $s \in (0,1]$, it follows that

$$\Big(\lambda x + (1-\lambda)y, \lambda^s\alpha + (1-\lambda)^s\beta + b(x,y,\lambda)\Big) \in X_i, \forall i \in I.$$

Thus,

$$\Big(\lambda x + (1-\lambda)y, \lambda^s\alpha + (1-\lambda)^s\beta + b(x,y,\lambda)\Big) \in \bigcap_{i\in I} X_i.$$

Hence, $\bigcap_{i\in I} X_i$ is a sub-*b*-*s*-convex set *i.s.s.* with respect to b and the conclusion obtains.

Proposition 2.15. *If $\{f_i | i \in I\}$ is a family of numerical functions i.s.s., and each f_i is a sub-b-s-convex function with respect to the same map $b(x,y,\lambda)$, then the numerical function $f = \sup_{i\in I} f_i(x)$ is a sub-b-s-convex function i.s.s. with respect to $b(x,y,\lambda)$.*

Proof Since f_i is a sub-*b*-*s*-convex function *i.s.s.* on S with respect to $b(x,y,\lambda)$, its epigraph $E(f_i) = \Big\{(x,\alpha)|x \in S, f_i(x) \le \alpha\Big\}$ is a sub-*b*-*s*-convex set *i.s.s.* with respect to b. Therefore, their intersection

$$\begin{aligned}\bigcap_{i\in I} E(f_i) &= \Big\{(x,\alpha)|x \in S, f_i(x) \le \alpha, i \in I\Big\}\\ &= \Big\{(x,\alpha)|x \in S, f(x) \le \alpha\Big\}\\ &= E(f),\end{aligned}$$

where $f(x)$=$\sup_{i\in I} f_i(x)$. By Theorem 2.13 and Proposition 2.14, we know that $f = \sup_{i\in I} f_i(x)$ is a sub-*b*-*s*-convex function *i.s.s.* with respect to $b(x,y,\lambda)$ and the conclusion follows.

3. Main Results

We consider continuously differentiable functions which are sub-*b*-*s*-convex functions with respect to a map $b(x,y,\lambda)$. For fixed $x,y \in S$, $b(x,y,\lambda)$ is a continuously decreasing function about λ. So, $\frac{b(x,y,\lambda)}{\lambda}$ is a continuously decreasing function about λ.

Furthermore, we assume that the limit $\lim_{\lambda\to 0_+} \frac{b(x,y,\lambda)}{\lambda}$ exists and the limit is the maximum of $\frac{b(x,y,\lambda)-o(\lambda)}{\lambda}$ for all $\lambda \in (0,1]$ and fixed $x,y \in S$.

Theorem 3.1. *Suppose that $f\colon S \to \mathbb{R}$ is a non-negative differentiable sub-b-s-convex function i.s.s. with respect to map $b(x,y,\lambda)$. Then*

$$(i)\ \nabla f(y)^T(x-y) \le \lambda^{s-1}\Big(f(x)+f(y)\Big) + \lim_{\lambda\to 0_+} \frac{b(x,y,\lambda)}{\lambda}, \tag{3.1}$$

$$(ii)\ \nabla f(y)^T(x-y) \le \lambda^{s-1}\Big(f(x)-f(y)\Big) + \frac{f(y)}{\lambda} + \lim_{\lambda\to 0_+} \frac{b(x,y,\lambda)}{\lambda}. \tag{3.2}$$

Proof (i) By the Taylor expansion and the sub-*b*-*s*-convexity of f, we have taht

$$\begin{aligned}f\Big(\lambda x + (1-\lambda)y\Big) &= f\Big(y + \lambda(x-y)\Big)\\ &= f(y) + \lambda\nabla f(y)^T(x-y) + o(\lambda).\end{aligned} \tag{3.3}$$

$$\begin{aligned} f\Big(\lambda x+(1-\lambda)y\Big) &\le \lambda^s f(x)+(1-\lambda)^s f(y)+b(x,y,\lambda) \\ &\le \lambda^s f(x)+(1+\lambda^s) f(y)+b(x,y,\lambda). \end{aligned} \tag{3.4}$$

Combining the equality (3.3) and inequality (3.4) yields that

$$\lambda\nabla f(y)^T(x-y)+o(\lambda)\le \lambda^s\Big(f(x)+f(y)\Big)+b(x,y,\lambda). \tag{3.5}$$

Dividing the inequality (3.5) above by λ and using the fact that $\lim_{\lambda\to 0_+}\frac{b(x,y,\lambda)}{\lambda}$ is the maximum of $\frac{b(x,y,\lambda)}{\lambda}-\frac{o(\lambda)}{\lambda}$, it yields that

$$\nabla f(y)^T(x-y)\le \lambda^{s-1}\Big(f(x)+f(y)\Big)+\lim_{\lambda\to 0_+}\frac{b(x,y,\lambda)}{\lambda},$$

which proves the first part of Theorem 3.1.

(ii) Combining the above equality (3.3) and inequality (3.4), it yields that

$$\begin{aligned} f(y)+\lambda\nabla f(y)^T(x-y)+o(\lambda) &\le \lambda^s f(x)+(1-\lambda)^s f(y)+b(x,y,\lambda) \\ &= \lambda^s f(x)+(1-\lambda)^s f(y)-\lambda^s f(y) \\ &+\lambda^s f(y)+b(x,y,\lambda) \\ &= \lambda^s\Big(f(x)-f(y)\Big)+b(x,y,\lambda) \\ &+\Big((1-\lambda)^s+\lambda^s\Big)f(y). \end{aligned} \tag{3.6}$$

Obviously, $\Big(\lambda^s+(1-\lambda)^s\Big)\le 2$ for $\lambda\in[0,1]$ and some fixed $s\in(0,1]$. By invoking the fact that f is a non-negative function, inequality (3.6) can be simplified to

$$f(y)+\lambda\nabla f(y)^T(x-y)+o(\lambda)\le \lambda^s\Big(f(x)-f(y)\Big)+2f(y)+b(x,y,\lambda).$$

Thus,

$$\lambda\nabla f(y)^T(x-y)+o(\lambda)\le \lambda^s\Big(f(x)-f(y)\Big)+f(y)+b(x,y,\lambda). \tag{3.7}$$

In the same way, dividing the inequality (3.7) above by λ and using the fact that $\lim_{\lambda\to 0_+}\frac{b(x,y,\lambda)}{\lambda}$ is the maximum of $\frac{b(x,y,\lambda)}{\lambda}-\frac{o(\lambda)}{\lambda}$, we have

$$\nabla f(y)^T(x-y)\le \lambda^{s-1}\Big(f(x)-f(y)\Big)+\frac{f(y)}{\lambda}+\lim_{\lambda\to 0_+}\frac{b(x,y,\lambda)}{\lambda},$$

which proves the second part of Theorem 3.1.

Theorem 3.2. *Suppose that $f\colon S\to\mathbb{R}$ is a negative differentiable sub-b-s-convex function i.s.s. with respect to map $b(x,y,\lambda)$. Then*

$$\nabla f(y)^T(x-y)\le \lambda^{s-1}\Big(f(x)-f(y)\Big)+\lim_{\lambda\to 0_+}\frac{b(x,y,\lambda)}{\lambda}. \tag{3.8}$$

Proof By the Taylor expansion and the sub-*b*-*s*-convexity of f, we have that

$$\begin{aligned} f\Big(\lambda x+(1-\lambda)y\Big) &= f\Big(y+\lambda(x-y)\Big) \\ &= f(y)+\lambda\nabla f(y)^T(x-y)+o(\lambda). \end{aligned} \tag{3.9}$$

$$f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda^s f(x) + (1-\lambda)^s f(y) + b(x,y,\lambda). \tag{3.10}$$

Since $\lambda \in [0,1]$ and some fixed $s \in (0,1]$, then we have $\Big(\lambda^s + (1-\lambda)^s\Big) \geq 1$. Furthermore, because f is a negative function, the inequality (3.10) can be simplified to

$$f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda^s f(x) + (1-\lambda^s) f(y) + b(x,y,\lambda). \tag{3.11}$$

Meanwhile, combining the above equality (3.9) and inequality (3.11) yields that

$$\lambda \nabla f(y)^T (x-y) + o(\lambda) \leq \lambda^s (f(x) - f(y)) + b(x,y,\lambda). \tag{3.12}$$

Dividing the inequality (3.12) above by λ and using the fact that $\lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda}$ is the maximum of $\frac{b(x,y,\lambda)}{\lambda} - \frac{o(\lambda)}{\lambda}$, we have

$$\nabla f(y)^T (x-y) \leq \lambda^{s-1}(f(x) - f(y)) + \lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda}.$$

The proof of Theorem 3.2 is completed.

Corollary 3.3. *Let $f\colon S \to \mathbb{R}$ be a differentiable sub-b-s-convex function i.s.s. with respect to map b. For $\lambda \in (0,1]$, if f is a non-negative function, then*

$$\nabla\Big(f(y) - f(x)\Big)^T (x-y) \leq \frac{f(y)}{\lambda} + \frac{f(x)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(y,x,\lambda)}{\lambda}. \tag{3.13}$$

If f is a negative function, then

$$\nabla\Big(f(y) - f(x)\Big)^T (x-y) \leq \lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(y,x,\lambda)}{\lambda}. \tag{3.14}$$

Proof If f is a non-negative function, by Theorem 3.1, we have that

$$\nabla f(y)^T (x-y) \leq \lambda^{s-1}\Big(f(x) - f(y)\Big) + \frac{f(y)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda},$$

$$\nabla f(x)^T (y-x) \leq \lambda^{s-1}\Big(f(y) - f(x)\Big) + \frac{f(x)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(y,x,\lambda)}{\lambda}.$$

Adding the two inequalities above, it is easy to show that

$$\nabla\Big(f(y) - f(x)\Big)^T (x-y) \leq \frac{f(y)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda} + \frac{f(x)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(y,x,\lambda)}{\lambda}.$$

In a similar way, if f is a negative function, by Theorem 3.2, we can also get

$$\nabla\Big(f(y) - f(x)\Big)^T (x-y) \leq \lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(y,x,\lambda)}{\lambda}.$$

The proof is completed.

Now, we apply the associated results above to the nonlinear programming. First, we consider the unconstraint problem (P).

$$(P) : \min\{f(x), x \in S\} \tag{3.15}$$

Theorem 3.4. *Let f: $S \to \mathbb{R}$ be a non-negative differentiable and sub-b-s-convex function i.s.s. with respect to b. If $\bar{x} \in S$ and the inequality*

$$\nabla f(\bar{x})^T(x-\bar{x}) \geq \frac{f(\bar{x})}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(x,\bar{x},\lambda)}{\lambda} \tag{3.16}$$

holds for each $x \in$ S, $\lambda \in (0,1]$ and some fixed $s \in (0,1]$, then $\bar{x}$ is the optimal solution to the optimal problem (P) with respect to f on S.

Proof For any $x \in S$, since f is a non-negative differentiable sub-b-s-convex function *i.s.s.*, by (3.2) of Theorem 3.1, we have that

$$\nabla f(\bar{x})^T(x-\bar{x}) - \frac{f(\bar{x})}{\lambda} - \lim_{\lambda \to 0_+} \frac{b(x,\bar{x},\lambda)}{\lambda} \leq \lambda^{s-1}\Big(f(x) - f(\bar{x})\Big)$$

holds for $\lambda \in (0,1]$ and some fixed $s \in (0,1]$, on the other hand, since

$$\nabla f(\bar{x})^T(x-\bar{x}) \geq \frac{f(\bar{x})}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(x,\bar{x},\lambda)}{\lambda},$$

we have $f(x) - f(\bar{x}) \geq 0$. Therefore, $\bar{x}$ is the optimal solution of f on S. This completes the proof.

Example 3.5. *Let $f : [0,+\infty) \to \mathbb{R}$ be defined as*

$$f(x) = (x^2+4x)^s,$$

and let $b(x,y,\lambda) = \lambda x^2 + 4\lambda y^2$, here s is a fixed constant on $(0,1)$. Then f is sub-b-s-convex function.

In fact, $g(x) = x^2+4x$ is a non-negative convex function on $[0,+\infty)$, combining Corollary 7 in [3](M. Alomari, M. Darus and S.S. Dragomir 2009), then $f(x)$ is an s-convex function on $[0,+\infty)$, $0 < s < 1$. So we have that

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y).$$

Since $b(x,y,\lambda) = \lambda x^2 + 4\lambda y^2 \geq 0$ for $x, y \in [0,2]$ and $\lambda \in (0,1]$, it is easy to show that

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) + b(x,y,\lambda).$$

Hence, f is sub-b-s-convex function.

Now we consider the following unconstraint sub-b-s-convex programming

$$P : \min\Big\{f(x), x \in [0,+\infty)\Big\},$$

where $f(x) = (x^2+4x)^s$, $b(x,y,\lambda) = \lambda x^2 + 4\lambda y^2$ and some fixed $s \in (0,1)$. Since $f(x)$ is a non-negative differentiable and sub-b-s-convex function *i.s.s.* with respect to b and the limit $\lim_{\lambda \to 0_+} \frac{b(x,y,\lambda)}{\lambda}$ exists for fixed $x, y \in [0,+\infty)$ and $\lambda \in (0,1]$. Followed by calculating, we have that

$$\begin{aligned}
\nabla f(\bar{x})^T(x-\bar{x}) &= s(\bar{x}^2+4\bar{x})^{s-1}(2\bar{x}+4)(x-\bar{x}) \\
\frac{f(\bar{x})}{\lambda} &= \frac{(\bar{x}^2+4\bar{x})^s}{\lambda} \\
\lim_{\lambda \to 0_+} \frac{b(x,\bar{x},\lambda)}{\lambda} &= x^2 + 4\bar{x}^2.
\end{aligned}$$

It can easily see that $\bar{x} = 0$, the inequality $\nabla f(\bar{x})^T(x-\bar{x}) \geq \frac{f(\bar{x})}{\lambda} + \lim_{\lambda \to 0_+} \frac{b(x,\bar{x},\lambda)}{\lambda}$ holds for all $x \in [0,+\infty)$, $\lambda \in (0,1]$ and some fixed $s \in (0,1)$. According to the theorem 3.4, the minimum value of $f(x)$ at zero. As shown in Figure 1, let $x \in [0,2]$ and s take 0.5, 0.6 and 0.7, respectively. We can obtain the same optimal value at (0,0).

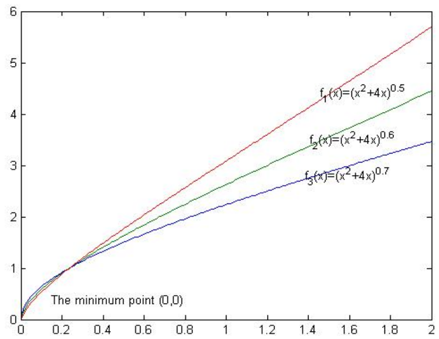

Figure 1: The optimal value of $f(x)$ with different s

Corollary 3.6. *Let f: $S \to \mathbb{R}$ be a strictly non-negative sub-b-s-convex function i.s.s. with respect to b. If $\bar{x} \in S$ and satisfies the condition (3.16), then $\bar{x}$ is the unique optimal solution of f on S.*

Proof From Theorem 3.1, if f is a strictly non-negative sub-b-s-convex function *i.s.s.* with respect to b, then we have

$$\nabla f(y)^T(x-y) < \lambda^{s-1}\Big(f(x)-f(y)\Big)+\frac{f(y)}{\lambda}+\lim_{\lambda\to 0_+}\frac{b(x,y,\lambda)}{\lambda}.$$

Suppose $x_1, x_2 \in S$ are two different optimal solutions of problem (P). Then $f(x_1) = f(x_2)$. Combining the above inequalities, we have

$$\nabla f(x_2)^T(x_1-x_2)-\frac{f(x_2)}{\lambda}-\lim_{\lambda\to 0_+}\frac{b(x_1,x_2,\lambda)}{\lambda} < \lambda^{s-1}\Big(f(x_1)-f(x_2)\Big).$$

Hence, from (3.16), we can see

$$\lambda^{s-1}\Big(f(x_1)-f(x_2)\Big) > 0.$$

Since $f(x_1) = f(x_2)$, so $x_1 = x_2 = \bar{x}$. Therefore, $\bar{x}$ is the unique optimal solution of f on S and the proof of Corollary 3.6 is completed.

Next, we apply the associated results to the nonlinear programming with inequality constraints as follows:

$$(P_s) : \min\Big\{f(x)|x \in \mathbb{R}^n, g_i(x) \le 0, i \in I\Big\}, I = \{1,2,\cdots,m\}. \tag{3.17}$$

Denote the feasible set of (P_s) by $F = \{x \in \mathbb{R}^n | g_i(x) \le 0, i \in I\}$. For the convenience of discussion, we assume that f and g_i are all differentiable and F is a nonempty set in $\mathbb{R}^n$.

Theorem 3.7. ***(Karush-Kuhn-Tucker Sufficient Conditions)*** *Suppose that the function f: $\mathbb{R}^n \to \mathbb{R}$ is non-negative differentiable sub-b-s-convex function i.s.s. with respect to b: $\mathbb{R}^n \times \mathbb{R}^n \times (0,1] \to \mathbb{R}$, g_i: $\mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ are differentiable sub-b-s-convex functions i.s.s. with respect to b_i: $\mathbb{R}^n \times \mathbb{R}^n \times (0,1] \to \mathbb{R}$ $(i \in I)$. Assume that $x^* \in F$ is a KKT point of (P_s), i.e., there exist multipliers $u_i \ge 0$ $(i \in I)$ such that*

$$\nabla f(x^*) + \sum_{i\in I} u_i \nabla g_i(x^*) = 0, u_i g_i(x^*) = 0. \tag{3.18}$$

If

$$\frac{f(x^*)}{\lambda} + \lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} \leq -\sum_{i\in I} u_i \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda}, \tag{3.19}$$

then x^* is an optimal solution of the problem (P_s).

Proof For any $x \in F$, we have

$$g_i(x) \leq 0 = g_i(x^*), i \in I(x^*) = \{i \in I | g_i(x^*) = 0\}.$$

Therefore, by the sub-b-s-convexity of g_i and the Theorem 3.2, for $i \in I(x^*)$, we obtain

$$\nabla g_i(x^*)^T(x - x^*) - \lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} \leq \lambda^{s-1}\Big(g_i(x) - g_i(x^*)\Big) \leq 0. \tag{3.20}$$

From (3.18), we have

$$\nabla f(x^*)^T(x - x^*) = -\sum_{i\in I} u_i \nabla g_i(x^*)^T(x - x^*) = -\sum_{i\in I(x^*)} u_i \nabla g_i(x^*)^T(x - x^*).$$

Using the condition (3.19), we have that

$$\begin{aligned}\nabla f(x^*)^T(x - x^*) - \frac{f(x^*)}{\lambda} - \lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} &\geq -\sum_{i\in I} u_i \nabla g_i(x^*)^T(x - x^*) + \sum_{i\in I} u_i \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda} \\ &= -\sum_{i\in I(x^*)} u_i\Big(\nabla g_i(x^*)^T(x - x^*) - \lim_{\lambda\to 0_+} \frac{b_i(x,x^*,\lambda)}{\lambda}\Big).\end{aligned}$$

Combining the inequality (3.20) with the above inequality, we have

$$\nabla f(x^*)^T(x - x^*) - \frac{f(x^*)}{\lambda} - \lim_{\lambda\to 0_+} \frac{b(x,x^*,\lambda)}{\lambda} \geq 0.$$

From Theorem 3.4, we can get $f(x) - f(x^*) \geq 0$ for each $x \in F$. Therefore x^* is an optimal solution of the problem (P_s). This ends the proof.

4. Conclusion

In this paper, we have introduced sub-b-s-convex functions and sub-b-s-convex sets *i.s.s.*. According to the definition, it is observed that sub-b-s-convex function can be reduced into sub-b-convex function on the condition that $s = 1$. Furthermore, it can be simplified into convex function on the conditions that $s = 1$ and $b(x, y, \lambda) \leq 0$. Therefore, the sub-b-s-convex function *i.s.s.* is a generalization of sub-b-convex and s-convex function. In the end, we have studied Karush-Kuhn-Tucker sufficient conditions for obtaining an optimal solution of sub-b-s-convex programming with unconstrained or inequality constrained conditions.

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