Filomat 30:2 (2016), 343–351
DOI 10.2298/FIL1602343N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Integral Inequalities for Two-dimensional *pq*-convex Functions

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Abstract. In this paper, we introduce the concept of two dimensional pq-convex functions. We establish several new Hermite-Hadamard inequalities for two dimensional pq-convex functions. Some special cases are also discussed. Results obtained in this paper can be viewed as significant extensions of the previously known results.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a convex function in the classical sense, i.e. a function that satisfies the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

whenever $x, y \in I$ and $t \in [0, 1]$. Here the domain of the function is a convex set. The notion of convex functions is very simple and natural and plays an important role in different fields of pure and applied sciences. Consequently the classical concept of convex functions has been extended and generalized in different directions using novel and innovative ideas, for example interested readers are referred to [1–4, 7, 8, 11, 13, 14, 16, 17, 19, 20, 29, 30]. It is well known that optimality condition for differentiable convex (nonconvex) functions can be characterized by some classes of variational inequalities, see for example [12, 13, 21]. Dragomir [4] considered the Hermite-Hadamard inequalities for convex functions on the coordinates in rectangle. Thus, it is possible to derive several integral inequalities for two dimensional (coordinated) convex functions, see [22–29] and the references therein. Many interesting inequalities in the literature are proved for convex functions. For example extensively studied result which is due to Hermite [10] and Hadamard [9] that is Hermite-Hadamard's inequality. This result can be considered as necessary and sufficient condition for a function to be convex. This inequality has also been extended and generalized for different classes of generalized convex functions, and refined under additional hypotheses, see [1, 3–8, 11, 14–20, 22–29].
Motivated and inspired by the research going on in this field, Zhang and Wan [30] introduced and studied a class of convex function, which is called p-convex functions. Using the concept of p-convex functions,

2010 *Mathematics Subject Classification*. 26D15; 26A51
Keywords. convex functions, two dimensional pq-convex functions, Hermite-Hadamard inequalities
Received: 04 March 2014; Accepted: 11 April 2014
Communicated by Predrag Stanimirović
This research is supported by HEC NRPU project No: 20-1966/R&D/11-2553.
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we introduce and study some new classes of two dimensional convex functions. This class is called *pq*-convex functions. It is shown that the two dimensional *pq*-convex functions include two dimensional convex functions [4] as special cases. We also establish several new Hermite-Hadamard inequalities for two dimensional *pq*-convex functions. Some special cases are also discussed. Results proved in this paper continue to hold for these classes of convex functions. The researchers are encouraged to find novel and innovative applications of *pq*-convex functions in pure and applied sciences.

2. Preliminaries

In this section, we recall some previously known concepts.

Definition 2.1 ([30]). *An interval I is said to be a p-convex set, if*

$$M_p(x,y;t) = [tx^p + (1-t)y^p]^{\frac{1}{p}} \in I, \quad \forall x,y \in I, t \in [0,1] \tag{2.1}$$

where $p = 2k+1$ or $p = \frac{n}{m}, n = 2r+1, m = 2t+1$ and $k,r,t \in \mathbb{N}$.

Definition 2.2 ([30]). *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be p-convex function or belongs to the class $PC(I)$, if*

$$f(M_p(x,y;t)) \leq tf(x) + (1-t)f(y), \ \forall x,y \in I, t \in [0,1].$$

It is obvious that for $p = 1$ Definition 2.2 reduces to the definition for convex functions.

Also note that for $t = \frac{1}{2}$ in Definition 2.2, we have Jensen p-convex functions or mid p-convex functions.

$$f\Big(M_p\big(x,y;1/2\big)\Big) \leq \frac{f(x)+f(y)}{2}, \ \forall x,y \in I, t \in [0,1].$$

Let us consider a bidimensional interval $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be convex function on Δ if the following inequality

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w),$$

holds, for all $(x,y),(z,w) \in \Delta$ and $t \in [0,1]$. This is definition is mainly due to Dragomir [4].

A function $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the partial functions $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex for all $x \in [a,b]$ and $y \in [c,d]$.

Definition 2.3 ([4]). *Let $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \to \mathbb{R}$ is said to be two dimensional (coordinated) convex function, if*

$$\begin{aligned} &f(tx + (1-t)y, ru + (1-r)w) \\ &\leq trf(x,u) + t(1-r)f(x,w) + r(1-t)f(y,u) + (1-t)(1-r)f(y,w), \end{aligned}$$

whenever $x,y \in [a,b]$, $u,w \in [c,d]$ and $t,r \in [0,1]$.

We now introduce the class of two dimensional *pq*-convex functions, which is one of the main motivations of this paper.

Definition 2.4. *Let $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \to \mathbb{R}$, is said to be two dimensional (coordinated) pq-convex function, if*

$$\begin{aligned} &f(M_p(x_1,x_2;t), M_q(y_1,y_2;r)) \\ &\leq trf(x_1,y_1) + t(1-r)f(x_1,y_2) + r(1-t)f(x_2,y_1) + (1-t)(1-r)f(x_2,y_2), \end{aligned} \tag{2.2}$$

whenever $x_1,x_2 \in [a,b]$, $y_1,y_2 \in [c,d]$ and $t,r \in [0,1]$.

We now discuss some special cases of Definition 2.4.

I. If $p = q$, then we have the following new concept.

Definition 2.5. *Let $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \to \mathbb{R}$, is said to be two dimensional (coordinated) p-convex function, if*

$$\begin{aligned}
&f(M_p(x_1, x_2; t), M_p(y_1, y_2; r)) \\
&\leq trf(x_1, y_1) + t(1-r)f(x_1, y_2) + r(1-t)f(x_2, y_1) + (1-t)(1-r)f(x_2, y_2).
\end{aligned}$$

II. If $p = 1 = q$, then Definition 2.4 and Definition 2.5 reduces to Definition 2.3.
This shows that the concept of two dimensional pq-convex functions is quite flexible and unifying one.

3. Main Results

In this section, we discuss our main results.

Theorem 3.1. *Let $f : \Delta \to \mathbb{R}$ be two dimensional pq-convex function, then*

$$\begin{aligned}
&f\Big(M_p\big(a, b; 1/2\big), M_q\big(c, d; 1/2\big)\Big) \\
&\leq \frac{pq}{(b^p - a^p)(d^q - c^q)} \int_a^b \int_c^d x^{p-1} y^{q-1} f(x, y) dy dx \\
&\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

Proof. Let f be two dimensional pq-convex function, then using $t = r = \frac{1}{2}$ and

$$x_1 = (t_1 a^p + (1 - t_1) b^p)^{\frac{1}{p}},\ x_2 = ((1 - t_1) a^p + t_1 b^p)^{\frac{1}{p}},$$

$$y_1 = (r_1 c^q + (1 - r_1) d^q)^{\frac{1}{q}},\ y_2 = ((1 - r_1) c^q + r_1 d^q)^{\frac{1}{q}},$$

in (2.2), we have

$$\begin{aligned}
&f\Big(M_p\big(a, b; 1/2\big), M_q\big(c, d; 1/2\big)\Big) \\
&\leq \frac{1}{4}\Big[f\Big((t_1 a^p + (1 - t_1) b^p)^{\frac{1}{p}}, (r_1 c^q + (1 - r_1) d^q)^{\frac{1}{q}}\Big) \\
&\quad + f\Big((t_1 a^p + (1 - t_1) b^p)^{\frac{1}{p}}, ((1 - r_1) c^q + r_1 d^q)^{\frac{1}{q}}\Big) \\
&\quad + f\Big(((1 - t_1) a^p + t_1 b^p)^{\frac{1}{p}}, (r_1 c^q + (1 - r_1) d^q)^{\frac{1}{q}}\Big) \\
&\quad + f\Big(((1 - t_1) a^p + t_1 b^p)^{\frac{1}{p}}, ((1 - r_1) c^q + r_1 d^q)^{\frac{1}{q}}\Big)\Big].
\end{aligned}$$

Integrating above inequality with respect to (t_1, r_1) on $[0, 1] \times [0, 1]$, we have

$$\begin{aligned}
&f\Big(M_p\big(a, b; 1/2\big), M_q\big(c, d; 1/2\big)\Big) \\
&\leq \frac{pq}{(b^p - a^p)(d^q - c^q)} \int_a^b \int_c^d x^{p-1} y^{q-1} f(x, y) dy dx.
\end{aligned}$$

Also for $x_1 = a$, $x_2 = b$, $y_1 = c$ and $y_2 = d$ and using the definition of two dimensional pq-convex function, we have

$$\begin{aligned}&f(M_p(a,b;t), M_q(c,d;r))\\ &\leq trf(a,c) + t(1-r)f(a,d) + r(1-t)f(b,c) + (1-t)(1-r)f(b,d),\end{aligned}$$

Integrating both sides of the above inequality with respect to (t,r) on $[0,1]\times[0,1]$, we have

$$\frac{pq}{(b^p-a^p)(d^q-c^q)}\int_a^b\int_c^d x^{p-1}y^{q-1}f(x,y)dydx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

This completes the proof. □

Note that if $p = q$ in Theorem 3.1, then we have the following result for two dimensional p-convex functions.

Corollary 3.1. *Let $f : \Delta \to \mathbb{R}$ be two dimensional p-convex function, then*

$$\begin{aligned}&f\Big(M_p\big(a,b;1/2\big), M_p\big(c,d;1/2\big)\Big)\\ &\leq \frac{p^2}{(b^p-a^p)(d^p-c^p)}\int_a^b\int_c^d x^{p-1}y^{p-1}f(x,y)dydx\\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}$$

Remark 3.1. *If $p = 1 = q$, then Theorem 3.1 reduces to a result for two dimensional convex functions, see [4].*

Lemma 3.2. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be partial differentiable function on $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t\partial r} \in L_1(\Delta)$, then*

$$\begin{aligned}&\Psi(t,r;p,q;\Delta)(f)\\ &= \frac{(b^p-a^p)(d^q-c^q)}{4pq}\int_0^1\int_0^1(1-2t)(1-2r)\frac{\partial^2 f}{\partial t\partial r}\Big(M_p(a,b;t), M_q(c,d;r)\Big)dtdr,\end{aligned}$$

where

$$\begin{aligned}&\Psi(t,r;p,q;\Delta)(f)\\ &= \frac{[b^p]^{\frac{1}{p}-1}[d^q]^{\frac{1}{q}-1}f(a,c) + [a^p]^{\frac{1}{p}-1}[d^q]^{\frac{1}{q}-1}f(b,c) + [b^p]^{\frac{1}{p}-1}[c^q]^{\frac{1}{q}-1}f(a,d) + [a^p]^{\frac{1}{p}-1}[c^q]^{\frac{1}{q}-1}f(b,d)}{4[a^p]^{\frac{1}{p}-1}[b^p]^{\frac{1}{p}-1}[c^q]^{\frac{1}{q}-1}[d^q]^{\frac{1}{q}-1}}\\ &- \frac{p}{2(b^p-a^p)}\left[\frac{1}{[c^q]^{\frac{1}{q}-1}}\int_a^b x^{2p-2}f(x,c)dx + \frac{1}{[d^q]^{\frac{1}{q}-1}}\int_a^b x^{2p-2}f(x,d)dx\right]\\ &- \frac{q}{2(b^q-a^q)}\left[\frac{1}{[a^p]^{\frac{1}{p}-1}}\int_c^d y^{2q-2}f(a,y)dy + \frac{1}{[b^p]^{\frac{1}{p}-1}}\int_c^d y^{2q-2}f(b,y)dy\right]\\ &+ \frac{pq}{(b^p-a^p)(d^q-c^q)}\int_a^b\int_c^d x^{2p-2}y^{2q-2}f(x,y)dxdy.\end{aligned}$$

Proof. Consider

$$\int_0^1\int_0^1(1-2t)(1-2r)\frac{\partial^2 f}{\partial t\partial r}\Big([ta^p+(1-t)b^p]^{\frac{1}{p}},[rc^q+(1-r)d^q]^{\frac{1}{q}}\Big)dtdr.$$

This implies

$$I=\int_0^1(1-2t)\left\{\int_0^1(1-2r)\frac{\partial^2 f}{\partial t\partial r}\Big([ta^p+(1-t)b^p]^{\frac{1}{p}},[rc^q+(1-r)d^q]^{\frac{1}{q}}\Big)dr\right\}dt. \tag{3.1}$$

Now integrating by parts, we have

$$\begin{aligned}I_1&=\int_0^1(1-2r)\frac{\partial^2 f}{\partial t\partial r}\Big([ta^p+(1-t)b^p]^{\frac{1}{p}},[rc^q+(1-r)d^q]^{\frac{1}{q}}\Big)dr\\&=\frac{\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},c)}{\frac{1}{q}[c^q]^{\frac{1}{q}-1}(d^q-c^q)}+\frac{\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},d)}{\frac{1}{q}[d^q]^{\frac{1}{q}-1}(d^q-c^q)}\\&\quad+\frac{2}{\frac{1}{q}(d^q-c^q)}\int_0^1\frac{\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},[rc^q+(1-r)d^q]^{\frac{1}{q}})}{[rc^q+(1-r)d^q]^{\frac{1}{q}-1}}dr.\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned}I_2&=\int_0^1(1-2t)\frac{\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},c)}{\frac{1}{q}[c^q]^{\frac{1}{q}-1}(d^q-c^q)}dt\\&=\frac{1}{\frac{1}{q}[c^q]^{\frac{1}{q}}(d^q-c^q)}\left[\left\{\frac{f(a,c)}{\frac{1}{p}[a^p]^{\frac{1}{p}-1}(b^p-a^a)}+\frac{f(b,c)}{\frac{1}{p}[b^p]^{\frac{1}{p}-1}(b^p-a^a)}\right\}\right.\\&\qquad\left.-\frac{2p^2}{(b^p-a^p)^2}\int_a^b x^{2p-2}f(x,c)dx\right].\end{aligned} \tag{3.3}$$

Similarly from (3.1) and (3.2), we have

$$\begin{aligned}I_3&=\int_0^1(1-2t)\frac{\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},d)}{\frac{1}{q}[d^q]^{\frac{1}{q}-1}(d^q-c^q)}dt\\&=\frac{1}{\frac{1}{q}[d^q]^{\frac{1}{q}}(d^q-c^q)}\left[\left\{\frac{f(a,d)}{\frac{1}{p}[a^p]^{\frac{1}{p}-1}(b^p-a^a)}+\frac{f(b,d)}{\frac{1}{p}[b^p]^{\frac{1}{p}-1}(b^p-a^a)}\right\}\right.\\&\qquad\left.-\frac{2p^2}{(b^p-a^p)^2}\int_a^b x^{2p-2}f(x,d)dx\right].\end{aligned} \tag{3.4}$$

Also

$$I_4 = \frac{2}{\frac{1}{q}(d^q - c^q)} \int_0^1 \int_0^1 (1-2t) \frac{\frac{\partial f}{\partial t}([ta^p + (1-t)b^p]^{\frac{1}{p}}, [rc^q + (1-r)d^q]^{\frac{1}{q}})}{[rc^q + (1-r)d^q]^{\frac{1}{q}-1}} drdt$$

$$= \frac{4p^2q^2}{(b^p - a^p)^2(d^q - c^q)^q} \int_a^b \int_c^d x^{2p-2} y^{2q-2} f(x,y) dxdy$$

$$- \frac{2pq^2}{[a^p]^{\frac{1}{p}-1}(b^p - a^p)(d^q - c^q)^2} \int_c^d y^{2q-2} f(a,y) dy$$

$$- \frac{2pq^2}{[b^p]^{\frac{1}{p}-1}(b^p - a^p)(d^q - c^q)^2} \int_c^d y^{2q-2} f(b,y) dy. \tag{3.5}$$

On summation of (3.2), (3.3), (3.4) and (3.5) and multiplying by $\frac{(b^p-a^p)(d^q-c^q)}{4pq}$ completes the proof. □

Remark 3.2. *For $p = q$ in Lemma 3.2, we have a new integral identity for partial differentiable functions on rectangle. If $p = 1 = q$, then Lemma 3.2 reduces to Lemma 1 [27].*

Theorem 3.3. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be partial differentiable function on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r \partial t}\right|$ is two dimensional pq-convex function, then*

$$|\Psi(t,r;p,q;\Delta)(f)|$$
$$\leq \frac{(b^p - a^p)(d^q - c^q)}{16pq} \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|}{4} \right].$$

Proof. Using Lemma 3.2 and the fact that $|\frac{\partial^2 f}{\partial t \partial r}|$ is two dimensional pq-convex, we have

$$|\Psi(t,r;p,q;\Delta)(f)|$$
$$= \left| \frac{(b^p - a^p)(d^q - c^q)}{4pq} \int_0^1 \int_0^1 (1-2t)(1-2r) \frac{\partial^2 f}{\partial t \partial r}\Big(M_p(a,b;t), M_q(c,d;r)\Big) dtdr \right|$$
$$\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} \int_0^1 \int_0^1 |1-2t||1-2r| \left| \frac{\partial^2 f}{\partial t \partial r}\Big(M_p(a,b;t), M_q(c,d;r)\Big) \right| dtdr$$
$$\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} \int_0^1 \int_0^1 |1-2t||1-2r|$$
$$\times \Bigg\{ tr\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right| + (1-t)r\left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right| + t(1-r)\left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right|$$
$$+ (1-t)(1-r)\left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right| \Bigg\} dtdr$$
$$= \frac{(b^p - a^p)(d^q - c^q)}{16pq} \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right| + \left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right| + \left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|}{4} \right].$$

This completes the proof. □

Theorem 3.4. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be partial differentiable function on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r \partial t}\right|^\beta$ is two dimensional pq-convex function, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha, \beta > 1$, then*

$$
\begin{aligned}
&|\Psi(t, r; p, q; \Delta)(f)| \\
&\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq(\alpha+1)^{\frac{2}{\alpha}}} \\
&\quad \times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|^\beta}{4} \right)^{\frac{1}{\beta}}
\end{aligned}
$$

Proof. Using Lemma 3.2, Holder's known inequality and the fact that $|\frac{\partial^2 f}{\partial r \partial t}|^\beta$ is two dimensional pq-convex function, we have

$$
\begin{aligned}
&|\Psi(t, r; p, q; \Delta)(f)| \\
&= \left| \frac{(b^p - a^p)(d^q - c^q)}{4pq} \int_0^1 \int_0^1 (1-2t)(1-2r) \frac{\partial^2 f}{\partial t \partial r}\Big(M_p(a,b;t), M_q(c,d;r)\Big) dt dr \right| \\
&\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} \int_0^1 \int_0^1 |1-2t||1-2r| \left| \frac{\partial^2 f}{\partial t \partial r}\Big(M_p(a,b;t), M_q(c,d;r)\Big) \right| dt dr \\
&\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} \left(\int_0^1 \int_0^1 |(1-2t)(1-2r)|^\alpha dt dr \right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial r}\Big([ta^p + (1-t)b^p]^{\frac{1}{p}}, [rc^q + (1-r)d^q]^{\frac{1}{q}}\Big) \right|^\beta dt dr \right)^{\frac{1}{\beta}} \\
&\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq(\alpha+1)^{\frac{2}{\alpha}}} \\
&\quad \times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|^\beta}{4} \right)^{\frac{1}{\beta}}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.5. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be partial differentiable function on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r \partial t}\right|^\beta$ is two dimensional pq-convex function, where $\beta > 1$, then*

$$
\begin{aligned}
&|\Psi(t, r; p, q; \Delta)(f)| \\
&\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} \left(\frac{1}{4}\right)^{1-\frac{1}{\beta}} \\
&\quad \times \left[\frac{\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right|^\beta + \left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|^\beta}{16} \right]^{\frac{1}{\beta}}.
\end{aligned}
$$

Proof. Using Lemma 3.2, power mean inequality and the fact that $|\frac{\partial^2 f}{\partial r \partial t}|^\beta$ is two dimensional pq-convex function, we have

$$\begin{aligned}
&|\Psi(t,r;p,q;\Delta)(f)| \\
&= \left|\frac{(b^p-a^p)(d^q-c^q)}{4pq}\int_0^1\int_0^1(1-2t)(1-2r)\frac{\partial^2 f}{\partial t\partial r}\Big(M_p(a,b;t),M_q(c,d;r)\Big)dtdr\right| \\
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\int_0^1\int_0^1|1-2t||1-2r|\left|\frac{\partial^2 f}{\partial t\partial r}\Big(M_p(a,b;t),M_q(c,d;r)\Big)\right|dtdr \\
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left(\int_0^1\int_0^1|(1-2t)(1-2r)|dtdr\right)^{1-\frac{1}{\beta}} \\
&\times\left(\int_0^1\int_0^1|(1-2t)(1-2r)|\left|\frac{\partial^2 f}{\partial t\partial r}\Big([ta^p+(1-t)b^p]^{\frac{1}{p}},[rc^q+(1-r)d^q]^{\frac{1}{q}}\Big)\right|^\beta dtdr\right)^{\frac{1}{\beta}} \\
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left(\int_0^1\int_0^1|(1-2t)(1-2r)|dtdr\right)^{1-\frac{1}{\beta}} \\
&\times\left(\int_0^1\int_0^1|(1-2t)(1-2r)|\left\{tr\left|\frac{\partial^2 f}{\partial t\partial r}(a,c)\right|^\beta+(1-t)r\left|\frac{\partial^2 f}{\partial t\partial r}(b,c)\right|^\beta\right.\right. \\
&\qquad\left.\left.+t(1-r)\left|\frac{\partial^2 f}{\partial t\partial r}(a,d)\right|^\beta+(1-t)(1-r)\left|\frac{\partial^2 f}{\partial t\partial r}(b,d)\right|^\beta\right\}dtdr\right)^{\frac{1}{\beta}} \\
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left(\frac{1}{4}\right)^{1-\frac{1}{\beta}} \\
&\quad\times\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial r}(a,c)\right|^\beta+\left|\frac{\partial^2 f}{\partial t\partial r}(b,c)\right|^\beta+\left|\frac{\partial^2 f}{\partial t\partial r}(a,d)\right|^\beta+\left|\frac{\partial^2 f}{\partial t\partial r}(b,d)\right|^\beta}{16}\right]^{\frac{1}{\beta}}
\end{aligned}$$

This completes the proof. □

Remark 3.3. *It can be seen that for $p = q$ in Theorems 3.3, 3.4 and 3.5, we have the result for two dimensional p-convex functions which appears to be new results. Also if $p = 1 = q$, Theorems 3.3, 3.4, 3.5 reduce to the results for two dimensional convex functions.*

Acknowledgements

The authors are thankful to the editor and anonymous referees for their valuable suggestions. The authors also would like to thank Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environment.

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