Filomat 30:14 (2016), 3897–3906
DOI 10.2298/FIL1614897D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Block Representations for the Drazin Inverse of Anti-Triangular Matrices

E. Dopazo[a], M.F. Martínez-Serrano[a], J. Robles[a]

[a] *ETS Ingenieros Informáticos, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain*

Abstract. The paper is devoted to the study of the Drazin inverse of some structured matrices that appear in applications. We focus mainly on deriving formulas for the Drazin inverse of an anti-triangular block matrix M in terms of its blocks. New representations for the Drazin inverse of M are given under some conditions, that extend recent results in the literature. Additionally, these results are applied to investigate the Drazin inverse of certain structured matrices, in particular the group inverse for Hermitian matrices, and to study additive properties of the Drazin inverse.

1. Introduction

The concept of Drazin inverse plays an important role in various fields like Markov chains, singular differential and difference equations, iterative methods, etc. (see [1, 8]). The Drazin inverse of a matrix $A \in \mathbb{C}^{n\times n}$ is the unique complex matrix A^D satisfying the relations:

$$A^D A A^D = A^D, \;\; AA^D = A^D A, \;\; A^{k+1}A^D = A^k,$$

where $k = \mathrm{ind}(A)$, called the index of A, is the smallest nonnegative integer such that $\mathrm{rank}(A^{k+1})=\mathrm{rank}(A^k)$. We will denote by $A^\pi = I - AA^D$. In the case $\mathrm{ind}(A) = 1$, A^D reduces to the group inverse of A, denoted by $A^\#$. Further, the Moore-Penrose inverse of $A \in \mathbb{C}^{n\times m}$, denoted by $A^\dagger$, is the unique solution of equations $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, and $(A^\dagger A)^* = A^\dagger A$, where $A^\star$ indicates the conjugate transpose of A. For a Hermitian matrix A, $A^\#$ exists and $A^\# = A^\dagger$ ([1, 8]).

A problem of great interest in this field is concerned with the Drazin inverse of matrices partitioned as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square matrices, in terms of the Drazin inverse of smaller size matrices A and D. It was posed as an open problem by Campbell and Meyer [8] in 1979, and it has received great attention. The most relevant case is concerned with block triangular matrices (either $B = 0$ or $C = 0$), solved by Meyer and Rose [21]. They proved the following expression, for the case $C = 0$,

$$M^{\mathrm{D}} = \begin{pmatrix} A^{\mathrm{D}} & X \\ 0 & D^{\mathrm{D}} \end{pmatrix}, \tag{1}$$

2010 *Mathematics Subject Classification*. *2000 MSC:* 15A09

Keywords. Drazin inverse, Group inverse, Anti-triangular matrix, Block matrix, Additive properties
Received: 06 November 2014; Accepted: 13 December 2014
Communicated by Dragana Cvetković-Ilić
The research has been partially supported by Project MTM2010-18057, "Ministerio de Ciencia e Innovación" of Spain.
Email addresses: edopazo@fi.upm.es (E. Dopazo), fmartinez@fi.upm.es (M.F. Martínez-Serrano), jrobles@fi.upm.es (J. Robles)

where $X = (A^{\mathrm{D}})^2 \left(\sum_{i=0}^{t-1} (A^{\mathrm{D}})^i B D^i \right) D^\pi + A^\pi \left(\sum_{i=0}^{r-1} A^i B (D^{\mathrm{D}})^i \right) (D^{\mathrm{D}})^2 - A^{\mathrm{D}} B D^{\mathrm{D}}$, $r = \mathrm{ind}(A)$, and $t = \mathrm{ind}(D)$.

Otherwise, the representation of the Drazin inverse of an anti-triangular matrix M, where $D = 0$, was posed as an open problem by Campbell [7] in 1983, in relation with the solution of singular second-order differential equations. Furthermore, these structured matrices appear in applications like graph theory, saddle-point problems, and optimization problems [2, 12, 15]. Additionally, it has been proved that any real symmetric indefinite matrix can be transformed into a block anti-triangular form by orthogonal similarity transformations [20]. In recent years, the problem has become an important issue and some results have been given under some conditions [3–6, 9, 10, 12–16], but it still remains open.

In this paper, we focus on deriving formulas for the Drazin inverse of an anti-triangular matrix M, in terms of the Drazin inverses of smaller order size matrices than M. The new expressions are given under more general settings than those given in the literature [12, 13, 15, 17, 19], in the matrix context. Additionally, these results are applied to investigate the Drazin inverse of certain structured matrices, in particular the group inverse for Hermitian matrices, and to study additive properties of the Drazin inverse.

Throughout the paper we concern with the upper anti-triangular block matrix

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \tag{2}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, and $C \in \mathbb{C}^{m\times n}$. Since Drazin inverse is preserved by similarity, analogous results may be obtained for the Drazin inverse of lower anti-triangular matrices, by noting

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix}^D = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & 0 \end{pmatrix}^D \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The paper is organized as follows. Section 2 is devoted to derive new formulas for the Drazin inverse of M in terms of the Drazin inverses of smaller size matrices than M. These results are applied to some structured matrices in Section 3. Finally, some additive properties of the Drazin inverse are obtained in Section 4.

2. Main Results

This section addresses the problem of expressing the Drazin inverse of an upper anti-triangular block matrix M in terms of Drazin inverses of smaller size matrices. The section is organized in three parts. First we present some expressions for M^D in terms of A^D and $(BC)^D$. Second, explicit formulas for M^D in terms of the Drazin inverse of the diagonal block A are given. Finally, we derive some formulas that extend a known expression for the ordinary inverse, to deal with the Drazin inverse. The results we provide recover some cases studied in the literature.

Theorem 2.1. *Let M be a matrix of the form* (2)*. If $ABCA^\pi = 0$ and $AA^DBC = 0$, then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} \Psi A & \Psi B \\ C\Psi & C\Psi A^D B \end{pmatrix} + \begin{pmatrix} (BC)^D \Phi ABC & 0 \\ 0 & C(BC)^D \Phi \left(A - BCA^D \right) B \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} \Psi &= (A^D)^2 + \sum_{j=0}^{s-1} (BC)^j (BC)^\pi \Gamma (A^D)^{2j+2} + \Phi(I - \Gamma) - (BC)^D \Phi ABCA^D, \\ \Gamma &= \sum_{n=0}^{r-1} A^n BC (A^D)^{n+2}, \qquad \Phi = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \left((BC)^D \right)^{k+1} A^{2k} A^\pi, \end{aligned} \tag{4}$$

$r = \mathrm{ind}(A)$*, $s = \mathrm{ind}(BC)$, and $\lfloor \frac{r}{2} \rfloor$ denotes the integer part of $\frac{r}{2}$.*

Proof. We consider the following splitting of matrix M

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ CAA^D & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix} := P + Q.$$

We have $Q^2 = 0$, and since $ABCA^{\pi} = 0$ we get $P^2Q = 0$. Hence matrices P and Q satisfy the conditions of Theorem 2.2 in [19], then we can write

$$\begin{aligned} (P+Q)^D &= \sum_{j=0}^{\nu_1-1} \left((PQ)^j(PQ)^{\pi} + Q(PQ)^j(PQ)^{\pi}P^D\right)(P^D)^{2j+1} \\ &+ \sum_{k=0}^{\nu_2-1} \left(((PQ)^D)^{k+1}P + Q((PQ)^D)^{k+1}\right)P^{2k}P^{\pi} \\ &= (X_1 + X_2)P + Q(X_1 + X_2), \end{aligned} \tag{5}$$

where

$$X_1 = \sum_{j=0}^{\nu_1-1} (PQ)^j(PQ)^{\pi}(P^D)^{2j+2}, \qquad X_2 = \sum_{k=0}^{\nu_2-1} \left((PQ)^D\right)^{k+1} P^{2k}P^{\pi}, \tag{6}$$

and $\nu_1 = \mathrm{ind}(PQ)$, $\nu_2 = \mathrm{ind}(P^2)$. Through the proof we will use the following relations derived directly from the condition $AA^DBC = 0$:

$$A^DBC = 0,\ A^D(BC)^D = 0,\ A^{\pi}BC = BC,\ A^{\pi}(BC)^D = (BC)^D.$$

Clearly PQ is the diagonal block matrix $PQ = \begin{pmatrix} BCA^{\pi} & 0 \\ 0 & 0 \end{pmatrix}$. Using Cline's formula, $(BCA^{\pi})^D = BC\left((A^{\pi}BC)^2\right)^D A^{\pi} = BC\left((BC)^2\right)^D A^{\pi} = (BC)^DA^{\pi}$, thus we have

$$((PQ)^D)^k = \begin{pmatrix} ((BC)^D)^kA^{\pi} & 0 \\ 0 & 0 \end{pmatrix}, \quad k \geq 1. \tag{7}$$

Furthermore, $(BCA^{\pi})^{\pi} = I - BC(BC)^DA^{\pi}$, hence we compute

$$(PQ)^{\pi} = \begin{pmatrix} I - BC(BC)^DA^{\pi} & 0 \\ 0 & I \end{pmatrix}, \quad (PQ)^j(PQ)^{\pi} = \begin{pmatrix} (BC)^j(BC)^{\pi}A^{\pi} & 0 \\ 0 & 0 \end{pmatrix}, \quad j \geq 1. \tag{8}$$

Next, we focus on obtaining P^D. We notice P is an anti-triangular matrix. Since $B\widetilde{C}A^{\pi} = 0$ and $AA^DB\widetilde{C} = 0$, where $\widetilde{C} = CAA^D$, P satisfies the conditions of Theorem 3.6 in [15] and

$$P^D = \begin{pmatrix} (I+\Gamma)A^D & (I+\Gamma)(A^D)^2B \\ C(A^D)^2 & C(A^D)^3B \end{pmatrix},$$

where Γ is defined as in (4). Furthermore, we prove by induction

$$(P^D)^j = \begin{pmatrix} (I+\Gamma)(A^D)^j & (I+\Gamma)(A^D)^{j+1}B \\ C(A^D)^{j+1} & C(A^D)^{j+2}B \end{pmatrix}, \quad j \geq 1. \tag{9}$$

After some computations, we have

$$
\begin{aligned}
P^{\pi} &= \begin{pmatrix} A^{\pi}-\Gamma & -(I+\Gamma)A^{D}B \\ -CA^{D} & I-C(A^{D})^{2}B \end{pmatrix}, \\
P^{2k}P^{\pi} &= \begin{pmatrix} A^{2k-1}(AA^{\pi}-\Gamma A) & A^{2k-2}(AA^{\pi}-\Gamma A)B \\ 0 & 0 \end{pmatrix}, \quad k \geq 1.
\end{aligned} \tag{10}
$$

Now substituting (8) and (9) in the expression of X_1 in (6), we get

$$
X_1 = \begin{pmatrix} (I+\Lambda)\,(A^{D})^{2} & (I+\Lambda)\,(A^{D})^{3}B \\ C(A^{D})^{3} & C(A^{D})^{4}B \end{pmatrix}, \tag{11}
$$

where $\Lambda = \sum_{j=0}^{s-1} (BC)^{j}(BC)^{\pi}\Gamma(A^{D})^{2j}$ and $s = \operatorname{ind}(BC)$.

Next, taking into account (7), (10), and $\Gamma A = BCA^{D} + A\Gamma$, we can rewrite X_2 in (6) in the form

$$
X_2 = \begin{pmatrix} \Phi(I-\Gamma) - (BC)^{D}\Phi ABCA^{D} & \left((BC)^{D}\Phi\left(A - (BC + ABCA^{D})A^{D}\right) - \Phi\Gamma A^{D}\right)B \\ 0 & 0 \end{pmatrix}, \tag{12}
$$

where Γ and Φ defined as in (4).

Additionally, from (11) and (12), since $\Phi A^{D} = 0$, we have

$$
X_1 + X_2 = \begin{pmatrix} \Psi & (\Psi - (BC)^{D}\Phi BC)A^{D}B + (BC)^{D}\Phi AB \\ C(A^{D})^{3} & C(A^{D})^{4}B \end{pmatrix}, \tag{13}
$$

where Ψ as in (4). Finally, using expression (13) in (5) and $(A^{D})^{2} + A^{\pi}\Psi = \Psi$, we get the formula (3) of the theorem. □

An explicit representation for the Drazin inverse of M, under the condition $BC = 0$, was given in [9]. This result was extended to the case $ABC = 0$ ([15, Theorem 3.3], [19, Corollary 3.9]), and to the case $BCA^{\pi} = 0$, $AA^{D}BC = 0$ ([15, Theorem 3.6]). These results can be obtained easily as corollaries of the previous theorem.

Corollary 2.2. *Let M be a matrix defined as in* (2).

(i) *If $ABC = 0$, then*

$$
\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{D} = \begin{pmatrix} \Psi A & \Psi B \\ C\Psi & C\Psi^{2}AB \end{pmatrix},
$$

where $\Psi = \sum_{j=0}^{s-1} (BC)^{j}(BC)^{\pi}(A^{D})^{2j+2} + \Phi$, and s, Φ are defined as in (4).

In particular if $BC = 0$, Ψ adopts the expression $\Psi = (A^{D})^{2}$.

(ii) *If BCA^{π} nilpotent, $ABCA^{\pi} = 0$, and $AA^{D}BC = 0$, then*

$$
\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{D} = \begin{pmatrix} \Psi A & \Psi B \\ C\Psi & C\Psi A^{D}B \end{pmatrix}, \tag{14}
$$

where $\Psi = (A^{D})^{2} + \sum_{j=0}^{s-1} (BC)^{j}\Gamma(A^{D})^{2j+2}$, and s, Γ are defined as in (4).

(iii) *If* $BCA^{\pi} = 0$ *and* $AA^DBC = 0$, *then* M^D *adopts the expression* (14), *where* $\Psi = (I + \Gamma)(A^D)^2$ *and* Γ *is defined as in* (4).

Parallel results can be obtained following similar steps that in the preceding theorem and corollaries. The proof of the next theorem is omitted.

Theorem 2.3. *Let M be a matrix of the form* (2). *If* $A^{\pi}BCA = 0$ *and* $BCAA^D = 0$, *then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} A\widehat{\Psi} & \widehat{\Psi}B \\ C\widehat{\Psi} & CA^D\widehat{\Psi}B \end{pmatrix} + \begin{pmatrix} BCA\widehat{\Phi}(BC)^D & 0 \\ 0 & C(A - A^DBC)\widehat{\Phi}(BC)^DB \end{pmatrix},$$

where $r = \mathrm{ind}(A)$, $s = \mathrm{ind}(BC)$,

$$\widehat{\Psi} = (A^D)^2 + \sum_{j=0}^{s-1}(A^D)^{2j+2}\widehat{\Gamma}(BC)^j(BC)^{\pi} + (I - \widehat{\Gamma})\widehat{\Phi} - A^DBCA\widehat{\Phi}(BC)^D,$$

$$\widehat{\Gamma} = \sum_{n=0}^{r-1}(A^D)^{n+2}BCA^n, \qquad \widehat{\Phi} = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} A^{2k}A^{\pi}\left((BC)^D\right)^{k+1}.$$

We notice that analogous results to those above stated may be given for lower anti-triangular matrices. They recover the cases studied in [17].

Now we focus on deriving explicit expressions of M^D in terms of the Drazin inverse of the diagonal block A, and blocks B and C.

Theorem 2.4. *Let M be defined as in* (2). *If* $BCAA^{\pi} = 0$, $BCA^{\pi}B = 0$ *and* $A^DBCA = 0$, *then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} \Psi A\left(I + (A^D)^2BC\right) & \Psi B \\ C\Psi\left(I + (A^D)^2BC\right) & C\Psi A^DB \end{pmatrix}, \tag{15}$$

where $r = \mathrm{ind}(A)$ *and*

$$\Psi = (A^D)^2 + \sum_{n=0}^{r-1} A^nBC(A^D)^{n+4}. \tag{16}$$

Proof. We split matrix M as follows

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ CAA^D & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix} := P + Q.$$

We notice $Q^2 = 0$. Since $BCAA^{\pi} = 0$ and $BCA^{\pi}B = 0$, we have $PQP = 0$. Therefore matrices P and Q satisfy the conditions of Theorem 2.1 in [23]. Then, for the particular case where Q is a nilpotent matrix, we get

$$M^D = P^D + Q(P^D)^2 + (P^D)^2Q + Q(P^D)^3Q. \tag{17}$$

Next, we focus on computing P^D. We notice that matrix P is in the conditions of Theorem 3.6 in [15] ($B\widetilde{C}A^{\pi} = 0$, $AA^DB\widetilde{C} = 0$, where $\widetilde{C} = CAA^D$). Hence, we obtain

$$P^D = \begin{pmatrix} \Psi A & \Psi B \\ C(A^D)^2 & C(A^D)^3B \end{pmatrix},$$

where $\Psi = (A^D)^2 + \sum_{n=0}^{r-1} A^nBC(A^D)^{n+4}$ and $\mathrm{ind}(A) = r$. Moreover, we have

$$(P^D)^j = \begin{pmatrix} \Psi A(A^D)^{j-1} & \Psi(A^D)^{j-1}B \\ C(A^D)^{j+1} & C(A^D)^{j+2}B \end{pmatrix}, \quad j \geq 1.$$

After computing, we get

$$Q(P^D)^2 = \begin{pmatrix} 0 & 0 \\ CA^\pi\Psi & CA^\pi\Psi A^D B \end{pmatrix}, \qquad (P^D)^2 Q = \begin{pmatrix} \Psi A^D BC & 0 \\ C(A^D)^4 BC & 0 \end{pmatrix},$$

$$Q(P^D)^3 Q = \begin{pmatrix} 0 & 0 \\ CA^\pi\Psi(A^D)^2 BC & 0 \end{pmatrix}.$$

Finally, substituting the above expressions into (17), we conclude the statement of the theorem. □

We notice that the previous result extends Theorem 3.6 in [15]. As direct consequence of Theorem 2.4, we get the following corollaries.

Corollary 2.5. *Let M be defined as in* (2)*.*

(i) *If $CAA^\pi = 0$, $CA^\pi B = 0$ and $A^D BCA = 0$, then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} \Psi A\left(I + (A^D)^2 BC\right) & \Psi B \\ C(A^D)^2\left(I + (A^D)^2 BC\right) & C(A^D)^3 B \end{pmatrix},$$

where Ψ is defined by (16)*.*

(ii) *If $CAA^\pi = 0$, $CA^\pi B = 0$ and $A^D BC = 0$ (see* [13]*), then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} \Psi A & \Psi B \\ C(A^D)^2 & C(A^D)^3 B \end{pmatrix},$$

where Ψ is defined by (16)*.*

(iii) *If $CA^\pi = 0$ and $AA^D B = 0$, then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} A^D + \sum_{n=0}^{r-1} A^n BC(A^D)^{n+3} & 0 \\ C(A^D)^2 & 0 \end{pmatrix}, \quad r = \mathrm{ind}(A).$$

Dually, using similar techniques that in Theorem 2.4 we obtain the following analogous result.

Theorem 2.6. *Let M be defined as in* (2)*. If $AA^\pi BC = 0$, $CA^\pi BC = 0$ and $ABCA^D = 0$, then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} \left(I + BC(A^D)^2\right) A\widehat{\Psi} & \left(I + BC(A^D)^2\right)\widehat{\Psi} B \\ C\widehat{\Psi} & CA^D\widehat{\Psi} B \end{pmatrix},$$

where $r = \mathrm{ind}(A)$ and $\widehat{\Psi} = (A^D)^2 + \sum_{n=0}^{r-1} (A^D)^{n+4} BCA^n$.

Our next purpose is to explore under which conditions the known formula [18] of the inverse of an anti-triangular matrix

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{pmatrix}, \tag{18}$$

can be extended to the Drazin inverse of the singular anti-triangular matrix with blocks B and C not necessarily square. First, we take into account the case of an anti-diagonal matrix, where $A = 0$. It was solved by Catral et al. [12] in the context of bipartite digraphs. They proved

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{pmatrix}, \tag{19}$$

where B and C are not assumed nonsingular, neither square matrices. We focus on extending the representation (19) to the general case where A is nonzero. We establish necessary and sufficient conditions under which the natural generalization of (18), in light of (19), holds.

Theorem 2.7. *Let M be a matrix of the form* (2). *Then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} 0 & (BC)^D B \\ C(BC)^D & -C(BC)^D A(BC)^D B \end{pmatrix} = \begin{pmatrix} 0 & B(CB)^D \\ (CB)^D C & -(CB)^D CAB(CB)^D \end{pmatrix} \tag{20}$$

if and only if

$$(BC)^\pi A(BC)^D B = 0, \quad C(BC)^D A(BC)^\pi = 0, \tag{21}$$

and $\begin{pmatrix} A(BC)^\pi & (BC)^\pi B \\ C(BC)^\pi & 0 \end{pmatrix}$ *is a nilpotent matrix.*

Proof. Let $X = \begin{pmatrix} 0 & (BC)^D B \\ C(BC)^D & -C(BC)^D A(BC)^D B \end{pmatrix}$. We will prove that matrix X verifies the conditions of the Drazin inverse of M. First, we compute

$$MX = \begin{pmatrix} BC(BC)^D & (BC)^\pi A(BC)^D B \\ 0 & C(BC)^D B \end{pmatrix}, \qquad XM = \begin{pmatrix} BC(BC)^D & 0 \\ C(BC)^D A(BC)^\pi & C(BC)^D B \end{pmatrix}.$$

Then, $MX = XM$ if and only if conditions in (21) hold. In this case, $MX = \begin{pmatrix} BC(BC)^D & 0 \\ 0 & C(BC)^D B \end{pmatrix}$. From the above expression, a simple computation shows that $XMX = X$.

Finally, under conditions in (21), we have

$$M - M^2 X = \begin{pmatrix} A(BC)^\pi & (BC)^\pi B \\ C(BC)^\pi & 0 \end{pmatrix}. \tag{22}$$

Therefore, $M - M^2X$ is a nilpotent matrix if and only if the matrix on the right side of (22) is nilpotent. So we conclude $X = M^D$.

Now, from expression X and attending to $(BC)^D B = B(CB)^D$ and $C(BC)^D = (CB)^D C$ (Cline's formula), we can derive the expression of M^D given in (20), in terms of $(CB)^D$. This completes the proof. □

As consequence of the Theorem 2.7, we can deduce the following result.

Corollary 2.8. *Let M be a matrix of the form* (2). *If $(BC)^\pi A = 0$ and $A(BC)^\pi = 0$, then M^D adopts the form* (20).

We notice that expression (19) can be obtained as direct application of Theorem 2.7 to the case $A = 0$. In addition, formula (18) is derived for the case of nonsingular square blocks B and C.

3. Applications to Special Structured Matrices

This section is devoted to study the Drazin inverse of several special structured matrices of interest in applications and in the literature.

First, we consider the class of Hermitian matrices. They arise in partial differential equations, optimization problems and variational problems, where they are linked for instance to a so-called saddle point problems. In many of these applications, the matrix is anti-triangular [2], and it is needed to calculate the generalized inverse. We recall that the group inverse of a Hermitian matrix exists and coincides with its Moore-Penrose inverse [8]. Furthermore, in [20] it has been proved that a real symmetric indefinite matrix can be reduced to a block anti-triangular form M, by orthogonal similarity transformations. Next we focus on deriving representations for the group inverse of Hermitian matrices in the form

$$M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}, \tag{23}$$

where A is a Hermitian matrix.

Proposition 3.1. *Let M be a matrix of the form* (23). *If $ABB^*=0$ then*

$$M^\# = \begin{pmatrix} (BB^*)^\pi A^\# & ((BB^*)^\pi (A^\#)^2 + (BB^*)^\# A^\pi)B \\ B^\dagger A^\pi & -B^\dagger A^\# B \end{pmatrix}.$$

Proof. M is a Hermitian matrix, then $M^\#$ exists and $M^\# = M^\dagger$. Because conditions of Theorem 2.1 are satisfied, $M^\#$ adopts the expression (3), where $C = B^*$. Now, since A is a Hermitian matrix, $AA^\pi = 0$, and Γ and Φ in (4) get reduced to

$$\Gamma = BB^*(A^\#)^2, \qquad \Phi = (BB^*)^\# A^\pi.$$

Now using that $\Phi A = 0$ and $(BB^*)^\pi \Gamma = 0$, after some computations, Ψ in (4) adopts the expression

$$\Psi = (BB^*)^\pi (A^\#)^2 + (BB^*)^\# A^\pi.$$

Therefore, substituting the above expressions of Γ, Φ, and Ψ in formula in (4), and taking into account the relations $B^*(BB^*)^\# = B^*(BB^*)^\dagger = B^\dagger$ and $B^*(BB^*)^\pi = 0$, we get the statement of the theorem. $\square$

A particular case of interest in applications is where B is a full row rank matrix.

Corollary 3.2. *Let M be a matrix of the form* (23), *where B is full row rank, then*

$$M^\# = \begin{pmatrix} 0 & (B^*)^\dagger \\ B^\dagger & -B^\dagger A (B^*)^\dagger \end{pmatrix}.$$

Proof. Since M is a Hermitian matrix, $M^\#$ exists. B is full row rank matrix, then BB^* is nonsingular and conditions of Theorem 2.7 are verified. Hence, applying this theorem, we obtain the representation (20) of $M^\#$, where $C = B^*$. Substituting in this expression $B^*(BB^*)^{-1} = B^\dagger$ and $(BB^*)^{-1}B = (B^*)^\dagger$, we get the final formula. $\square$

Next, we examine the special case of the anti-triangular matrix M, where $C = -I$, and matrices A and B are square. Campbell [7] showed how the Drazin inverse of this matrix can be used to express the solution of certain second-order differential equations. It has been studied in [5] and [22] under assumption $AB = BA$. As direct application of Theorem 2.1, we get the following explicit representation of M^D in terms of A^D and B^D.

Corollary 3.3. *Let A and B be square complex matrices. If $ABA^\pi = 0$ and $AA^DB = 0$, then*

$$\begin{pmatrix} A & B \\ -I & 0 \end{pmatrix}^D = \begin{pmatrix} \Psi A + B^D \Phi AB & \Phi B \\ -\Psi & B^D \Phi AB \end{pmatrix},$$

where $r = \mathrm{ind}(A)$, $s = \mathrm{ind}(B)$,

$$\Psi = (A^D)^2 + \sum_{j=0}^{s-1} (-1)^{j+1} B^j B^\pi \Gamma (A^D)^{2j+2} + \Phi(I + \Gamma) - B^D \Phi ABA^D,$$

$$\Gamma = \sum_{n=0}^{r-1} A^n B (A^D)^{n+2}, \qquad \Phi = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^{k+1} \left(B^D\right)^{k+1} A^{2k} A^\pi.$$

Now, we are concerned with the special case of an anti-triangular matrix M, where $A = A^2$, widely studied in the literature [6, 15, 16]. Applying Theorem 2.4 and noting that $A^D = A^\# = A$ and $A^\pi = I - A$, we get the following reduced expression of M^D.

Corollary 3.4. *Let M be defined as in* (2), *where $A = A^2$. If $BCA^\pi B = 0$ and $ABCA = 0$, then*

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D = \begin{pmatrix} (A + BC)^2 & (A + BC)B \\ C(A + BC)^2 & C(A + BC)B \end{pmatrix}.$$

4. Application to Additive Properties of the Drazin Inverse

A topic closely related to the representation of the Drazin inverse of block matrices, is to express the Drazin inverse of the sum of two matrices P and Q in terms of P^D, Q^D, and $(PQ)^D$. It is useful for analyzing perturbation properties, in iterative methods and for computing the Drazin inverse for block matrices. This topic has been largely studied and some results have been obtained in some special cases (see, e.g., [10, 23] and references given there). In this section, we apply our main results to investigate the additive problem for the Drazin inverse of matrices. The following result extends Theorem 2.1 in [23] and Corollary 4.3 in [11].

Theorem 4.1. *Let $P, Q \in \mathbb{C}^{n\times n}$ such that $PQ^2 = 0$. Then*

$$(P+Q)^D = \begin{pmatrix} I & Q \end{pmatrix}\left[\sum_{n=0}^{t-1}\begin{pmatrix} 0 & 0 \\ 0 & Q^nQ^\pi \end{pmatrix}\left(\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^D\right)^{n+1} + \sum_{n=0}^{v-1}\begin{pmatrix} 0 & 0 \\ 0 & (Q^D)^{n+1} \end{pmatrix}\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^n\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^\pi\right]^2\begin{pmatrix} P \\ I \end{pmatrix},$$

where $t = \operatorname{ind}(Q)$ and $v = \operatorname{ind}(\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix})$. Moreover,

(i) *If $P^2QP^\pi = 0$, $P^DQ = 0$, then*

$$\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^D = \begin{pmatrix} \widetilde{\Psi}P & \Phi PQ \\ \widetilde{\Psi} - (PQ)^D\Phi P^2QP^D & (PQ)^D\Phi P^2Q \end{pmatrix}, \tag{24}$$

where $r = \operatorname{ind}(P)$, $s = \operatorname{ind}(PQ)$,

$$\widetilde{\Psi} = (P^D)^2 + \sum_{j=0}^{s-1}(PQ)^j(PQ)^\pi\Gamma(P^D)^{2j+2} + \Phi(I-\Gamma),$$

$$\Gamma = \sum_{n=0}^{r-2}P^{n+1}Q(P^D)^{n+2}, \qquad \Phi = \sum_{k=0}^{\lfloor \frac{r}{2}\rfloor}\left((PQ)^D\right)^{k+1}P^{2k}P^\pi.$$

(ii) *If $PQPP^\pi = 0$, $P^DQP = 0$, then*

$$\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^D = \begin{pmatrix} \Psi(P+Q) & \Psi PQ \\ \Psi(I+P^DQ) & \Psi Q \end{pmatrix}, \tag{25}$$

where $\Psi = (P^D)^2 + \sum_{n=0}^{r-2}P^{n+1}Q(P^D)^{n+4}$ and $r = \operatorname{ind}(P)$.

Proof. Using the Cline's formula,

$$(P+Q)^D = \left(\begin{pmatrix} I & Q \end{pmatrix}\begin{pmatrix} P \\ I \end{pmatrix}\right)^D = \begin{pmatrix} I & Q \end{pmatrix}\left(\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}^D\right)^2\begin{pmatrix} P \\ I \end{pmatrix}.$$

Then, we notice it is sufficient to calculate $\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}^D$. To do this, we consider the following splitting

$$\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix} = \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} := E + F.$$

Since $PQ^2 = 0$, we get $EF = 0$, hence we apply the well-known formula

$$(E+F)^D = \sum_{n=0}^{t-1} F^n F^\pi (E^D)^{n+1} + \sum_{n=0}^{v-1} (F^D)^{n+1} E^n E^\pi,$$

where $v = \mathrm{ind}(E)$ and $t = \mathrm{ind}(F) = \mathrm{ind}(Q)$.

Attending F is a diagonal matrix, $F^D = \begin{pmatrix} 0 & 0 \\ 0 & Q^D \end{pmatrix}$. Then, substituting in the above expression, we get

$$\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}^D = \sum_{n=0}^{t-1} \begin{pmatrix} 0 & 0 \\ 0 & Q^n Q^\pi \end{pmatrix} \left(\begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^D \right)^{n+1} + \sum_{n=0}^{v-1} \begin{pmatrix} 0 & 0 \\ 0 & (Q^D)^{n+1} \end{pmatrix} \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^n \begin{pmatrix} P & PQ \\ I & 0 \end{pmatrix}^\pi.$$

Now, we focus on obtaining the Drazin inverse of the anti-triangular matrix E. If conditions in (i) are verified, matrix E satisfies conditions of Theorem 2.1. Applying conveniently this result, we get the expression (24) for E^D. Otherwise, if we assume conditions in (ii), matrix E satisfies the conditions of Theorem 2.4, that yields the formula (25) for E^D. This completes the proof. □

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions that greatly improved the original manuscript.

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