Filomat 30:2 (2016), 367–372
DOI 10.2298/FIL1602367R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Planar Torsion Graph of Modules

P. Malakooti Rad[a]

[a] *Department of Mathematics, Islamic Azad University, Qazvin Branch, Qazvin, Iran.*

Abstract. Let R be a commutative ring with identity. Let M be an R-module and $T(M)^*$ be the set of nonzero torsion elements. The set $T(M)^*$ makes up the vertices of the corresponding torsion graph, $\Gamma_R(M)$, with two distinct vertices $x, y \in T(M)^*$ forming an edge if $Ann(x) \cap Ann(y) \neq 0$. In this paper we study the case where the torsion graph $\Gamma_R(M)$ is planar.

1. Introduction

The idea of associating a graph with the zero-divisors of a commutative ring was introduced by Beck in [10], where the author talked about the colorings of such graphs. He lets every elements of R is a vertex in the graph, and two vertices x, y are adjacent if and only if $xy = 0$. In [6], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are non-zero zero-divisors while $x - y$ is an edge whenever $xy = 0$. Anderson and Badawi also introduced and investigated total graph of commutative ring in [2, 3]. The concept of zero-divisor graph has been extended to non-commutative rings by Redmond [18], and has been extended to module by Ghalandarzadeh and Malakooti in [13]. The zero-divisor graph of a commutative ring and has been studied extensively by several authors [4, 5, 7, 9, 14–16].

Let $x \in M$. The residual of Rx by M denoted by $[x : M] = \{r \in R | rM \subseteq Rx\}$. The annihilator of an R-module M, denoted by $Ann_R(M) = [0 : M]$. If $m \in M$, then $Ann(m) = \{r \in R | rm = 0\}$. Let $T(M) = \{m \in M | Ann(m) = 0\}$. It is clear that if R is an integral domain, then $T(M)$ is a submodule of M, which is called torsion submodule of M. If $T(M) = 0$, then the module M is said torsion-free, and it is called a torsion module if $T(M) = M$.

An R-module M is a multiplication module if for every R-submodule K of M there is an ideal I of R such that $K = IM$. Note that $I \subseteq [N : M]$, hence $N = IM \subseteq [N : M]M \subseteq N$. So $N = [N : M]M$. An R-module M is called a cancellation module if $IM = JM$ for any ideals I and J of R implies that $I = J$. Also, an R-module M is a weak-cancellation module if $IM = JM$ for any ideals I and J of R implies that $I + Ann(M) = J + Ann(M)$. Finitely generated multiplication modules are weak cancellation, Theorem 3 [1].

Let R be a commutative ring with identity and M be a unitary R-module. In this paper, we investigate the concept of torsion-graph for module that was introduced by Malakooti and Yassemi in [17]. Here the torsion graph $\Gamma_R(M)$ of M is a simple graph whose vertices are non-zero torsion elements of M and two different elements x, y are adjacent if and only if $Ann(x) \cap Ann(y) \neq 0$. Thus $\Gamma_R(M)$ is an empty graph if and only if M is a torsion-free R-module. Clearly if R is a domain or $Ann(M) \neq 0$, then $\Gamma_R(M)$ is complete. This study helps to illuminate the structure of $T(M)$, for example, let $M \cong M_1 \times M_2$, if $\Gamma_R(M)$ is a planar graph,

2010 *Mathematics Subject Classification*. 13A99; 05C99;13C99
Keywords. (Multiplication modules, Planar Graph, Torsion Graph)
Received: 05 March 2014; Accepted: 19 April 2014
Communicated by Miroslav Ćirić
Email address: pmalakooti@dena.kntu.ac.ir and pmalakoti@gmail.com (P. Malakooti Rad)

then $|T(M)| = 4$. Also, If M is a torsion module and $\Gamma_R(M)$ is a planar graph, then M is both Noetherian and Artinian.

Recall that a simple graph is finite if its vertices set is finite, and we use the symbol $|\Gamma_R(M)|$ to denote the number of vertices in graph $\Gamma_R(M)$. Also, a graph G is connected if there is a path between any two distinct vertices. The distance, $d(x, y)$ between connected vertices x, y is the length of the shortest path from x to y, ($d(x, y) = \infty$ if there is no such path). An isolated vertex is a vertex that has no edges incident to it. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n for the complete graph with n vertices. The complement $\overline{G}$ of G is the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. The complement of a complete graph is the null graph. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [11], p.153. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

One may address two major problems in this area: characterization of the planar torsion graphs and realization of the connection between the structures of a module and the corresponding graph. The organization of this paper is as follows:

In section 2, we study the planar torsion graph of multiplication module, and show that if the torsion graph of multiplication R-module M is planar, then M is both Noetherian and Artinian.

In section 3, we study the number of maximal submodule of multiplication modules. It is shown that if $\Gamma_R(M)$ is a planar graph, then $|Max(M)| \leq 4$. Also, we show that, if M be a multiplication R-module with $|Max(M)| \neq 1$ and $\Gamma_R(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$

Throughout the paper, $Max(M)$ is a set of the maximal submodules H of M, we use symbol $|Max(M)|$ to denote the number of maximal submodule of M. As a consequence of Theorem 2.5 [12], for any non-zero multiplication R-module $Max(M) \neq \emptyset$. Also, let $J(R)$ be the Jacobson radical of R and

$$J(M) := \cap_{H \in Max(M)} H.$$

We follow standard notation and terminology from graph theory [11] and module theory [8].

2. Planar Torsion Graph

This section is concerned with some basic and important results in the theory of planar torsion graphs over a module.

Lemma 2.1. *Let M be a multiplication R-module. If $\Gamma_R(M)$ is a planar graph, then $Ann(N) \neq 0$ for all prime submodules N of M.*

Proof. Let N be a prime submodule of M such that $Ann(N) = 0$. So there exists $0 \neq x \in M$ such that $x \notin H$. If $x = \alpha x$ for some $\alpha \in [N : M]$, then $(1 - \alpha)x = 0 \in N$. Thus $(1 - \alpha) \in [N : M]$, which is a contradiction. Hence $x \neq \alpha x$ for all $\alpha \in [N : M]$. Suppose $[N : M]^i x = [N : M]^j x$ for all integer $0 < i < j$, then $R = [N : M]^{j-i} + Ann([N : M]^i x)$. Let $r[N : M]^i x = 0$, thus $r[N : M]^{i-1}[x : M]N = 0$. Since $[N : M]M = N$ and $Ann(N) = 0$, we have $r[x : M] = 0$. Therefore

$$Rx = [N : M]^{j-i}x + Ann([N : M]^i x)x = [N : M]^{j-i}x \subseteq [N : M]x \subseteq Rx.$$

Which is a contradiction, and so $[N : M]^i x \neq [N : M]^j x$ for all $0 < i < j$. Hence

$$[N : M]^4 x \subset [N : M]^3 x \subset [N : M]^2 x \subset [N : M]x \subset Rx,$$

then there are five distinct vertices that form K_5 as an induced subgraph, which is a contradiction. This contradiction leads to the conclusion that $Ann(N) \neq 0$. □

Proposition 2.2. *Let M be an R-module with $[x : M] \neq 0$ for some $x \in T(M)^*$. $\Gamma_R(M)$ is null if and only if $M \cong M_1 \oplus M_2$ with $|M| \leq 4$.*

Proof. Let x be a vertex of $\Gamma_R(M)$ such that $[x : M] \neq 0$. Since $\Gamma_R(M)$ is null, one can easily check that $Rx = \{0, x\}$. Hence $x = \alpha x$ or $\alpha x = 0$ for all non-zero elements $\alpha \in [x : M]$. If $x = \alpha x$, then $R = R\alpha + Ann(x)$. Thus $M = Rx + Ann(x)M$. Suppose that $y \in Rx \cap Ann(x)M$. Then $y = rx = \sum_{i=1}^{n} \alpha_i m_i$ for some $r \in R$, $\alpha_i \in Ann(x)$ and $m_i \in M$. Hence

$$y = rx = r\alpha x = \sum_{i=1}^{n} \alpha_i m_i \alpha \subseteq Ann(x)x = 0.$$

Therefore $M = Rx \oplus Ann(x)M$ with $|Rx| = 2$. Let $y, z \in Ann(x)M$. So $0 \neq \alpha \in Ann(z) \cap Ann(y)$, implies that $y = z$ or $y = 0$ or $z = 0$. Therefore $|Ann(x)M| = 2$. Suppose $\alpha x = 0$ and $0 \neq m \in M$. If $\alpha m = 0$, then $\alpha \in Ann(x) \cap Ann(m)$. Hence $m = x \in Rx$. Now if $\alpha m \neq 0$, since $\alpha m \in Rm = \{0, m\}$, we have $m = \alpha m \in Rx = \{0, x\}$, so $M = Rx$ with $|Rx| = 2$. □

Corollary 2.3. *Let M be a multiplication R-module. $\Gamma_R(M)$ is null if and only if $M \cong M_1 \oplus M_2$ with $|M_1| \leq 2$ and $|M_2| \leq 2$.*

Let M_1 be an R_1-module and M_2 an R_2-module; then $M = M_1 \times M_2$ is an $R = R_1 \times R_2$ module with this multiplication $R \times M \longrightarrow M$, defined by $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$.

Theorem 2.4. *$\Gamma_R(M_1 \times M_2)$ is planar if and only if one of $\Gamma_R(M_1)$ or $\Gamma_R(M_2)$ is empty and another is null.*

Proof. Let $\Gamma_R(M_1)$ not be null and $\Gamma_R(M_2)$ not be empty. So there exist $x_1, x_2 \in T(M_1)^*$ and $y \in T(M_2)^*$ such that x_1 is adjacent to x_2. Hence there is $0 \neq s \in Ann(x_1) \cap Ann(x_2)$. It follows that

$$(0, s) \in Ann((x_1, 0)) \cap Ann((x_2, 0)) \cap Ann((x_1, y)) \cap Ann((x_2, y)) \cap Ann((0, y)).$$

So $\Gamma_R(M_1 \times M_2)$ has a K_5 as an induced subgraph, which is a contradiction. Therefore one of $\Gamma_R(M_1)$ or $\Gamma_R(M_2)$ is empty and another is null. □

As an immediate consequence, we obtain the following results.

Corollary 2.5. *If $\Gamma_R(M_1 \times M_2)$ is planar, then $|T(M)| = 4$.*

Corollary 2.6. *$\Gamma_R(M_1 \times M_2 \times M_3)$ is planar if and only if M_i is a simple R_i module for $i \in \{1, 2, 3\}$.*

Proof. Let $\Gamma_R(M_1 \times M_2 \times M_3)$ be a planar graph and M_3 not be a simple R_3- module. So there exists $0 \neq N < M_3$. Suppose $0 \neq x \in M_3$ such that $x \notin N$ and let $y \in N$. By Theorem 2.4, $\Gamma_R(M_2 \times M_3)$ is null or empty. But $(1, 0) \in Ann((0, x)) \cap Ann((0, y))$, which is a contradiction. Therefore M_i is a simple R_i module for $i \in \{1, 2, 3\}$. □

Theorem 2.7. *Let M be a multiplication R-module. If $\Gamma_R(M)$ is a planar graph, then M is both Noetherian and Artinian.*

Proof. Let $N_1 \subset N_2 \subset N_3 \subset N_4 \subset N_5$ be a chine of nontrivial proper submodule of M. Then there is distinct element $x_i \in N_i$, $1 \leq i \leq 5$. By Theorem 2.5 of [12], M has a maximal submodule H such that $N_5 \subseteq H$. Then $Ann(H) \subseteq Ann(N_5)$ and by Lemma 2.1, $0 \neq Ann(H) \subseteq Ann(N_5)$. Thus $Ann(x_i) \cap Ann(x_j) \neq 0$ for all distinct element $i, j \in \{1, 2, \ldots, 5\}$. So x_i, $1 \leq i \leq 5$ form K_5 as an induced subgraph, which is a contradiction. Therefore M is both Noetherian and Artinian. □

Corollary 2.8. *Let M be a multiplication R-module. If $\Gamma_R(M)$ is a planar graph, then M is cyclic.*

Proof. Let $\Gamma_R(M)$ be a planar graph. By Proposition 2.7, M is an Artinian module. And so by Corollary 2.9 of [12], M is a cyclic R-module. □

3. $\Gamma_R(M)$ and Maximal Submodules of Multiplication Module

Our theorems in this section are somewhat more delicate in their characterization of a multiplication R-module.

Proposition 3.1. *Let M be a multiplication R-module. If $\Gamma_R(M)$ is a planar graph, then $1 \leq |Max(M)| \leq 4$.*

Proof. Let $\Gamma_R(M)$ be a planar graph. Suppose $|Max(M)| \geq 5$ and $H_1, H_2, \dots H_5$ be distinct maximal submodules of M, such that $H_1 \cap H_2 \cap H_3 \cap H_4 = 0$. Then $[H_1 : M][H_2 : M][H_3 : M]H_4 = 0 \subseteq H_5$. Sine every maximal submodule of multiplication modules is prime, we have $[H_1 : M][H_2 : M][H_3 : M] \subseteq [H_5 : M]$. One can easily check that $[H_5 : M]$ is a prime ideal of R. Hence $[H_i : M] = [H_5 : M]$ for some $i \in \{1, 2, 3, 4\}$. It follows that $H_i = H_5$ for some $i \in \{1, 2, 3, 4\}$, which is a contradiction. Therefore $H_1 \cap H_2 \cap H_3 \cap H_4 \neq 0$. Hence

$$H_1 \cap H_2 \cap H_3 \cap H_4 \subset H_1 \cap H_2 \cap H_3 \subset H_1 \cap H_2 \subset H_1$$

and

$$H_1 \cap H_2 \cap H_3 \cap H_4 \subset H_1 \cap H_2 \cap H_4 \subset H_1 \cap H_2 \subset H_1.$$

Thus there are distinct elements $x_1 \in H_1$, $x_2 \in H_1 \cap H_2$, $x_3 \in H_1 \cap H_2 \cap H_3$, $x_4 \in H_1 \cap H_2 \cap H_4$ and $x_5 \in H_1 \cap H_2 \cap H_3 \cap H_4$. By Lemma 2.1, $Ann(H_1) \neq 0$. It implies that x_i, $1 \leq i \leq 5$ form K_5 as an induced subgraph, which is a contradiction. Therefore $|Max(M)| \leq 4$. □

Proposition 3.2. *Let M be a multiplication R-module with $|Max(M)| = 4$ and $\Gamma_R(M)$ be planar then $M \cong M_1 \oplus M_2$.*

Proof. Let H_i, $1 \leq i \leq 4$ be distinct maximal submodules of M. Suppose that $H_1 \cap H_2 \cap H_3 \cap H_4 \neq 0$. It is clear that

$$H_1 \cap H_2 \cap H_3 \cap H_4 \subset H_1 \cap H_2 \cap H_3 \subset H_1 \cap H_2 \subset H_1$$

and

$$H_1 \cap H_2 \cap H_3 \cap H_4 \subset H_1 \cap H_2 \cap H_4 \subset H_1 \cap H_2 \subset H_1.$$

Thus there are distinct elements $x_1 \in H_1$, $x_2 \in H_1 \cap H_2$, $x_3 \in H_1 \cap H_2 \cap H_3$, $x_4 \in H_1 \cap H_2 \cap H_4$ and $x_5 \in H_1 \cap H_2 \cap H_3 \cap H_4$. By Lemma 2.1, $Ann(H_1) \neq 0$. It follows that x_i, $1 \leq i \leq 5$ form K_5 as an induced subgraph, which is a contradiction. So $H_1 \cap H_2 \cap H_3 \cap H_4 = 0$. Let $H_1 \cap H_2 \cap H_3 \subseteq H_4$. It follows that $[H_1 : M][H_2 : M]H_3 \subseteq H_4$. Since H_3 is a maximal submodule of M, we have $[H_1 : M] \subseteq [H_4 : M]$ or $[H_1 : M] \subseteq [H_4 : M]$. Therefore $H_1 = H_4$ or $H_2 = H_4$, which is a contradiction. Hence $H_1 \cap H_2 \cap H_3 \nsubseteq H_4$. Consequently $M = H_1 \cap H_2 \cap H_3 \oplus H_4$. □

Corollary 3.3. *Let M be a multiplication R-module with $|Max(M)| = 4$. Then $\Gamma_R(M)$ is a planar graph if and only if $M \cong M_1 \times M_2 \times M_3$ where M_i is a simple R module for $i \in \{1, 2, 3\}$*

Proof. Let $\Gamma_R(M)$ is a planar graph. By Proposition 3.7, $M \cong M_1 \times M_2$. And by Theorem 2.4, M_1 or M_2 is empty another is null. Then by Corollary 2.3, $M \cong M_1 \times M_2 \times M_3$ and by Corollary 2.6, the result follows. □

Theorem 3.4. *Let M be a multiplication R-module with $|Max(M)| = 3$. Then $\Gamma_R(M)$ is a planar graph if and only if $M \cong M_1 \oplus M_2$ such that $\Gamma_R(M_1)$ or $\Gamma_R(M_2)$ is empty another is null.*

Proof. Let H_i, $1 \leq i \leq 3$ be distinct maximal submodules of M. First suppose that $[H_1 : M]H_1 \cap [H_2 : M]H_2 \cap [H_3 : M]H_3 \neq 0$. Then $H_1 \cap H_2 \cap H_3 \neq 0$. It is clear that $H_1 \cap H_2 \cap H_3 \subset H_1 \cap H_2 \subset H_1$ and $H_1 \cap H_2 \cap H_3 \subset H_1 \cap H_3 \subset H_1$. So there are distinct elements $x_1 \in H_1$, $x_2 \in H_1 \cap H_2$, $x_3 \in H_1 \cap H_3$ and $x \in H_1 \cap H_2 \cap H_3$ such that $x_1, x_2, x_3 \notin H_1 \cap H_2 \cap H_3$. If $[x : M]x = Rx$, then $R = [x : M] + Ann(x)$. One can easily check that $M = Rx \oplus Ann(x)M$ and by Theorem 2.4, $\Gamma_R(Rx)$ or $\Gamma_R(Ann(x)M)$ is empty another is null. Let $[x : M]x \neq Rx$. Then $x \neq \alpha x$ for all $\alpha \in [x : M]$. It follows that $\alpha x \notin \{x, x_1, x_2, x_3\}$. By Lemma 2.1, $Ann(H_1) \neq 0$. Therefore $x, \alpha x, x_1, x_2, x_3$ form K_5 as an induced subgraph, which is a contradiction. So $[H_1 : M]H_1 \cap [H_2 : M]H_2 \cap [H_3 : M]H_3 = 0$. Assume $[H_1 : M]H_1 + [H_2 : M]H_2 \cap [H_3 : M]H_3 \neq M$. By Theorem 2.5 of [12], M has a maximal submodule H such that $[H_1 : M]H_1 + [H_2 : M]H_2 \cap [H_3 : M]H_3 \subseteq H$. One can easily check that $H \neq H_2$ and $H \neq H_3$. Let $H = H_1$. Then $[H_1 : M]H_1 + [H_2 : M]H_2 \cap [H_3 : M]H_3 \subseteq H_1$, that is $[H_2 : M]^2[H_3 : M]^2 \subseteq [H_1 : M]$. Since $[H_1 : M]$ is a prime ideal of R, we have $[H_2 : M] \subseteq [H_1 : M]$ or $[H_3 : M] \subseteq [H_1 : M]$, that is $H_2 = H_1$ or $H_3 = H_1$, which is a contradiction. So $[H_1 : M]H_1 + [H_2 : M]H_2 \cap [H_3 : M]H_3 = M$. Consequently $M \cong M_1 \oplus M_2$ and by Theorem 2.4, the result follows. □

Lemma 3.5. *Let M be a faithful finitely generated multiplication R-module. Then $J(R)M = J(M)$.*

Proof. Let M be a faithful finitely generated multiplication R-module and H be a maximal submodule of M. By Theorem 3.1 of [12], $hM \neq M$ for all maximal ideal h of M. Also, by Theorem 2.5 of [12], $H = hM$ for some maximal ideal h of M. On the other hand by Lemma 3.5,

$$J(M) = \cap_{H \in Max(M)} H = \cap_{h \in Max(R)} (hM) = (\cap_{h \in Max(R)} h)M = J(R)M$$

□

Theorem 3.6. *Let M be a multiplication R-module with $|Max(M)| = 2$. Then $\Gamma_R(M)$ is a planar graph if and only if $M \cong [H_1 : M]^4M \oplus [H_2 : M]^4M$ such that $\Gamma_R([H_1 : M]^4M)$ or $\Gamma_R([H_1 : M]^4M)$ is empty another is null, where H_1, H_2 are maximal submodule of M.*

Proof. Let H_1 and H_2 be distinct maximal submodules of M. Suppose that $[H_1 : M]^4M + [H_2 : M]^4M \neq M$. By Theorem 2.5 of [12], there is a maximal submodule H of M such that $[H_1 : M]^4M + [H_2 : M]^4M \subseteq H$. Since $|Max(M)| = 2$, we have $H = H_1$ or $H = H_2$. It follows that $[H_1 : M]^4M \subseteq H_2$ or $[H_2 : M]^4M \subseteq H_1$. Thus $H_1 = H_2$, which is a contradiction. So $M = [H_1 : M]^4M + [H_2 : M]^4M$. Assume $[H_1 : M]^4M \cap [H_2 : M]^4M \neq 0$. Hence $H_1 \cap H_2 \neq 0$. If M is not a faithful. Then $\Gamma_R(M)$ is a complete graph and by Corollary 2.8, there are non-zero distinct elements $x, y, z, w \in M$ such that $M = Rx$, $H_1 = Ry$, $H_2 = Rz$ and $H_1 \cap H_2 = Rw$. It is clear that $y + w \notin \{x, y, z, w\}$, thus $x, y, z, w, y + w$ form K_5 as an induced subgraph, which is a contradiction. Therefore M is a faithful R-module. On the other hand By Theorem 1.6 [12],

$$[H_1 : M]^iM \cap [H_2 : M]^iM = ([H_1 : M]^i \cap [H_2 : M]^i)M,$$

for all positive integer i. Since M is a cyclic faithful multiplication module, by Lemma 3.5, we have $J(R)M = J(M)$. Now Nakayama's lemma follows that

$$([H_1 : M]^4 \cap [H_2 : M]^4)M \subset \ldots \subset ([H_1 : M] \cap [H_2 : M])M \subset H_1.$$

Hence there exist distinct elements $x_1 \in H_1$, $x_2 \in [H_1 : M]M \cap [H_2 : M]M$, $x_3 \in [H_1 : M]^2M \cap [H_2 : M]^2M$, $x_4 \in [H_1 : M]^3M \cap [H_2 : M]^3M$ and $x_5 \in [H_1 : M]^4M \cap [H_2 : M]^4M$. By Lemma 2.1, $Ann(H_1) \neq 0$. It follows that x_i, $1 \leq i \leq 5$ form K_5 as an induced subgraph, which is a contradiction. Therefore $[H_1 : M]^4M \cap [H_2 : M]^4M = 0$. Consequently $M \cong [H_1 : M]^4M \oplus [H_2 : M]^4M$ and by Theorem 2.4, the result follows. □

Proposition 3.7. *Let M be a multiplication R-module with $|Max(M)| = 1$. If $\Gamma_R(M)$ is a planar graph then $|M| \leq 5$ or $[H : M]^5M = 0$ where H is a maximal submodule of M.*

Proof. Suppose M be a faithful multiplication R-module. By Lemma 3.5, R is a local ring with unique maximal ideal $[H : M]$. By Nakayama's lemma, we have $[H : M]^iM \neq [H : M]^jM$ for all positive integer $i \neq j$. Since $\Gamma_R(M)$ is a planar graph then $[H : M]^5M = 0$. If M is not faithful, then $\Gamma_R(M)$ is a complete graph. Hence $|M| \leq 5$. □

Now we obtain the central results of this section.

Corollary 3.8. *Let M be a multiplication R-module with $|Max(M)| \neq 1$. If $\Gamma_R(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$*

References

[1] D. D. Anderson, Multiplication ideals, multiplication rings and the ring R(X), Canad. J. Math 28 (1976), 260–768.
[2] D. F. Anderson, A. Badawi, On the zero-divisor graph of a ring, Comm. Algebra 36 (2008) 3073-3092.
[3] D. F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008) 2706-2719.
[4] D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston, The zero-divisor graph of a commutative ring, II. in, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York 220 (2001) 61-72.
[5] D. F. Anderson, Sh. Ghalandarzadeh, S. Shirinkam, P. Malakooti Rad, On the diameter of the graph $\Gamma_{Ann(M)}(R)$, Filomat 26 (2012) 623–629.
[6] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999) 434–447.
[7] D. F. Anderson, S. Shirinkam, Some Remarks on the Graph $\Gamma_I(R)$, Comm. Algebra To appear.
[8] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra Addison-Wesley, Reading, MA, 1969.
[9] A. Badawi, D. F. Anderson, Divisibility conditions in commutative rings with zero divisors, Comm. Algebra 38 (2002) 4031-4047.
[10] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988) 208–226.
[11] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, American Elsevier, New York, 1976
[12] Z. A. El-Bast, P. F. Smith, Multiplication modules, comm. Algebra 16 (1988) 755–779.
[13] Sh. Ghalandarzadeh, P. Malakooti Rad, Torsion graph over multiplication modules, Extracta Mathematicae 24 (2009) 281–299.
[14] Sh. Ghalandarzadeh, S. Shirinkam, P. Malakooti Rad, Annihilator Ideal-Based Zero-Divisor Graphs Over Multiplication Modules, Communications in Algebra 41 (2013) 1134–1148.
[15] D. C. Lu, T. S. Wu, On bipartite zero-divisor graphs, Discrete Math 309 (2009) 755-762.
[16] P. Malakooti Rad, Sh. Ghalandarzadeh, S. Shirinkam, On The Torsion Graph and Von Numann Regular Rings, Filomat 26 (2012) 47–53.
[17] P. Malakooti Rad, S. Yassemi, Sh. Ghalandarzadeh, P. Safari, Diameter and girth of Torsion Graph, Analele Stiintifice ale Universitatii Ovidius Constanta 22(3) (2014) 127–136.
[18] S. P. Redmond, The zero-divisor graph of a non-commutative ring, Internat. J. Commutative Rings 1 (2002) 203-211.