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Fixed Point Results for Weakly α-Admissible Pairs

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Abstract. In this paper, we introduce the concepts of weakly and partially weakly α-admissible pair of mappings and obtain certain coincidence and fixed point theorems for classes of weakly α-admissible contractive mappings in a b-metric space. As an application, we derive some new coincidence and common fixed point results in a b-metric space endowed with a binary relation or a graph. Moreover, an example is provided here to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

The concept of a weakly contractive mapping ($d(fx, fy) \leq d(x, y) - \varphi(d(x, y))$ for all $x, y \in X$, where φ is an altering distance function) was introduced by Alber and Guerre-Delabrere [5] in the setup of Hilbert spaces. Rhoades [34] proved that every weakly contractive mapping defined on a complete metric space has a unique fixed point.

Self mappings f and g on a metric space X are called generalized weakly contractions, if there exists a lower semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(fx, gy) \leq N(x, y) - \varphi(N(x, y)),$$

where,

$$N(x, y) = \max\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, gy) + d(y, fx)]\},$$

for all $x, y \in X$ ([33]).

Theorem 1.1. *[33] Let (X, d) be a complete metric space. If $f, g : X \to X$ are generalized weakly contractions, then there exists a unique point $u \in X$ such that $u = fu = gu$.*

For more results in this direction we refer the reader to [8, 15].

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Many researchers have obtained fixed point results in complete metric spaces endowed with a partial order (See, e.g., [1, 3, 9, 11, 23–27, 30]).

In 2012, Samet et al. [32] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards, Salimi et al. [31] and Hussain et al. [16–18] modified the notion of α-admissible mapping and established certain (common) fixed point theorems.

Definition 1.2. *[32] Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.3. *Let f and g be two self-maps on a set X and let $\alpha : X \times X \to [0, \infty)$ be a function. A pair (f, g) is said to be,*

(i) weakly α-admissible if $\alpha(fx, gfx) \geq 1$ and $\alpha(gx, fgx) \geq 1$ for all $x \in X$,

(ii) partially weakly α-admissible if $\alpha(fx, gfx) \geq 1$ for all $x \in X$.

Let X be a non-empty set and $f : X \to X$ be a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

Definition 1.4. *Let X be a set, $f, g, h : X \to X$ are mappings such that $fX \cup gX \subseteq hX$ and let $\alpha : X \times X \to [0, \infty)$ be a function. The ordered pair (f, g) is said to be:*
(a) weakly α-admissible with respect to h if and only if for all $x \in X$, $\alpha(fx, gy) \geq 1$ for all $y \in h^{-1}(fx)$ and $\alpha(gx, fy) \geq 1$ for all $y \in h^{-1}(gx)$,
(b) partially weakly α-admissible with respect to h if $\alpha(fx, gy) \geq 1$ for all $y \in h^{-1}(fx)$.

Remark 1.5. *In the above definition: (i) if $g = f$, we say that f is weakly α-admissible (partially weakly α-admissible) with respect to h, (ii) if $h = I_X$ (the identity mapping on X), then the above definition reduces to the concepts of weakly α-admissible (partially weakly α-admissible) mapping.*

Definition 1.6. *Let f and g be two self-maps on a set X and let $\alpha : X \times X \to [0, \infty)$ be a function. The weakly α-admissible (partially weakly α-admissible) pair (f, g) is said to be triangular weakly α-admissible (triangular partially weakly α-admissible) if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$ for all $x, y, z \in X$.*

Definition 1.7. *Let X be a set, $f, g, h : X \to X$ are mappings such that $fX \cup gX \subseteq hX$ and let $\alpha : X \times X \to [0, \infty)$ be a function. The ordered pair (f, g) is said to be triangular weakly α-admissible (triangular partially weakly α-admissible) with respect to h if it is weakly α-admissible (partially weakly α-admissible) with respect to h and if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$ for all $x, y, z \in X$.*

Example 1.8. *Let $X = [0, \infty)$,*

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 1, & 1 \leq x \leq \infty, \end{cases} \qquad g(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 1, \\ 1, & 1 \leq x \leq \infty, \end{cases}$$

$$R(x) = \begin{cases} x^3, & 0 \leq x \leq 1, \\ 1, & 1 \leq x \leq \infty, \end{cases} \qquad S(x) = \begin{cases} x^2, & 0 \leq x \leq 1, \\ 1, & 1 \leq x \leq \infty, \end{cases}$$

and let $\alpha(x, y) = e^{y-x}$ for all $x, y \in [0, \infty)$. Then (f, g) is triangular weakly α-admissible with respect to R, and, (g, f) is a triangular weakly α-admissible pair with respect to S. Indeed, if $\begin{cases} \alpha(x,z) \geq 1 \\ \alpha(z,y) \geq 1 \end{cases}$, then $\begin{cases} x - z \leq 0, \\ z - y \leq 0, \end{cases}$ that is, $x - y \leq 0$ and so, $\alpha(x, y) = e^{y-x} \geq 1$.

To prove that (f, g) is partially weakly α-admissible with respect to R, let $x, y \in X$ be such that $y \in R^{-1}fx$, that is, $Ry = fx$. So, we have $x = y^3$ and hence, $y = \sqrt[3]{x}$. As $gy = g(\sqrt[3]{x}) = \sqrt{\sqrt[3]{x}} = \sqrt[6]{x} \geq x = fx$, for all $x \in [0, 1]$, therefore, $\alpha(fx, gy) = e^{gy-fx} = e^{\sqrt[6]{x}-x} \geq 1$. Hence, (f, g) is partially weakly α-admissible with respect to R.

Also, (g, f) is partially weakly α-admissible with respect to S. Indeed, let $x, y \in X$ be such that $y \in S^{-1}gx$, that is, $Sy = gx$. Hence, we have $y^2 = \sqrt{x}$. As $fy = f(\sqrt[4]{x}) = \sqrt[4]{x} \geq \sqrt{x} = gx$, for all $x \in [0, 1]$, therefore, $\alpha(gx, fy) = e^{fy-gx} = e^{\sqrt[4]{x}-\sqrt{x}} \geq 1$. Hence, (g, f) is partially weakly α-admissible with respect to S.

Recently, Hussain et al. [16] introduced the concept of α-completeness for a metric space which is weaker than the concept of completeness.

Definition 1.9. *[16] Let (X, d) be a metric space and let $\alpha : X \times X \to [0, \infty)$ be a mapping. The metric space X is said to be α-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.*

Remark 1.10. *If X is a complete metric space, then X is also an α-complete metric space. But, the converse is not true(see, Example 1.17 of [37]).*

Definition 1.11. *[16] Let (X, d) be a metric space and let $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. We say that T is an α-continuous mapping on (X, d), if, for given $x \in X$ and sequence $\{x_n\}$,*

$$x_n \to x \text{ as } n \to \infty \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \implies Tx_n \to Tx.$$

Example 1.12. *[16] Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume that $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be defined by*

$$Tx = \begin{cases} x^5, & \text{if } x \in [0, 1], \\ \sin \pi x + 2, & \text{if } (1, \infty), \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, T is not continuous, but T is α-continuous on (X, d).

Motivated by [19] we introduce the following concept.

Definition 1.13. *[19] Let (X, d) be a metric space and $f, g : X \to X$. The pair (f, g) is said to be α-compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Remark 1.14. *If (f, g) is a compatible pair, then (f, g) is also an α-compatible pair. But, the converse is not true. The following example which is adapted from example 1.2 of [7] illustrates this fact.*

Example 1.15. *Let $X = [1, \infty)$ and $d(x, y) = |x - y|$. Assume that $f, g : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be defined by*

$$fx = \begin{cases} 2, & \text{if } x \in [1, 2], \\ 6, & \text{if } (2, \infty), \end{cases} \quad gx = \begin{cases} 6 - 2x, & \text{if } x \in [1, 2], \\ 7, & \text{if } (2, \infty), \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} 1, & \text{if } x = y = 2, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, (f, g) is not compatible, but it is an α-compatible pair. Indeed, let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$. Then, $x_n = 2$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2$ and $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = 2$. Again, if we consider the sequence $y_n = 2 - \frac{1}{n}$, then $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = 2$, $\lim_{n\to\infty} gfy_n = 2$ and $\lim_{n\to\infty} fgy_n = 6$. Thus, f and g are α-compatible but not compatible.

Definition 1.16. *[20] Let $f, g : X \to X$ be given self-mappings on X. The pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points (i.e., $fgx = gfx$, whenever $fx = gx$).*

Definition 1.17. *Let (X, d) be a metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. We say that (X, d) is α-regular if the following conditions hold:*
if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

The concept of b-metric space was introduced by Czerwik in [10]. Since then, several papers have been published on the fixed point theory of various classes of operators in b-metric spaces (see, also, [4, 6, 12–14, 21, 28, 29]).

Definition 1.18. *[10] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric iff, for all $x, y, z \in X$, the following conditions are satisfied:*
b_1. $d(x, y) = 0$ iff $x = y$,
b_2. $d(x, y) = d(y, x)$,
b_3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

Definition 1.19. *Let X be a nonempty set. Then $(X, d, \preceq)$ is called a partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X, \preceq)$.*

Recently, Hussain et al. have presented an example of a b-metric which is not continuous (see, example 3 in [12]).

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 1.20. *[2] Let (X, d) be a b-metric space with $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then we have,*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \longrightarrow \infty} d(x_n, y_n) \leq \limsup_{n \longrightarrow \infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n \longrightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s} d(x, z) \leq \liminf_{n \longrightarrow \infty} d(x_n, z) \leq \limsup_{n \longrightarrow \infty} d(x_n, z) \leq s d(x, z).$$

Motivated by the works in [11, 17, 18, 23, 24], we prove some coincidence point results for weakly α-admissible (ψ, φ)-contractive mappings in b-metric and partially ordered b-metric spaces. Our results extend and generalize certain recent results in the literature and provide main results in [23, 24] as corollaries.

2. Main Results

Let (X, d) be a b-metric space and let $f, g, R, S : X \to X$ be four self mappings. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M(x, y) \in \{d(Sx, Ry), \frac{d(Sx, fx) + d(Ry, gy)}{2s}, \frac{d(Sx, gy) + d(Ry, fx)}{2s}\}$$

and

$$N(x, y) = \min\{d(Sx, fx), d(Sx, gy), d(Ry, fx), d(Ry, gy)\}.$$

Throughout this paper, $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions and $\phi : [0, \infty) \to [0, \infty)$ is a bounded function. Recall that a function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if φ is continuous and nondecreasing and $\varphi(t) = 0$ if and only if $t = 0$ [22].

Theorem 2.1. *Let (X, d) be an α-complete b-metric space and let $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$ and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that for every $x, y \in X$ with $\alpha(Sx, Ry) \geq 1$,*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y). \tag{1}$$

Assume that f, g, R and S are α-continuous, the pairs (f, S) and (g, R) are α-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to R and S, respectively. Then, the pairs (f, S) and (g, R) have a coincidence point z in X. Moreover, if $\alpha(Sz, Rz) \geq 1$, then z is a coincidence point of f, g, R and S.

Proof. Let x_0 be an arbitrary point of X. Choose $x_1 \in X$ such that $fx_0 = Rx_1$ and $x_2 \in X$ such that $gx_1 = Sx_2$. Continuing this way, construct a sequence $\{z_n\}$ defined by:

$$z_{2n+1} = Rx_{2n+1} = fx_{2n}$$

and

$$z_{2n+2} = Sx_{2n+2} = gx_{2n+1}$$

for all $n \geq 0$.

As $x_1 \in R^{-1}(fx_0)$ and $x_2 \in S^{-1}(gx_1)$ and the pairs (f, g) and (g, f) are partially weakly α-admissible with respect to R and S, respectively, we have,

$$\alpha(Rx_1 = fx_0, gx_1 = Sx_2) \geq 1$$

and

$$\alpha(gx_1 = Sx_2, fx_2 = Rx_3) \geq 1.$$

Repeating this process, we obtain $\alpha(Rx_{2n+1}, Sx_{2n+2}) = \alpha(z_{2n+1}, z_{2n+2}) \geq 1$ for all $n \geq 0$.

We will complete the proof in three steps.

Step I. We will prove that $\lim_{k\to\infty} d(z_k, z_{k+1}) = 0$.

Define $d_k = d(z_k, z_{k+1})$. Suppose that $d_{k_0} = 0$ for some k_0. Then, $z_{k_0} = z_{k_0+1}$. If $k_0 = 2n$, then $z_{2n} = z_{2n+1}$ gives $z_{2n+1} = z_{2n+2}$. Indeed,

$$\begin{aligned} \psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) &= \psi\Big(sd(fx_{2n}, gx_{2n+1})\Big) \\ &\leq \psi\Big(M(x_{2n}, x_{2n+1})\Big) - \varphi\Big(M(x_{2n}, x_{2n+1})\Big) + \phi(N(x_{2n}, x_{2n+1}))N(x_{2n}, x_{2n+1}), \end{aligned} \tag{2}$$

where,

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &\in \{d(Sx_{2n}, Rx_{2n+1}), \frac{d(Sx_{2n}, fx_{2n}) + d(Rx_{2n+1}, gx_{2n+1})}{2s}, \frac{d(Sx_{2n}, gx_{2n+1}) + d(Rx_{2n+1}, fx_{2n})}{2s}\} \\ &= \{d(z_{2n}, z_{2n+1}), \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}, \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\} \\ &= \{0, \frac{d(z_{2n+1}, z_{2n+2})}{2s}, \frac{d(z_{2n}, z_{2n+2})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} &N(x_{2n}, x_{2n+1}) \\ &= \min\{d(Sx_{2n}, fx_{2n}), d(Sx_{2n}, gx_{2n+1}), d(Rx_{2n+1}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1})\} \\ &= \min\{d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+2}), d(z_{2n+1}, z_{2n+1}), d(z_{2n+1}, z_{2n+2})\} = 0. \end{aligned}$$

If $M(x_{2n}, x_{2n+1}) = \frac{d(z_{2n+1}, z_{2n+2})}{2s}$, then (2) will be,

$$\begin{aligned} \psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) &\leq \psi\Big(\frac{d(z_{2n+1}, z_{2n+2})}{2s}\Big) - \varphi\Big(\frac{d(z_{2n+1}, z_{2n+2})}{2s}\Big) + \phi(0) \times 0 \\ &\leq \psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) - \varphi\Big(\frac{d(z_{2n+1}, z_{2n+2})}{2s}\Big), \end{aligned} \tag{3}$$

which implies that $\varphi(\frac{d(z_{2n+1}, z_{2n+2})}{2s}) = 0$, that is, $z_{2n} = z_{2n+1} = z_{2n+2}$. Similarly, if $k_0 = 2n + 1$, then $z_{2n+1} = z_{2n+2}$ gives $z_{2n+2} = z_{2n+3}$. Continuing this process, we find that z_k is a constant sequence for $k \geq k_0$. Hence, $\lim_{k\to\infty} d(z_k, z_{k+1}) = 0$ holds true.

Now, suppose that

$$d_k = d(z_k, z_{k+1}) > 0 \tag{4}$$

for each k. We claim that

$$d(z_{k+1}, z_{k+2}) \leq d(z_k, z_{k+1}) \tag{5}$$

for each $k = 1, 2, 3, \cdots$.

Let $k = 2n$ and for an $n \geq 0$, $d(z_{2n+1}, z_{2n+2}) \geq d(z_{2n}, z_{2n+1}) > 0$. Then, as $\alpha(Sx_{2n}, Rx_{2n+1}) \geq 1$, using (1) we obtain that

$$\begin{aligned} \psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) &= \psi\Big(sd(fx_{2n}, gx_{2n+1})\Big) \\ &\leq \psi\Big(M(x_{2n}, x_{2n+1})\Big) - \varphi\Big(M(x_{2n}, x_{2n+1})\Big) + \phi(N(x_{2n}, x_{2n+1}))N(x_{2n}, x_{2n+1}), \end{aligned} \tag{6}$$

where,

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &\in \{d(Sx_{2n}, Rx_{2n+1}), \frac{d(Sx_{2n}, fx_{2n}) + d(Rx_{2n+1}, gx_{2n+1})}{2s}, \frac{d(Sx_{2n}, gx_{2n+1}) + d(Rx_{2n+1}, fx_{2n})}{2s}\} \\ &= \{d(z_{2n}, z_{2n+1}), \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}, \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} &N(x_{2n}, x_{2n+1}) \\ &= \min\{d(Sx_{2n}, fx_{2n}), d(Sx_{2n}, gx_{2n+1}), d(Rx_{2n+1}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1})\} \\ &= \min\{d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+2}), d(z_{2n+1}, z_{2n+1}), d(z_{2n+1}, z_{2n+2})\} = 0. \end{aligned}$$

If

$$M(x_{2n}, x_{2n+1}) = \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s} \leq \frac{d(z_{2n+1}, z_{2n+2})}{s},$$

as $d(z_{2n+1}, z_{2n+2}) \geq d(z_{2n}, z_{2n+1})$, then from (6), we have,

$$\begin{aligned} &\psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) \\ &\leq \psi\Big(\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}\Big) - \varphi\Big(\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}\Big) \\ &\leq \psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) - \varphi\Big(\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}\Big), \end{aligned} \tag{7}$$

which implies that, $\varphi\Big(\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}\Big) \leq 0$, this is possible only if

$$\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s} = 0,$$

that is, $d(z_{2n}, z_{2n+1}) = 0$, a contradiction to (4). Hence, $d(z_{2n+1}, z_{2n+2}) \leq d(z_{2n}, z_{2n+1})$ for all $n \geq 0$.

Therefore, (5) is proved for $k = 2n$.

Similarly, it can be shown that,

$$d(z_{2n+2}, z_{2n+3}) \leq d(z_{2n+1}, z_{2n+2}) \tag{8}$$

for all $n \geq 0$.

Analogously, for other values of $M(x_{2n}, x_{2n+1})$, we can see that $\{d(z_k, z_{k+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an $r \geq 0$ such that

$$\lim_{k\to\infty} d(z_k, z_{k+1}) = r. \tag{9}$$

We know that,

$$M(x_{2n}, x_{2n+1}) \in \{d(z_{2n}, z_{2n+1}), \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}, \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\}.$$

Substituting the values of $M(x_{2n}, x_{2n+1})$ in (6) and then taking the limit as $n \to \infty$ in (6), we obtain that $r = 0$. For instance, let

$$M(x_{2n}, x_{2n+1}) = \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}.$$

So, from (6) we have

$$\begin{aligned} &\psi\Big(sd(z_{2n+1}, z_{2n+2})\Big) \\ &\leq \psi\Big(\frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\Big) - \varphi\Big(\frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\Big) \\ &= \psi\Big(\frac{d(z_{2n}, z_{2n+2})}{2s}\Big) - \varphi\Big(\frac{d(z_{2n}, z_{2n+2})}{2s}\Big) \\ &\leq \psi\Big(\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\Big) - \varphi\Big(\frac{d(z_{2n}, z_{2n+2})}{2s}\Big). \end{aligned} \tag{10}$$

Letting $n \to \infty$ in (10), using (9) and the continuity of ψ and φ, we have,

$$\varphi\Big(\lim_{n\to\infty} \frac{d(z_{2n}, z_{2n+2})}{2s}\Big) = 0.$$

Hence, $\lim_{n\to\infty} \frac{d(z_{2n}, z_{2n+2})}{2s} = 0$, from our assumptions about φ.

Now, taking into account (10) and letting $n \to \infty$, we find that $\psi\big(sr\big) \leq \psi\big(0\big) - \varphi\big(0\big)$. Hence, $r = 0$. In general, for the other values of $M(x_{2n}, x_{2n+1})$ we can show that,

$$r = \lim_{k\to\infty} d(z_k, z_{k+1}) = \lim_{n\to\infty} d(z_{2n}, z_{2n+1}) = 0. \tag{11}$$

Step II. We will show that $\{z_n\}$ is a b-Cauchy sequence in X. Assume on contrary that, there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{2m(k)}\}$ and $\{z_{2n(k)}\}$ of $\{z_{2n}\}$ such that $n(k) > m(k) \geq k$ and

$$d(z_{2m(k)}, z_{2n(k)}) \geq \varepsilon \tag{12}$$

and $n(k)$ is the smallest number such that the above condition holds; *i.e.*,

$$d(z_{2m(k)}, z_{2n(k)-1}) < \varepsilon. \tag{13}$$

From triangle inequality and (12) and (13), we have,

$$\varepsilon \leq d(z_{2m(k)}, z_{2n(k)}) \leq s[d(z_{2m(k)}, z_{2n(k)-1}) + d(z_{2n(k)-1}, z_{2n(k)})]. \tag{14}$$

Taking the limit as $k \to \infty$ in (14), from (11) we obtain that,

$$\varepsilon \leq \limsup_{k\to\infty} d(z_{2m(k)}, z_{2n(k)}) \leq s\varepsilon. \tag{15}$$

Using triangle inequality again we have,

$$d(z_{2m(k)}, z_{2n(k)}) \leq s[d(z_{2m(k)}, z_{2m(k)+1}) + d(z_{2m(k)+1}, z_{2n(k)})].$$

Making $k \to \infty$ in the above inequality, we have,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(z_{2m(k)+1}, z_{2n(k)}). \tag{16}$$

Finally,

$$d(z_{2m(k)+1}, z_{2n(k)-1}) \leq s[d(z_{2m(k)+1}, z_{2m(k)}) + d(z_{2m(k)}, z_{2n(k)-1})].$$

Letting $k \to \infty$, and using (15), we have,

$$\limsup_{k\to\infty} d(z_{2m(k)+1}, z_{2n(k)-1}) \leq s\varepsilon. \tag{17}$$

We know that $2n(k) - 1 \geq 2m(k)$ and $\alpha(Sx_{2n+2}, Rx_{2n+1}) = \alpha(gx_{2n+1}, fx_{2n}) \geq 1$ for all $n \in \mathbb{N}$. On the other hand, the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to R and S, respectively. So, $\alpha(Rx_{2n(k)-1}, Sx_{2n(k)-2}) \geq 1$ and $\alpha(Sx_{2n(k)-2}, Rx_{2n(k)-3}) \geq 1$ implies $\alpha(Rx_{2n(k)-1}, Rx_{2n(k)-3}) \geq 1$. Also, $\alpha(Rx_{2n(k)-1}, Rx_{2n(k)-3}) \geq 1$ and $\alpha(Rx_{2n(k)-3}, Sx_{2n(k)-4}) \geq 1$ implies that $\alpha(Rx_{2n(k)-1}, Sx_{2n(k)-4}) \geq 1$. Continuing this manner, we obtain that $\alpha(Rx_{2n(k)-1}, Sx_{2m(k)}) \geq 1$. Now we can apply (1), to obtain that

$$\begin{aligned} \psi\Big(sd(z_{2m(k)+1}, z_{2n(k)})\Big) &= \psi\Big(sd(fx_{2m(k)}, gx_{2n(k)-1})\Big) \\ &\leq \psi\Big(M(x_{2m(k)}, x_{2n(k)-1})\Big) - \varphi\Big(M(x_{2m(k)}, x_{2n(k)-1})\Big) \\ &+\phi(N(x_{2m(k)}, x_{2n(k)-1}))N(x_{2m(k)}, x_{2n(k)-1}), \end{aligned} \tag{18}$$

where,

$$\begin{aligned} &M(x_{2m(k)}, x_{2n(k)-1}) \\ &\in \{d(Sx_{2m(k)}, Rx_{2n(k)-1}), \frac{d(Sx_{2m(k)}, fx_{2m(k)}) + d(Rx_{2n(k)-1}, gx_{2n(k)-1})}{2s}, \\ &\frac{d(Sx_{2m(k)}, gx_{2n(k)-1}) + d(Rx_{2n(k)-1}, fx_{2m(k)})}{2s}\} \\ &= \{d(z_{2m(k)}, z_{2n(k)-1}), \frac{d(z_{2m(k)}, z_{2m(k)+1}) + d(z_{2n(k)-1}, z_{2n(k)})}{2s}, \\ &\frac{d(z_{2m(k)}, z_{2n(k)}) + d(z_{2n(k)-1}, z_{2m(k)+1})}{2s}\} \end{aligned}$$

and

$$\begin{aligned} &N(x_{2m(k)}, x_{2n(k)-1}) \\ &= \min\{d(Sx_{2m(k)}, fx_{2m(k)}), d(Sx_{2m(k)}, gx_{2n(k)-1}), d(Rx_{2n(k)-1}, fx_{2m(k)}), d(Rx_{2n(k)-1}, gx_{2n(k)-1})\} \\ &= \min\{d(z_{2m(k)}, z_{2m(k)+1}), d(z_{2m(k)}, z_{2n(k)}), d(z_{2n(k)-1}, z_{2m(k)+1}), d(z_{2n(k)-1}, z_{2n(k)})\}. \end{aligned}$$

From (11), clearly $N(x_{2m(k)}, x_{2n(k)-1}) \longrightarrow 0$.

If

$$M(x_{2m(k)}, x_{2n(k)-1}) = \frac{d(z_{2m(k)}, z_{2m(k)+1}) + d(z_{2n(k)-1}, z_{2n(k)})}{2s},$$

then from (11), we get that $\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1}) = 0$. Hence, according to (18) we have, $\lim_{k\to\infty} d(z_{2m(k)+1}, z_{2n(k)}) = 0$, which contradicts (16). If

$$M(x_{2m(k)}, x_{2n(k)-1}) = \frac{d(z_{2m(k)}, z_{2n(k)}) + d(z_{2n(k)-1}, z_{2m(k)+1})}{2s},$$

then from (15) and (17), we get that,

$$\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1}) \le \frac{s\varepsilon + s\varepsilon}{2s} = \varepsilon.$$

Taking the limit as $k \to \infty$ in (18), we have,

$$\begin{aligned}
\psi(\varepsilon) &= \psi(s \cdot \frac{\varepsilon}{s}) \\
&\le \psi\Big(s \limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)})\Big) \\
&\le \psi\Big(\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})\Big) - \varphi\Big(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})\Big) \\
&+ \limsup_{k\to\infty} \phi(N(x_{2m(k)}, x_{2n(k)-1}))N(x_{2m(k)}, x_{2n(k)-1}) \\
&\le \psi(\varepsilon) - \varphi\Big(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})\Big) + 0,
\end{aligned} \tag{19}$$

which implies that $\varphi(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})) \le 0$. Hence, $\liminf_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = 0$, a contradiction to (15).

If

$$M(x_{2m(k)}, x_{2n(k)-1}) = d(x_{2m(k)}, x_{2n(k)-1}),$$

then from (13), by taking the limit as $k \to \infty$ in (18), we have,

$$\begin{aligned}
\psi(\varepsilon) &= \psi(s \cdot \frac{\varepsilon}{s}) \\
&\le \psi\Big(s \limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)})\Big) \\
&\le \psi\Big(\limsup_{k\to\infty} d(z_{2m(k)}, z_{2n(k)-1})\Big) - \varphi\Big(\liminf_{k\to\infty} d(z_{2m(k)}, z_{2n(k)-1})\Big) \\
&\le \psi(\varepsilon) - \varphi(\liminf_{k\to\infty} d(z_{2m(k)}, z_{2n(k)-1})),
\end{aligned} \tag{20}$$

which implies that $\varphi(\liminf_{k\to\infty} d(z_{2m(k)}, z_{2n(k)-1})) \le 0$. Hence, $\liminf_{k\to\infty} d(z_{2m(k)}, z_{2n(k)-1}) = 0$. Therefore, from triangular inequality we can conclude that $\liminf_{k\to\infty} d(z_{2m(k)}, z_{2n(k)}) = 0$ which contradicts (15).

Hence $\{z_n\}$ is a b-Cauchy sequence.

Step III. We will show that f, g, R and S have a coincidence point.

Since $\{z_n\}$ is a b-Cauchy sequence in the α-complete b-metric space X and $\alpha(z_k, z_{k+1}) \ge 1$, then there exists $z \in X$ such that,

$$\lim_{n\to\infty} d(z_{2n+1}, z) = \lim_{n\to\infty} d(Rx_{2n+1}, z) = \lim_{n\to\infty} d(fx_{2n}, z) = 0 \tag{21}$$

and

$$\lim_{n\to\infty} d(z_{2n}, z) = \lim_{n\to\infty} d(Sx_{2n}, z) = \lim_{n\to\infty} d(gx_{2n-1}, z) = 0. \tag{22}$$

Hence,

$$Sx_{2n} \to z \text{ and } fx_{2n} \to z, \quad \text{as } n \to \infty. \tag{23}$$

As (f, S) is α-compatible and $\alpha(z_{2n}, z_{2n+2}) \ge 1$, so,

$$\lim_{n\to\infty} d(Sfx_{2n}, fSx_{2n}) = 0. \tag{24}$$

Moreover, from $\lim_{n\to\infty} d(fx_{2n}, z) = 0$, $\lim_{n\to\infty} d(Sx_{2n}, z) = 0$ and the α-continuity of S and f, we obtain that

$$\lim_{n\to\infty} d(Sfx_{2n}, Sz) = 0 = \lim_{n\to\infty} d(fSx_{2n}, fz). \tag{25}$$

By the triangle inequality, we have,

$$\begin{aligned} d(Sz, fz) &\leq s[d(Sz, Sfx_{2n}) + d(Sfx_{2n}, fz)] \\ &\leq sd(Sz, Sfx_{2n}) + s^2[d(Sfx_{2n}, fSx_{2n}) + d(fSx_{2n}, fz)]. \end{aligned} \tag{26}$$

Taking the limit as $n \to \infty$ in (26), we obtain that

$$d(Sz, fz) \leq 0,$$

which yields that $fz = Sz$, that is, z is a coincidence point of f and S.

Similarly, it can be proved that $gz = Rz$. Now, let $\alpha(Rz, Sz) \geq 1$. From (1) we have,

$$\psi\Big(sd(fz, gz)\Big) \leq \psi\Big(M(z, z)\Big) - \varphi\Big(M(z, z)\Big) + \phi(N(z, z))N(z, z), \tag{27}$$

where,

$$\begin{aligned} M(z, z) &\in \{d(Sz, Rz), \frac{d(Sz, fz) + d(Rz, gz)}{2s}, \frac{d(Sz, gz) + d(Rz, fz)}{2s}\} \\ &= \{d(fz, gz), 0, \frac{d(fz, gz)}{s}\} \end{aligned}$$

and

$$N(z, z) = \min\{d(Sz, fz), d(Sz, gz), d(Rz, fz), d(Rz, gz)\} = 0.$$

In all three cases, (27) yields that $fz = gz = Sz = Rz$. □

In the following theorem, we omit the assumption of α-continuity of f, g, R and S and replace the α-compatibility of the pairs (f, S) and (g, R) by weak compatibility of the pairs.

Theorem 2.2. *Let (X, d) be an α-regular α-complete b-metric space, $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ and RX and SX are b-closed subsets of X. Suppose that*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y), \tag{28}$$

for all x and y with $\alpha(Sx, Ry) \geq 1$. Then, the pairs (f, S) and (g, R) have a coincidence point z in X provided that the pairs (f, S) and (g, R) are weakly compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to R and S, respectively. Moreover, if $\alpha(Sz, Rz) \geq 1$, then $z \in X$ is a coincidence point of f, g, R and S.

Proof. Following the proof of Theorem 2.1, there exists $z \in X$ such that:

$$\lim_{k\to\infty} d(z_k, z) = 0. \tag{29}$$

Since $R(X)$ is b-closed and $\{z_{2n+1}\} \subseteq R(X)$, therefore $z \in R(X)$. Hence, there exists $u \in X$ such that $z = Ru$ and

$$\lim_{n\to\infty} d(z_{2n+1}, Ru) = \lim_{n\to\infty} d(Rx_{2n+1}, Ru) = 0. \tag{30}$$

Similarly, there exists $v \in X$ such that $z = Ru = Sv$ and

$$\lim_{n\to\infty} d(z_{2n}, Sv) = \lim_{n\to\infty} d(Sx_{2n}, Sv) = 0. \tag{31}$$

Now, we prove that v is a coincidence point of f and S.

Since $Rx_{2n+1} \to z = Sv$, as $n \to \infty$, from α-regularity of X, $\alpha(Rx_{2n+1}, Sv) \geq 1$. Therefore, from (28), we have

$$\psi\Big(sd(fv, gx_{2n+1})\Big) \leq \psi\Big(M(v, x_{2n+1})\Big) - \varphi\Big(M(v, x_{2n+1})\Big) + \phi(N(v, x_{2n+1}))N(v, x_{2n+1}), \tag{32}$$

where,

$$\begin{aligned} &M(v, x_{2n+1}) \\ &\in \{d(Sv, Rx_{2n+1}), \frac{d(Sv, fv) + d(Rx_{2n+1}, gx_{2n+1})}{2s}, \frac{d(Sv, gx_{2n+1}) + d(Rx_{2n+1}, fv)}{2s}\} \\ &= \{d(z, z_{2n+1}), \frac{d(z, fv) + d(z_{2n+1}, z_{2n})}{2s}, \frac{d(z, z_{2n}) + d(z_{2n+1}, fv)}{2s}\} \end{aligned}$$

and

$$\begin{aligned} &N(v, x_{2n+1}) \\ &= \min\{d(Sv, fv), d(Sv, gx_{2n+1}), d(Rx_{2n+1}, fv), d(Rx_{2n+1}, gx_{2n+1})\} \\ &= \min\{d(z, fv), d(z, z_{2n}), d(z_{2n+1}, fv), d(z_{2n+1}, z_{2n})\} \to 0. \end{aligned}$$

From Lemma 1.20,

$$\frac{d(z, fv)}{2s^2} \leq \liminf_n M(v, x_{2n+1}) \leq \limsup_n M(v, x_{2n+1}) \leq \frac{d(z, fv)}{2}.$$

Taking the limit as $n \to \infty$ in (32), using Lemma 1.20 and the continuity of ψ and φ, we can obtain that $fv = z = Sv$.

As f and S are weakly compatible, we have $fz = fSv = Sfv = Sz$. Thus, z is a coincidence point of f and S.

Similarly, it can be shown that z is a coincidence point of the pair (g, R). The rest of the proof follows from similar arguments as in Theorem 2.1. □

Taking $S = R$ in Theorem 2.1, we obtain the following result.

Corollary 2.3. *Let (X, d) be an α-complete b-metric space and let $f, g, R : X \to X$ be three mappings such that $f(X) \cup g(X) \subseteq R(X)$ and R is α-continuous. Suppose that for every $x, y \in X$ with $\alpha(Rx, Ry) \geq 1$, we have,*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y), \tag{33}$$

where,

$$M(x, y) \in \{d(Rx, Ry), \frac{d(Rx, fx) + d(Ry, gy)}{2s}, \frac{d(Rx, gy) + d(Ry, fx)}{2s}\}$$

and

$$N(x, y) = \min\{d(Rx, fx), d(Rx, gy), d(Ry, fx), d(Ry, gy)\}.$$

Then, f, g and R have a coincidence point in X provided that the pair (f, g) is triangular weakly α-admissible with respect to R and either,

a. the pair (f, R) is α-compatible and f is α-continuous, or,

b. the pair (g, R) is α-compatible and g is α-continuous.

Taking $R = S$ and $f = g$ in Theorem 2.1, we obtain the following coincidence point result:

Corollary 2.4. *Let (X, d) be an α-complete b-metric space and let $f, R : X \to X$ be two mappings such that $f(X) \subseteq R(X)$. Suppose that for every $x, y \in X$ with $\alpha(Rx, Ry) \geq 1$, we have,*

$$\psi\Big(sd(fx, fy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y), \tag{34}$$

where,

$$M(x,y) \in \{d(Rx,Ry), \frac{d(Rx,fx)+d(Ry,fy)}{2s}, \frac{d(Rx,fy)+d(Ry,fx)}{2s}\}$$

and

$$N(x,y) = \min\{d(Rx,fx), d(Rx,fy), d(Ry,fx), d(Ry,fy)\}.$$

Then, the pair (f,R) has a coincidence point in X provided that f and R are α-continuous, the pair (f,R) is α-compatible and f is triangular weakly α-admissible with respect to R.

Example 2.5. *Let $X=[0,\infty)$, the metric d on X be given by $d(x,y)=\left|x-y\right|^2$, for all $x,y\in X$ and $\alpha : X\times X\to[0,\infty)$ be given by $\alpha(x,y)=e^{x-y}$. Define self-maps f, g, S and R on X by*

$$fx=\ln(1+x),\quad Rx=e^x-1,$$
$$gx=\ln(1+\frac{x}{2}),\quad Sx=e^{2x}-1.$$

To prove that (f,g) is partially weakly α-admissible with respect to R, let $x,y\in X$ be such that $y\in R^{-1}fx$, that is, $Ry=fx$. By the definition of f and R, we have $e^y-1=\ln(1+x)$ and so, $y=\ln(1+\ln(1+x))$. Therefore,

$$fx=\ln(1+x)\geq\ln(1+\frac{\ln(1+\ln(1+x))}{2})=\ln(1+\frac{y}{2})=gy.$$

Therefore, $\alpha(fx,gy)\geq 1$. Hence (f,g) is partially weakly α-admissible with respect to R.

To prove that (g,f) is partially weakly α-admissible with respect to S, let $x,y\in X$ be such that $y\in S^{-1}gx$, that is, $Sy=gx$. Hence, we have $e^{2y}-1=\ln(1+\frac{x}{2})$ and so, $y=\frac{\ln(1+\ln(1+\frac{x}{2}))}{2}$. Therefore,

$$gx=\ln(1+\frac{x}{2})\geq\ln(1+\frac{\frac{\ln(1+\ln(1+\frac{x}{2}))}{2}}{2})=\ln(1+y)=fy.$$

Therefore, $\alpha(gx,fy)\geq 1$.

Furthermore, $fX=gX=SX=RX=[0,\infty)$.

Define $\psi,\varphi:[0,\infty)\to[0,\infty)$ as $\psi(t)=bt$ and $\varphi(t)=(b-1)t$ for all $t\in[0,\infty)$, where $1<b\leq 22$.

Using the mean value theorem, for all x and y with $\alpha(Sx,Ry)\geq 1$ we have,

$$\begin{aligned}
\psi(2d(fx,gy)) &= 2b\left|fx-gy\right|^2\\
&= 2b\left|\ln(1+x)-\ln(1+\frac{y}{2})\right|^2\\
&\leq 2b\left|x-\frac{y}{2}\right|^2\\
&\leq 2b\frac{\left|2x-y\right|^2}{4}\\
&\leq \frac{2b}{4}\left|e^{2x}-1-(e^y-1)\right|^2\\
&\leq \left|Sx-Ry\right|^2\\
&= d(Sx,Ry)\\
&= \psi(d(Sx,Ry))-\varphi(d(Sx,Ry))+\phi(N(x,y))N(x,y).
\end{aligned}$$

Thus, (1) is true for all $x,y\in X$ and $M(x,y)=d(Sx,Ry)$. Therefore, all the conditions of Theorem 2.1 are satisfied. Moreover, 0 is a coincidence point of f, g, R and S. □

Corollary 2.6. *Let (X, d) be an α-regular b-metric space, $f, g, R : X \to X$ be three mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq R(X)$ and RX is a b-closed subset of X. Suppose that for all elements x and y with $\alpha(Rx, Ry) \geq 1$, we have,*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y), \tag{35}$$

where

$$M(x, y) \in \{d(Rx, Ry), \frac{d(Rx, fx) + d(Ry, gy)}{2s}, \frac{d(Rx, gy) + d(Ry, fx)}{2s}\}$$

and

$$N(x, y) = \min\{d(Rx, fx), d(Rx, gy), d(Ry, fx), d(Ry, gy)\}.$$

Then, the pairs (f, R) and (g, R) have a coincidence point z in X provided that the pairs (f, R) and (g, R) are weakly compatible and the pair (f, g) is triangular weakly α-admissible with respect to R. Moreover, if $\alpha(Rz, Rz) \geq 1$, then $z \in X$ is a coincidence point of f, g and R.

Corollary 2.7. *Let (X, d) be an α-regular b-metric space, $f, R : X \to X$ be two mappings such that $f(X) \subseteq R(X)$ and RX is a b-closed subset of X. Suppose that for all elements x and y with $\alpha(Rx, Ry) \geq 1$, we have,*

$$\psi\Big(sd(fx, fy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y), \tag{36}$$

where

$$M(x, y) \in \{d(Rx, Ry), \frac{d(Rx, fx) + d(Ry, fy)}{2s}, \frac{d(Rx, fy) + d(Ry, fx)}{2s}\}$$

and

$$N(x, y) = \min\{d(Rx, fx), d(Rx, fy), d(Ry, fx), d(Ry, fy)\}.$$

Then, the pair (f, R) have a coincidence point z in X provided that the pair (f, R) is weakly compatible and f is triangular weakly α-admissible with respect to R.

Taking $R = S = I_X$ (the identity mapping on X) in Theorems 2.1 and 2.2, we obtain the following common fixed point result.

Corollary 2.8. *Let (X, d) be an α-complete b-metric space and let $f, g : X \to X$ be two mappings. Suppose that for every elements $x, y \in X$ with $\alpha(x, y) \geq 1$,*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y), \tag{37}$$

where,

$$M(x, y) \in \{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\},$$

and

$$N(x, y) = \min\{d(x, fx), d(x, gy), d(y, fx), d(y, gy)\}.$$

Then, the pair (f, g) have a common fixed point z in X provided that the pair (f, g) is triangular weakly α-admissible and either,

a. f or g is α-continuous, or,

b. X is α-regular.

Remark 2.9. *1. In all obtained results in this paper, we can replace $M(x, y)$ by $O(x, y)$, where,*

$$O(x, y) = \max\{d(Sx, Ry), d(Sx, fx), d(Ry, gy), \frac{d(Sx, gy) + d(Ry, fx)}{2s}\}.$$

2.In all obtained results in this paper, we can replace $N(x, y)$ by $P(x, y)$, where,

$$P(x, y) = d(Rx, fx) \times d(Rx, gy) \times d(Ry, fx) \times d(Ry, gy).$$

3. Consequences in Partially Ordered b-Metric Spaces

In this section, we give some common fixed point results on metric spaces endowed with an arbitrary binary relation, specially a partial order relation which can be regarded as consequences of the results presented in the previous section.

In the sequel, let (X, d) be a metric space and let $\mathcal{R}$ be a transitive binary relation over X.

Definition 3.1. *Let f and g be two selfmaps on X and $\mathcal{R}$ be a binary relation over X. A pair (f, g) is said to be,*
(i) weakly $\mathcal{R}$-increasing if $fx\mathcal{R}gfx$ and $gx\mathcal{R}fgx$ for all $x \in X$,
(ii) partially weakly $\mathcal{R}$-increasing if $fx\mathcal{R}gfx$ for all $x \in X$.

Definition 3.2. *Let $\mathcal{R}$ be a binary relation over X and let $f, g, h : X \to X$ are mappings such that $fX \cup gX \subseteq hX$. The ordered pair (f, g) is said to be:*
(a) weakly $\mathcal{R}$-increasing with respect to h if and only if for all $x \in X$, $fx\mathcal{R}gy$ for all $y \in h^{-1}(fx)$ and $gx\mathcal{R}fy$ for all $y \in h^{-1}(gx)$,
(b) partially weakly $\mathcal{R}$-increasing with respect to h if $fx\mathcal{R}gy$ for all $y \in h^{-1}(fx)$.

Let $\mathcal{R}$ be a binary relation over X and let

$$\alpha(x, y) = \begin{cases} 1, & x\mathcal{R}y, \\ 0, & \textit{otherwise}. \end{cases}$$

By this assumption, we see that the above definitions are special cases from the definition of weak α-admissibility and partially weak α-admissibility.

Definition 3.3. *[37] Let (X, d) be a metric space. The metric space X is said to be $\mathcal{R}$-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}$, converges in X.*

Definition 3.4. *[37] Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We say that T is an $\mathcal{R}$-continuous mapping on (X, d), if, for given $x \in X$ and sequence $\{x_n\}$ with $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}$,*

$$x_n \to x \text{ as } n \to \infty \text{ for all } n \in \mathbb{N} \implies Tx_n \to Tx.$$

Definition 3.5. *Let (X, d) be a metric space and let $f, g : X \to X$. The pair (f, g) is said to be $\mathcal{R}$-compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Definition 3.6. *Let $\mathcal{R}$ be a binary relation over X and let d be a metric on X. We say that $(X, d, \mathcal{R})$ is $\mathcal{R}$-regular if the following condition hold:*
if a sequence $x_n \to x$ where where $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n\mathcal{R}x$ for all $n \in \mathbb{N}$.

Taking $\mathcal{R} = \preceq$ where $\preceq$ is a partial order on the non-empty set X, we have

Corollary 3.7. *a) Theorem 2.1 of [24] is a special case of Corollary 2.3.*
b) Theorem 2.2 of [24] is a special case of Corollary 2.6.
c) Corollary 2.1 of [24] is a special case of Corollary 2.8.
d) Corollary 2.2 of [24] is a special case of Corollary 2.8.
e) Theorem 2.4 of [23] is a special case of Corollary 2.4.
f) Theorem 2.6 of [23] is a special case of Corollary 2.7.
g) Corollary 2.7 of [23] is a special case of Corollary 2.3 with $R = I_X$.

4. Contractive Mappings on b-Metric Spaces Endowed with a Graph

Consistent with Jachymski [35], let (X,d) be a b-metric space and Δ denotes the diagonal of the Cartesian product $X\times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G)\supseteq\Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph (see [36], p. 309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N\in\mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0=x$, $x_N=y$ and $(x_{i-1},x_i)\in E(G)$ for $i=1,\dots,N$.

Recently, some results have appeared in the setting of metric spaces which are endowed with a graph. The first result in this direction was given by Jachymski [35].

Definition 4.1. *Let f and g be two selfmaps on graphic b-metric space (X,d). The pair (f,g) is said to be,*
(i) weakly G-increasing if $(fx,gfx)\in E(G)$ and $(gx,fgx)\in E(G)$ for all $x\in X$,
(ii) partially weakly G-increasing if $(fx,gfx)\in E(G)$ for all $x\in X$.

Definition 4.2. *Let (X,d) be a graphic b-metric space and let $f,g,h:X\to X$ are mappings such that $fX\cup gX\subseteq hX$. The ordered pair (f,g) is said to be:*
(a) weakly G-increasing with respect to h if and only if for all $x\in X$, $(fx,gy)\in E(G)$ for all $y\in h^{-1}(fx)$ and $(gx,fy)\in E(G)$ for all $y\in h^{-1}(gx)$,
(b) partially weakly G-increasing with respect to h if $(fx,gy)\in E(G)$ for all $y\in h^{-1}(fx)$.

Let (X,d) be a graphic b-metric space and let

$$\alpha(x,y)=\begin{cases}1, & (x,y)\in E(G),\\ 0, & otherwise.\end{cases}$$

By this assumption, we see that the above definitions are special cases from the definition of weak α-admissibility and partially weak α-admissibility.

Definition 4.3. *[37] Let (X,d) be a graphic metric space. (X,d) is said to be G-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, converges in X.*

Definition 4.4. *[37] Let (X,d) be a graphic metric space and let $T:X\to X$ be a mapping. We say that T is an G-continuous mapping on (X,d), if, for given $x\in X$ and sequence $\{x_n\}$ with $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$,*

$$x_n\to x \text{ as } n\to\infty \text{ for all } n\in\mathbb{N} \implies Tx_n\to Tx.$$

Definition 4.5. *Let (X,d) be a graphic metric space and let $f,g:X\to X$. The pair (f,g) is said to be G-compatible if $\lim_{n\to\infty}d(fgx_n,gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=t$ for some $t\in X$.*

Definition 4.6. *Let $\mathcal{R}$ be a binary relation over X and let d be a metric on X. We say that $(X,d,\mathcal{R})$ is $\mathcal{R}$-regular if the following condition hold:*
if a sequence $x_n\to x$ where where $x_n\mathcal{R}x_{n+1}$ for all $n\in\mathbb{N}$, then $x_n\mathcal{R}x$ for all $n\in\mathbb{N}$.

Definition 4.7. *Let (X,d) be a graphic b-metric space. We say that (X,d) is G-regular if the following condition holds:*
if a sequence $x_n\to x$ with $(x_n,x_{n+1})\in E(G)$, then $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$.

In the following theorems, we assume that:
for all $(x,y)\in E(G)$ and $(y,z)\in E(G)$, we have $(x,z)\in E(G)$.

Theorem 4.8. *Let (X, G, d) be a G-complete graphic b-metric space. Let $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$. Suppose that for every $x, y \in X$ such that $(Sx, Ry) \in E(G)$, we have,*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y).$$

Let f, g, R and S are G-continuous, the pairs (f, S) and (g, R) are G-compatible and the pairs (f, g) and (g, f) are partially weakly G-increasing with respect to R and S, respectively. Then, the pairs (f, S) and (g, R) have a coincidence point z in X. Moreover, if $(Sz, Rz) \in E(G)$, then z is a coincidence point of f, g, R and S.

Theorem 4.9. *Let (X, G, d) be a G-regular G-complete graphic b-metric space, $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ and RX and SX are b-closed subsets of X. Suppose that*

$$\psi\Big(sd(fx, gy)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(M(x, y)\Big) + \phi(N(x, y))N(x, y),$$

for all x and y for which $(Sx, Ry) \in E(G)$. Then, the pairs (f, S) and (g, R) have a coincidence point z in X provided that the pairs (f, S) and (g, R) are weakly compatible and the pairs (f, g) and (g, f) are partially weakly G-increasing with respect to R and S, respectively. Moreover, if $(Sz, Rz) \in E(G)$, then $z \in X$ is a coincidence point of f, g, R and S.

5. Conclusion

As we know, the concepts of α-complete metric space, α-continuity of a mapping and α-compatibility of a pair of mappings are weaker than the concepts of complete metric space, continuity of a mapping and compatibility of a pair of mappings, respectively. Therefore, Theorems 2.1 and 2.2 are more general than the corresponding results in [38].

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