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Sensitivity Functionals in Convex Optimization Problem

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Abstract. We consider sensitivity functionals and Lagrange multiplier method for solving finite dimensional convex optimization problem.An analysis based on this property is also applied for semicoercive infinite dimensional variational inequality in mechanics.

1. Introduction

The Lagrange multiplier method based on modified Lagrangian functions is one of the main methods for solving finite-dimensional convex optimization problems (cf. [1, 5, 6]). In the past time, the Lagrange multiplier method is successfully applied to find the solutions of infinite-dimensional variational inequalities in mechanics, see [9, 11, 12].

Convergence analysis of Lagrange multiplier method is in many respects provided with the help of property of lower semicontinuity of sensitivity function. With the help of lower semicontinuity of sensitivity functions, it is possible to prove continuous differentiability of dual functions, which allows to solve the modified dual problems by applying effective iterative methods.

In Section 2 and 3 we consider sensitivity function and Lagrange multiplier method for solving finite-dimensional convex optimization problem. In Section 4 the Lagrange multiplier method is considered in semicoercive infinite-dimensional variational inequality in mechanics.

2. Finite-Dimensional Convex Optimization Problem

We consider the convex optimization problem

$$\begin{cases} f(x) - \min, \\ x \in \Omega = \{z \in \mathbb{R}^n : g^j(z) \le 0, j = 1, 2, \dots, m\}, \end{cases} \tag{1}$$

where f and g^j are given convex functions defined on $\mathbb{R}^n$. Assume that there exists $\tilde{x} \in \Omega$ such that $g^j(\tilde{x}) < 0$ for all $j = 1, 2, \dots, m$, which is called a Slater's condition. In this case, the Lagrange function

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$L(x,y) = f(x) + \sum_{i=1}^{m} y_j g^j(x)$ has a saddle point $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^m_+$; that is,

$$L(x^*, y) \le L(x^*, y^*) \le L(x, y^*), \qquad \forall x \in \mathbb{R}^n, \quad \forall y \in \mathbb{R}^m_+.$$

Moreover, x^* is an optimal solution of initial problem(1) and y^* is optimal solution of the dual problem, see[1, 5, 6]

$$\begin{cases} \underline{L}(y) - \max, \\ y \in \Omega^* = \{\omega \in \mathbb{R}^m_+ : \underline{L}(\omega) > -\infty\}, \end{cases} \tag{2}$$

where $\underline{L}(y) = \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{j=1}^{m} y_j g^j(x) \right\}$.

The effective domain of the concave dual function $\underline{L}(y)$ cannot coincide with the space $\mathbb{R}^m$, and so it complicates searching for the optimal solution of problem(2). Assuming that Ω is a compact set, it can be seen that the set $\Omega_v = \{x : g^j(x) \le v_j, j = 1, 2, \ldots, m\}$ is a compact set for every $v = (v_1, ..., v_m) \in R^m$ if $\Omega_v \neq \emptyset$. Let us consider the following function defined on the space $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$:

$$K(x,y,v) = \begin{cases} f(x) + \sum_{j=1}^{m} y_j v_j + \frac{r}{2} \sum_{j=1}^{m} v_j^2, & \text{if } g(x) \le v, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $g(x) = (g^1(x), ..., g^m(x))$.

We now introduce the modified Lagrangian function

$$M(x,y) = \inf_{v \in \mathbb{R}^m} K(x,y,v).$$

We then have the equation

$$\begin{aligned} \inf_{v \in \mathbb{R}^m} K(x,y,v) &= \inf_{v \ge g(x)} \left\{ f(x) + \sum_{j=1}^{m} y_j v_j + \frac{r}{2} \sum_{j=1}^{m} v_j^2 \right\} \\ &= f(x) + \frac{1}{2r} \inf_{v \ge g(x)} \sum_{j=1}^{m} \left(\left(y_j + r v_j \right)^2 - y_j^2 \right) = f(x) + \frac{1}{2r} \sum_{j=1}^{m} \inf_{v_j \ge g^j(x)} \left(\left(y_j + r v_j \right)^2 - y_j^2 \right) \\ &= f(x) + \frac{1}{2r} \sum_{j=1}^{m} \left(\left((y_j + r v_j)^+ \right)^2 - y_j^2 \right), \end{aligned}$$

where $(y_j + r v_j(x))^+ = \max\{0, y_j + r v_j(x)\}$.

Let us introduce the modified dual function

$$\underline{M}(y) = \inf_{x \in \mathbb{R}^n} M(x,y) = \inf_{x \in \mathbb{R}^n} \inf_{v \in \mathbb{R}^m} K(x,y,v).$$

Since $\inf_{x \in \mathbb{R}^n} \inf_{v \in \mathbb{R}^m} K(x,y,v) = \inf_{v \in \mathbb{R}^m} \inf_{x \in \mathbb{R}^n} K(x,y,v)$, we get

$$\underline{M}(y) = \inf_{v \in \mathbb{R}^m} \inf_{g(x) \le v} \left\{ f(x) + \sum_{j=1}^{m} y_j v_j + \frac{r}{2} \sum_{j=1}^{m} v_j^2 \right\} = \inf_{v \in \mathbb{R}^m} \left\{ \chi(v) + \sum_{j=1}^{m} y_j v_j + \frac{r}{2} \sum_{j=1}^{m} v_j^2 \right\},$$

where the function χ is given by

$$\chi(v) = \begin{cases} \inf_{g(x) \le v} f(x), & \text{if } \{x : g(x) \le v\} \neq \emptyset, \\ +\infty, & \text{otherwise,} \end{cases}$$

which is called a sensitivity function (cf. [1, 5, 6]).

Thus, the function $\underline{M}(y)$ may have the following two representation:

$$\underline{M}(y) = \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \frac{1}{2r}\sum_{j=1}^{m}\left(\left((y_j + rv_j)^+\right)^2 - y_j^2\right)\right\}, \tag{3}$$

$$\underline{M}(y) = \inf_{v\in\mathbb{R}^m} \left\{ \chi(v) + \sum_{j=1}^{m} y_j v_j + \frac{r}{2}\sum_{j=1}^{m} v_j^2 \right\}. \tag{4}$$

Definition 2.1. *A point $(\overline{x}, \overline{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ is called a saddle point of the modified Lagrangian function $M(x, y)$, if the following inequalities hold:*

$$M(\overline{x}, y) \le M(\overline{x}, \overline{y}) \le M(x, \overline{y}), \qquad \forall x \in \mathbb{R}^n, \quad \forall y \in \mathbb{R}^m.$$

It has been known that the sets of saddle points of the functions $L(x, y)$ and $M(x, y)$ are the same, see [6, 10], and hence it allows to use the modified Lagrangian function $M(x, y)$ instead of classical Lagrangian function $L(x, y)$ for searching saddle points, and a question of solvability of problem (3) or (4) can be arisen. In well-known monographs the question of solvability of problem (3) has been considered by assuming $f(x)$ is strongly convex function, see [5, 6, 10]. We will consider the question of solvability of problem (4).

Since Ω is a compact set, it can be shown that $\chi(v)$ is a proper convex function [1, 3].

Theorem 2.2. *The sensitivity function $\chi(v)$ is a lower semicontinuous function.*

Proof. Let us take an arbitrary sequence $\{v^k\} \subset \operatorname{dom}\chi$ such that $\lim_{k\to\infty} v^k = \hat{v}$, and let $x(v^k) = \operatorname{argmin}_{x\in\Omega_{v^k}} f(x)$. It is obvious that $f(x(v^k)) = \chi(v^k)$ and $\{x(v^k)\}$ is a bounded sequence. Without lose of generality, we can assume that $\{x(v^k)\}$ is a convergent sequence. Let $\hat{x} = \lim_{k\to\infty} x(v^k)$. We then have $f(\hat{x}) = f(\lim_{k\to\infty} x(v^k)) = \lim_{k\to\infty} f(x(v^k)) = \lim_{k\to\infty} \chi(v^k)$. Since $g^j(x(v^k)) \le v_j^k$ for $j = 1, 2, \ldots, m$, we get $g^j(\hat{x}) \le \hat{v}_j$ for $j = 1, 2, \ldots, m$ as $k \to +\infty$. Therefore, $\Omega_{\hat{v}} = \varnothing$ and $\lim_{k\to\infty} \chi(v^k) = f(\hat{x}) \ge \chi(\hat{v})$, which means that $\chi(v)$ is a lower semicontinuous function. □

It can be easily seen that $\operatorname{epi}\chi = \{(v, a) \in \mathbb{R}^m \times \mathbb{R} \colon \chi(v) \le a\}$ is a convex closed set in $\mathbb{R}^m \times \mathbb{R}$ and, the function $F_y(v) = \chi(v) + \sum_{j=1}^{m} y_j v_j + \frac{r}{2}\sum_{j=1}^{m} v_j^2$ is a lower semicontinuous function for any fixed $y \in \mathbb{R}^m$. From the separation theorem, one can see that there exist $\psi \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that

$$\langle \psi, v\rangle_{\mathbb{R}^m} + \chi(v) + \alpha \ge 0, \qquad \forall v \in \operatorname{dom}\chi,$$

where $\langle \psi, v\rangle_{\mathbb{R}^m}$ is the usual inner product on $\mathbb{R}^m$.

Therefore, $F_y(v) \longrightarrow +\infty$ under $\|v\|_{\mathbb{R}^m} \longrightarrow \infty$, which implies that $F_y(v)$ is a coercive function. Hence, there is a unique point $v(y) = \operatorname{argmin}_{v\in\mathbb{R}^m} F_y(v)$. This means that (4) is a solvable problem.

3. Modified Duality Scheme

Let us consider the modified dual problem

$$\begin{cases} \underline{M}(y) - \max, \\ y \in \mathbb{R}^m. \end{cases} \tag{5}$$

It is known that (2) and (5) are equivalent (cf. [1, 6]). But unlike $\underline{L}(y)$ in Problem (2), the function $\underline{M}(y)$ in (5) is a smooth function.

Theorem 3.1. *The function* $\underline{M}(y)$ *is differentiable in* $\mathbb{R}^m$ *and its derivative* $\nabla\underline{M}(y)$ *is equal to* $v(y) = \operatorname*{argmin}_{v\in\mathbb{R}^m} F_y(v)$. *Moreover,* $v(y)$ *satisfies the Lipschitz condition with a constant* $\frac{1}{r}$; *that is,*

$$\|v(\hat{y}) - v(\hat{\hat{y}})\|_{\mathbb{R}^m} \le \frac{1}{r}\|\hat{y} - \hat{\hat{y}}\|_{\mathbb{R}^m}, \qquad \forall \hat{y}, \hat{\hat{y}} \in \mathbb{R}^m.$$

Proof. Let $\hat{y}, \hat{\hat{y}} \in \mathbb{R}^m$, $\hat{v} = v(\hat{y})$, and let $\hat{\hat{v}} = v(\hat{\hat{y}})$. We then have the equalities

$$\chi(\hat{v}) + \sum_{j=1}^m \hat{y}_j\hat{v}_j + \frac{r}{2}\sum_{j=1}^m \hat{v}_j^{\,2} + \frac{r}{2}\|\hat{\hat{v}} - \hat{v}\|^2_{\mathbb{R}^m} \le \chi(\hat{\hat{v}}) + \sum_{j=1}^m \hat{y}_j\hat{\hat{v}}_j + \frac{r}{2}\sum_{j=1}^m \hat{\hat{v}}_j^{\,2}, \tag{6}$$

$$\chi(\hat{\hat{v}}) + \sum_{j=1}^m \hat{\hat{y}}_j\hat{\hat{v}}_j + \frac{r}{2}\sum_{j=1}^m \hat{\hat{v}}_j^{\,2} + \frac{r}{2}\|\hat{v} - \hat{\hat{v}}\|^2_{\mathbb{R}^m} \le \chi(\hat{v}) + \sum_{j=1}^m \hat{\hat{y}}_j\hat{v}_j + \frac{r}{2}\sum_{j=1}^m \hat{v}_j^{2}. \tag{7}$$

Adding two inequalities above, we obtain the inequality

$$r\|\hat{v} - \hat{\hat{v}}\|^2_{\mathbb{R}^m} \le \sum_{j=1}^m (\hat{y}_j - \hat{\hat{y}}_j)(\hat{\hat{v}}_j - \hat{v}_j),$$

$$r\|\hat{v} - \hat{\hat{v}}\|^2_{\mathbb{R}^m} \le \langle \hat{y} - \hat{\hat{y}}, \hat{\hat{v}} - \hat{v}\rangle. \tag{8}$$

It follows from (8) that

$$\|\hat{v} - \hat{\hat{v}}\|_{\mathbb{R}^m} \le \frac{1}{r}\|\hat{y} - \hat{\hat{y}}\|_{\mathbb{R}^m}. \tag{9}$$

From (6) and (7), it also follows that

$$\sum_{j=1}^m \hat{\hat{y}}_j(\hat{\hat{v}}_j - \hat{v}_j) + \frac{r}{2}\sum_{j=1}^m (\hat{\hat{v}}_j^{\,2} - \hat{v}_j^{2}) \le \chi(\hat{v}) - \chi(\hat{\hat{v}}) \le \sum_{j=1}^m \hat{y}_j(\hat{\hat{v}}_j - \hat{v}_j) + \frac{r}{2}\sum_{j=1}^m (\hat{\hat{v}}_j^{\,2} - \hat{v}_j^{2}),$$

and hence we have

$$\lim_{\hat{\hat{y}}\to\hat{y}} \chi(\hat{\hat{v}}) = \chi(\hat{v}).$$

This means that the dual function $\underline{M}(y)$ is continuous in $\mathbb{R}^m$. It follows from the continuity of the concave function $\underline{M}(y)$ that the subdifferential $\partial(-\underline{M}(y))$ of the convex function $-\underline{M}(y)$ is not empty for any $y \in \mathbb{R}^m$. Let $t \in \partial(-\underline{M}(y))$. Then for any $\xi \in \mathbb{R}^m$ we have the inequality

$$\underline{M}(\xi) \le \underline{M}(y) + \sum_{j=1}^m t_j(\xi_j - y_j);$$

that is,

$$\begin{aligned}
&\chi(v(\xi)) + \sum_{j=1}^m \xi_j v_j(\xi) + \frac{r}{2}\sum_{j=1}^m v_j^2(\xi) \\
&\le \chi(v(y)) + \sum_{j=1}^m y_j v_j(y) + \frac{r}{2}\sum_{j=1}^m v_j^2(y) + \sum_{j=1}^m t_j(\xi_j - y_j) \\
&\le \chi(v(\xi)) + \sum_{j=1}^m y_j v_j(\xi) + \frac{r}{2}\sum_{j=1}^m v_j^2(\xi) + \sum_{j=1}^m t_j(\xi_j - y_j).
\end{aligned}$$

Therefore, for any $\xi \in \mathbb{R}^m$, $\beta > 0$, one can see that

$$\beta^{-1}\sum_{j=1}^{m}(v_j(\xi) - t_j)(\xi_j - y_j) \le 0.$$

Setting $\xi = y + \beta p$, where $p \in \mathbb{R}^m$ is arbitrary, and letting β tend to zero, we can obtain the following inequality by (9):

$$\sum_{j=1}^{m}(v_j(y) - t_j)p_j \le 0, \qquad \forall p \in \mathbb{R}^m.$$

This means that $t = v(y)$. Hence, due to the uniqueness of the element $v(y)$, the function $\underline{M}(y)$ is differentiable in $\mathbb{R}^m$ and $\nabla\underline{M}(y) = v(y)$, see [3]. This completes the proof. □

Theorem 3.1 helps us to solve the dual problem (5) by using an iterative gradient method given below (cf. [5, 6]) :

$$y^{k+1} = y^k + rv(y^k), \quad k = 0, 1, 2, ..., \text{where } y^0 \text{ is a given initial vector,} \tag{10}$$

$v(y^k) = \underset{v\in\mathbb{R}^m}{\operatorname{argmin}}\left\{\chi(v) + \sum\limits_{j=1}^{m} y_j^k v_j + \frac{r}{2}\sum\limits_{j=1}^{m} v_j^2\right\}$, and $r > 0$ is a constant.

Theorem 3.2. ([5, 6]) *The sequence $\{y^k\}$, generated by (10), converges to the solution of dual problem (5).*

The iterative method (10) can be copied as follows (cf. [5, 6, 10]) :

(i) $x^{k+1} = \underset{x}{\operatorname{argmin}}\, M(x, y^k)$

(ii) $y^{k+1} = (y^k + rg(x^{k+1}))^+$

Theorem 3.3. ([6, 10]) *Any limit point of the sequence $\{(x^k, y^k)\}$ is a saddle point of the modified Lagrangian function $M(x, y)$.*

4. Infinite-Dimensional Convex Optimization Problem in Mechanics

Consider the Signorini problem in mechanics (cf. [2, 4, 9, 11])

$$\begin{cases} J(v) = \frac{1}{2}\int\limits_{\Omega}|\nabla v|^2 d\Omega - \int\limits_{\Omega} fv d\Omega - \min, \\ v \in \mathbf{K} = \{\omega \in \mathbf{H}^1(\Omega) \colon\ -\gamma\omega \le 0 \text{ on } \Gamma\}, \end{cases} \tag{11}$$

where $\Omega \subset \mathbb{R}^m (n = 2, 3)$ is a bounded domain with sufficiently smooth boundary Γ, $f \in \mathbf{L}_2(\Omega)$ is a given function, and $\gamma v \in \mathbf{H}^{1/2}(\Gamma)$ is the trace of a function $v \in \mathbf{H}^1(\Omega)$ on Γ.

Since the functional $J(v)$ is not strongly convex on $\mathbf{H}^1(\Omega)$, problem (11) may have no solution. However, if the condition

$$\int_{\Omega} f d\Omega < 0 \tag{12}$$

is satisfied, then for any $v \in \mathbf{K}$ we have $J(v) \longrightarrow +\infty$ as $\|v\|_{\mathbf{H}^1(\Omega)} \longrightarrow \infty$, and, hence, problem (11) is solvable. Moreover, condition (12) provides a unique solution of problem (11). In this section, we assume that (12) is satisfied.

For simplicity, the trace operator symbol γ will be omitted. For an arbitrary $m \in \mathbf{L}_2(\Gamma)$, we introduce the set

$$\mathbf{K}_m = \{v \in \mathbf{H}^1(\Omega) \colon\ -v \le m \text{ on } \Gamma\},$$

and define the sensitivity functional

$$\widetilde{\chi}(v) = \begin{cases} \inf\limits_{v \in \mathbf{K}_m} J(v), & \text{if } \mathbf{K}_m \neq \emptyset, \\ +\infty, & \text{otherwise,} \end{cases}$$

for any function $m \in \mathbf{L}_2(\Gamma)$.

It is easy to see that if a function $m \in \mathbf{L}_2(\Gamma)$ is bounded below on Γ, the corresponding set $\mathbf{K}_m$ is not empty and $\inf\limits_{v \in K_m} J(v) > -\infty$, see [2]. The set $\mathbf{K}_m$ can be empty if $m \in \mathbf{L}_2(\Gamma) \setminus \mathbf{H}^{1/2}(\Gamma)$ and m is not bounded below on Γ, see [7, 8]. Then $\chi(m)$ is a proper convex functional on $\mathbf{L}_2(\Gamma)$, but its effective domain $\operatorname{dom}\chi = \{m \in \mathbf{L}_2(\Gamma) \colon \chi(m) < +\infty\}$ does not coincide with $\mathbf{L}_2(\Gamma)$. Notice that $\operatorname{dom}\chi$ is a convex but not closed set. In this case, $\overline{\operatorname{dom}\chi} = \mathbf{L}_2(\Gamma)$.

We define the following functional on the space $\mathbf{H}^1(\Omega) \times \mathbf{L}_2(\Gamma) \times \mathbf{L}_2(\Gamma)$:

$$\widetilde{K}(v, l, m) = \begin{cases} J(v) + \frac{1}{2r} \int\limits_{\Gamma} ((l + m)^2 - l^2) d\Gamma, & \text{if } -\gamma v \leq m \text{ on } \Gamma, \\ +\infty, & \text{otherwise,} \end{cases}$$

and the modified Lagrangian functional $\widetilde{M}(v, l)$ on the space $\mathbf{H}^1(\Omega) \times \mathbf{L}_2(\Gamma)$

$$\widetilde{M}(v, l) = \inf_m \widetilde{K}(v, l, m) = J(v) + \frac{1}{2r} \int\limits_{\Gamma} \Big(\big((l - rv)^+ \big)^2 - l^2 \Big) d\Gamma.$$

Let us introduce the modified dual functional

$$\underline{\widetilde{M}}(l) = \inf_v \widetilde{M}(v, l) = \inf_v \left\{ J(v) + \frac{1}{2r} \int\limits_{\Gamma} \Big(\big((l - rv)^+ \big)^2 - l^2 \Big) d\Gamma \right\}. \tag{13}$$

The functional $\underline{\widetilde{M}}(l)$ can have another presentation given by (cf. [9, 11])

$$\underline{\widetilde{M}}(l) = \inf_m \Big\{ \chi(m) + \int\limits_{\Gamma} l m d\Gamma + \frac{r}{2} \int\limits_{\Gamma} m^2 d\Gamma \Big\}. \tag{14}$$

Definition 4.1. *A point $(\bar{v}, \bar{l}) \in \mathbf{H}^1(\Omega) \times \mathbf{L}_2(\Gamma)$ is called a saddle point of the modified Lagrangian functional $\widetilde{M}(v, l)$ if the following inequalities hold:*

$$\widetilde{M}(\bar{v}, l) \leq \widetilde{M}(\bar{v}, \bar{l}) \leq \widetilde{M}(v, \bar{l}), \qquad \forall v \in \mathbf{H}^1(\Omega), \quad \forall l \in \mathbf{L}_2(\Gamma).$$

If $(\bar{v}, \bar{l})$ is a saddle point of $\widetilde{M}(v, l)$, then $\bar{v}$ is a solution of Signorini problem (11) and $\bar{l}$ is a solution of the dual problem (cf. [9])

$$\begin{cases} \underline{\widetilde{M}}(l) - \max, \\ l \in \mathbf{L}_2(\Gamma). \end{cases} \tag{15}$$

Theorem 4.2. ([11]) *The sensitivity functional $\widetilde{\chi}(m)$ is a weakly lower semicontinuous on* $\mathbf{L}_2(\Gamma)$.

For an arbitrary fixed $l \in \mathbf{L}_2(\Gamma)$, consider the functional

$$\widetilde{F}_l(m) = \widetilde{\chi}(m) + \int\limits_{\Gamma} l m d\Gamma + \frac{r}{2} \int\limits_{\Gamma} m^2 d\Gamma,$$

where $r > 0$ is a constant.

The functional $\widetilde{F}_l(m)$ is very important in studying the duality methods based on modified Lagrangian functionals (cf. [9, 11, 12]). It is obvious that $\widetilde{F}_l(m)$ is also a weakly lower semicontinuous on $L_2(\Gamma)$.

Theorem 4.3. ([9]) *For any $l \in \mathbf{L}_2(\Gamma)$, there exists a unique element*

$$m(l) = \operatorname*{argmin}_{m \in \mathbf{L}_2(\Gamma)} \widetilde{F}_l(m).$$

Theorem 4.4. ([9, 11]) *The functional $\underline{\widetilde{M}}(l)$ is Gateaux differentiable on $\mathbf{L}_2(\Gamma)$ and its derivative $\nabla \underline{\widetilde{M}}(l)$ is equal to $m(l)$. Moreover, $m(l)$ satisfies the Lipschitz condition with a constant $\frac{1}{r}$; that is,*

$$\|m(l_1) - m(l_2)\|_{\mathbf{L}_2(\Gamma)} \le \frac{1}{r}\|l_1 - l_2\|_{\mathbf{L}_2(\Gamma)}, \qquad \forall l_1, l_2 \in \mathbf{L}_2(\Gamma).$$

To solve problem (15) one can use the iterative method (cf. [9, 11, 12])

$$\begin{aligned} &\text{(i)}\ \ u^{k+1} = \operatorname*{argmin}_{v \in \mathbf{H}^1(\Omega)} \left\{ J(v) + \tfrac{1}{2r} \int_\Gamma \left\{ \left((l^k - rv)^+ \right)^2 - (l^k)^2 \right\} d\Gamma \right\}, \qquad l^0 \in \mathbf{L}_2(\Gamma), \\ &\text{(ii)}\ \ l^{k+1} = (l^k + ru^{k+1})^+. \end{aligned} \tag{16}$$

Algorithm (16) converges with respect to the functional; that is,

$$\lim_{k \to \infty} J(u^k) = \min_{v \in \mathbf{K}} J(v) = J(u),$$

where u is a solution of problem (11).

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