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Computing {2,4} and {2,3}-inverses Using SVD-like Factorizations and QR Factorization

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Abstract. We derive conditions for the existence and investigate representations of $\{2,4\}$ and $\{2,3\}$-inverses with prescribed range T and null space S. A general computational algorithm for $\{2,4\}$ and $\{2,3\}$ generalized inverses with given rank and prescribed range and null space is derived. The algorithm is derived generating the full-rank representations of these generalized inverses by means of various complete orthogonal matrix factorizations. More precisely, computational algorithm for $\{2,4\}$ and $\{2,3\}$-inverses of a given matrix A is defined using an unique approach on SVD, QR and URV matrix decompositions of appropriately selected matrix W.

1. Introduction and Preliminaries

Let $\mathbb{C}^{m\times n}$ and $\mathbb{C}_r^{m\times n}$ denote the set of all complex $m\times n$ matrices and all complex $m\times n$ matrices of rank r, respectively. I denotes the unit matrix of an appropriate order. By A^*, $\mathcal{R}(A)$, $\operatorname{rank}(A)$ and $\mathcal{N}(A)$ we denote the conjugate transpose, the range, the rank and the null space of $A\in\mathbb{C}^{m\times n}$.

The problem of calculating generalized pseudoinverses is narrowly related with the four Penrose equations

$$(1)\quad AXA = A \qquad (2)\quad XAX = X \qquad (3)\quad (AX)^* = AX \qquad (4)\quad (XA)^* = XA.$$

It is usual convention to denote by $A\{\mathcal{S}\}$ the set of all matrices obeying the conditions contained in $\mathcal{S}$. In this regard, any matrix from $A\{\mathcal{S}\}$ is called $\mathcal{S}$-inverse of A and it is denoted by $A^{(\mathcal{S})}$. By $A\{\mathcal{S}\}_s$ we denote the set of all $\mathcal{S}$-inverses of A with rank s. The Moore-Penrose inverse of A is a single element in the set $A\{1,2,3,4\}$ and it is denoted by $A^\dagger$. For other important properties of generalized inverses see [1, 21].

The set of outer (or $\{2\}$-inverses) with prescribed range and null space is very important and frequently investigated. If $A\in\mathbb{C}_r^{m\times n}$, T is a subspace of $\mathbb{C}^n$ of dimension $s\le r$ and S is a subspace of $\mathbb{C}^m$ of dimension $m-s$, then A has a $\{2\}$-inverse X such that $\mathcal{R}(X)=T$ and $\mathcal{N}(X)=S$ if and only if $AT\oplus S=\mathbb{C}^m$, in which case X is unique and we denote it by $A^{(2)}_{T,S}$.

Full-rank representation of $\{2\}$-inverses with prescribed range and null space is determined in the next proposition, which is originated in [12].

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Proposition 1.1. [12] *Let $A \in \mathbb{C}_r^{m\times n}$, T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and let S be a subspace of $\mathbb{C}^m$ of dimensions $m-s$. In addition, suppose that $W \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(W) = T, \mathcal{N}(W) = S$. Let W has an arbitrary full-rank decomposition, that is $W = FG$. If A has a $\{2\}$-inverse $A^{(2)}_{T,S}$, then:*

(1) *GAF is an invertible matrix;*

(2) $A^{(2)}_{T,S} = F(GAF)^{-1}G$.

There exist many full-rank representations for different generalized inverses of prescribed rank [4, 6, 11–14, 17]. A generalization of known representations for $\{2,3\}$, $\{2,4\}$-inverses of prescribed rank from [1] is introduced in [20]. In Proposition 1.2 we reduce known results from [20] to constant complex matrices.

Proposition 1.2. [20] *Let $A \in \mathbb{C}_r^{m\times n}$ and $0 < s \le r$, $m_1, n_1 \ge s$ be chosen integers. Then the following general representations for $\{2,4\}$ and $\{2,3\}$-inverses of prescribed rank are valid:*

(a) $A\{2,4\}_s = \left\{(GA)^\dagger G |\, G \in \mathbb{C}^{n_1\times m},\ GA \in \mathbb{C}_s^{n_1\times n}\right\}$.

(b) $A\{2,3\}_s = \left\{F(AF)^\dagger |\, F \in \mathbb{C}^{n\times m_1},\ AF \in \mathbb{C}_s^{m\times m_1}\right\}$.

Proposition 1.3 from [16] exactly distinguishes sets $A\{2,4\}_s$ and $A\{2,3\}_s$ as two proper subsets of the set $A\{2\}_s$.

Proposition 1.3. [16] *Let $A \in \mathbb{C}_r^{m\times n}$ be the given matrix and $0 < s \le r$ a chosen integer. Assume that $G \in \mathbb{C}_s^{s\times m}$ and $F \in \mathbb{C}_s^{n\times s}$ are two arbitrary matrices satisfying* $\mathrm{rank}(GA) = \mathrm{rank}(G)$ *and* $\mathrm{rank}(AF) = \mathrm{rank}(F)$. *Then the following statements are valid:*

(a) $A\{2,4\}_s = \left\{(GA)^* \left(GA(GA)^*\right)^{-1} G \mid G \in \mathbb{C}_s^{s\times m},\ GA \in \mathbb{C}_s^{s\times n}\right\}$;

(b) $A\{2,3\}_s = \left\{F\left((AF)^* AF\right)^{-1} (AF)^* \mid F \in \mathbb{C}_s^{n\times s},\ AF \in \mathbb{C}_s^{m\times s}\right\}$;

The Moore-Penrose inverse $A^\dagger$, The weighted Moore-Penrose inverse $A^\dagger_{M,N}$, The Drazin inverse A^D and the group inverse $A^\#$ are particular appearances of the generalized inverses $A^{(2)}_{T,S}$ for appropriate choices of the matrix W which is exploited in Proposition 1.1. More precisely, these pseudo-inverses can be derived in particular cases $W = A^*$, $W = A^\sharp = N^{-1}A^*M$, $W = A^l$, $l \ge \mathrm{ind}(A)$ and $W = A$, respectively (see, for example [1]). In [16] we investigate representations and computation of $\{2,4\}$ and $\{2,3\}$-inverses with prescribed range and null space.

Proposition 1.4. [16] *For arbitrary matrix $A \in \mathbb{C}_r^{m\times n}$ and arbitrary integer s satisfying $0 < s \le r$ we have*

$$\text{(a)}\quad A\{2,4\}_s = \left\{A^{(2,4)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} |\, G \in \mathbb{C}_s^{s\times m},\ \mathrm{rank}(GA) = \mathrm{rank}(G)\right\} \tag{1.1}$$

$$\text{(b)}\quad A\{2,3\}_s = \left\{A^{(2,3)}_{\mathcal{R}(F),\mathcal{R}(AF)^\perp} |\, F \in \mathbb{C}_s^{n\times s},\ \mathrm{rank}(AF) = \mathrm{rank}(F)\right\}. \tag{1.2}$$

Various representations of $\{2,3\}$ and $\{2,4\}$-inverses with prescribed range and null space has been investigated [2, 5, 12, 24, 25]. The expressions for $\{2,3\}$ and $\{2,4\}$-inverses of a normal matrix by its Schur decomposition are discussed in [26]. But, these representations are not exploited in developing of some effective computational procedures. Additionally, the general representations of $\{2,4\}$ and $\{2,3\}$-inverses of the form $(GA)^\dagger G$ and $F(AF)^\dagger$, respectively, are not widely exploited in the literature. we can emphasize the effective numerical methods from [13], which are based on several modifications of the hyper-power method. Effective full-rank representations of the sets $A\{2,4\}_s$ and $A\{2,3\}_s$ as particular cases of the full-rank representation of the set $A\{2\}_s$ are derived in [16]. Introduced full-rank representations enable adaptation of well-known algorithms for computing outer inverses with prescribed range and null space into corresponding algorithms for computing $\{2,4\}$ and $\{2,3\}$-inverses. Corresponding adaptation of the successive matrix squaring algorithm from [15] is developed in the paper [16]. In the present paper we continue and expand the results from [18, 19].

The results derived in the present paper can be divided in two parts. Firstly, we derive several additional results concerning representations of $\{2,4\}$ and $\{2,3\}$-inverses. Later, using these representations, we derive numerical algorithms for computing $\{2,4\}$ and $\{2,3\}$-inverses. More precisely, we observe that derived representations are matrix expressions $(GA)^{\dagger}G$, $F(AF)^{\dagger}$ involving generalized inverse of a matrix product. For this purpose we define and exploit SVD, QR and URV matrix decompositions adopted for the matrix product and later compute its generalized inverse. In this way, we define algorithms for computing $\{2,4\}$ and $\{2,3\}$-inverses based on these SVD, QR and URV decompositions.

The paper is organized as follows. Necessary and sufficient conditions which ensure that an arbitrary $\{2,4\}$ and $\{2,3\}$-inverse represents an inverse with prescribed range T and null space S are considered in the second section. Various representations of $\{2,4\}$ and $\{2,3\}$-inverses arising from matrix decompositions are introduced and investigated in the third section. Algorithm arising from defined representations is given in Section 4. Numerical examples are presented in the last section.

2. Representations of {2,4} and {2,3}-inverses with Prescribed Range and Null Space

The next statements from [1] are used as auxiliary results.

Proposition 2.1. *1. If the matrix F in the matrix product $A = FBG$ is of full-column rank and G is of full-row rank, then* $\operatorname{rank}(A) = \operatorname{rank}(B)$.

2. $\mathcal{R}(AB) = \mathcal{R}(A)$ *if and only if* $\operatorname{rank}(AB) = \operatorname{rank}(A)$ *and* $\mathcal{N}(AB) = \mathcal{N}(B)$ *if and only if* $\operatorname{rank}(AB) = \operatorname{rank}(B)$.

3. $A^{\dagger} = (A^*A)^{\dagger} A^* = A^* (AA^*)^{\dagger}$.

4. $\mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}$, $\mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$.

In Lemma 2.1 we derive conditions for the existence and representations of $\{2,4\}$ and $\{2,3\}$-inverses which continue corresponding results from [3, Lemma 3.3].

Lemma 2.1. *Let $A \in \mathbb{C}_r^{m\times n}$ be given $m \times n$ matrix of rank r and $0 < s \leq r$ is selected integer. Assume that T is a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and S is a subspace of $\mathbb{C}^m$ of dimensions $m - s$. Let $G \in \mathbb{C}_s^{s\times m}$ be an arbitrary matrix satisfying $\mathcal{N}(G) = S$ and* $\operatorname{rank}(GA) = \operatorname{rank}(G) = s$. *Then $\{2,4\}$-inverse $X := (GA)^{\dagger}G$ satisfies $X = A^{(2,4)}_{T,S}$ with $\mathcal{R}(X) = T$, $\mathcal{N}(X) = S$ if and only if $\mathcal{N}(GA) = T^{\perp}$ or $T = \mathcal{R}(A^*G^*)$.*

Proof. According to Proposition 1.3 and Proposition 1.1 we get

$$(GA)^{\dagger}G = (GA)^* \left(GA(GA)^*\right)^{-1} G = A^{(2,4)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)}.$$

Now, we conclude $X := A^{(2,4)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} = A^{(2,4)}_{T,S}$ if and only if $\mathcal{N}(GA)^{\perp} = T$ or $\mathcal{N}(GA) = T^{\perp}$. The proof can be completed using part 4 from Proposition 2.1. □

The proof of dual Lemma 2.2 is similar and will be omitted.

Lemma 2.2. *Let $A \in \mathbb{C}_r^{m\times n}$ be given $m \times n$ matrix of rank r and $0 < s \leq r$ is selected integer. Assume that T is a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and S is a subspace of $\mathbb{C}^m$ of dimensions $m - s$. Let $F \in \mathbb{C}_s^{n\times s}$ be an arbitrary matrix satisfying $\mathcal{R}(F) = T$ and* $\operatorname{rank}(AF) = \operatorname{rank}(F) = s$. *Then $\{2,3\}$-inverse $X := F(AF)^{\dagger}$ satisfies $X = A^{(2,3)}_{T,S}$ with $\mathcal{R}(X) = T$, $\mathcal{N}(X) = S$ if and only if $\mathcal{R}(AF) = S^{\perp}$ or $S = \mathcal{N}(F^*A^*)$.*

Comparing Proposition 1.1 with Proposition 1.3 we conclude :

- $\{2,4\}$-inverses are subset of outer inverses generated by the choice $W = (GA)^*G$, i.e. $F = (GA)^*$.
- $\{2,3\}$-inverses are subset of outer inverses generated by the choice $W = F(AF)^*$, i.e. $G = (AF)^*$.

In the next statement it is shown that $\{2,4\}$-inverses can be generated using only the matrix G and $\{2,3\}$-inverses can be generated using only the matrix F.

Lemma 2.3. *Let $A \in \mathbb{C}_r^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(W) = T, \mathcal{N}(W) = S$.*

(a) *In the case $W = (GA)^*G$ we have*

$$(WA)^\dagger W = (GA)^\dagger G.$$

(b) *In the case $W = F(AF)^*$ we have*

$$W(AW)^\dagger = F(AF)^\dagger.$$

Proof. The proof can be derived using the property 3 of the Moore-Penrose inverse from Proposition 2.1. □

3. Computing {2,4} and {2,3}-inverses by Matrix Factorizations

In this section we derive numerical methods for computation of $\{2,4\}$ and $\{2,3\}$-inverses using their general representations from the previous section and various matrix factorizations.

A complete orthogonal factorization of an $m\times n$ matrix A of rank r is any factorization of the form

$$A = U\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}V^*,$$

where T is $r\times r$ square nonsingular matrix [9]. In the present paper we exploit three appearances of the complete orthogonal factorization: the singular value decomposition (*SVD*), QR decomposition (*QRD*) and the URV decomposition (*URVD*).

Let $A \in \mathbb{C}_r^{m\times n}$ be given and $0 < s \le r$ be a chosen integer. Suppose that the matrix $W \in \mathbb{C}_s^{n\times m}$ is chosen using the rules from Proposition 1.1. Also, let the SVD factorization of W is of the general form

$$W = U\Sigma V^*, \tag{3.1}$$

where $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ are column-orthogonal and $\Sigma \in \mathbb{C}_s^{n\times m}$ is a diagonal matrix with the singular values of W in descent order on the main diagonal. Suppose that the nonzero singular values of W are ordered as

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_s. \tag{3.2}$$

As it is known, *SVD* decomposition (3.1) is not a full-rank factorization of W. In order to generate the full-rank factorization of W arising from (3.1), let us consider the matrices $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{m\times m}$ and $\Sigma \in \mathbb{C}^{n\times m}$ partitioned in appropriate blocks

$$U = \begin{bmatrix} U_s & U_R \end{bmatrix}, \quad V = \begin{bmatrix} V_s & V_R \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_s & O \\ O & O \end{bmatrix}, \tag{3.3}$$

where $U_s \in \mathbb{C}^{n\times s}$, $V_s \in \mathbb{C}^{m\times s}$ and $\Sigma_s = \operatorname{diag}\{\sigma_1, \dots, \sigma_s\}$. Then the compact SVD of W

$$W = U_s\Sigma_s V_s^* \tag{3.4}$$

is its full-rank factorization.

Suppose that the *QR* factorization of W as in Theorem 3.3.11 from [22] is of the form

$$W = QRP^*, \tag{3.5}$$

where P is an $m\times m$ permutation matrix, $Q \in \mathbb{C}^{n\times n}$, $Q^*Q = I_n$ and $R \in \mathbb{C}_s^{n\times m}$ is an upper triangular matrix. Assume that P is chosen so that Q and R can be partitioned as

$$Q = \begin{bmatrix} Q_s & Q_R \end{bmatrix}, \quad R = \begin{bmatrix} R_s \\ O \end{bmatrix}, \tag{3.6}$$

where Q_s (resp. R_s) consists of the first s columns of the matrix Q (resp. R). The columns of Q_s form an orthonormal basis for $\mathcal{R}(W)$ and the columns of Q_R an orthonormal basis for $\mathcal{N}(W^*)$ (see [10, 21]).

URV decomposition of the matrix $W \in \mathbb{C}_s^{n\times m}$, $s \le r$:

$$W = URV^* = U\begin{bmatrix} C & O \\ O & O \end{bmatrix}V^*, \tag{3.7}$$

where

$$U = \begin{bmatrix} U_s & U_R \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad V = \begin{bmatrix} V_s & V_R \end{bmatrix} \in \mathbb{C}^{m\times m}$$

are orthogonal matrices and $C \in \mathbb{C}^{s\times s}$ is invertible. The matrices U_s and V_s is generated from the first s columns of U and V, respectively. The columns of U_s form an orthonormal basis for $\mathcal{R}(W)$ and the columns of U_R an orthonormal basis for $\mathcal{N}(W^*)$. The columns of V_s form an orthonormal basis for $\mathcal{R}(W^*)$ and the columns of V_R an orthonormal basis for $\mathcal{N}(W)$ (see [8] pages 406, 407). Therefore, using known properties of the matrix Q_s from the QR decomposition, it is possible to conclude $Q_s \equiv U_s$.

Since, as said above, the matrix U satisfies $U^*U = I$ we have

$$\begin{bmatrix} U_s^* \\ U_R^* \end{bmatrix}\begin{bmatrix} U_s & U_R \end{bmatrix} = \begin{bmatrix} U_s^*U_s & U_s^*U_R \\ U_R^*U_s & U_R^*U_R \end{bmatrix} = I$$

and so, $U_s^*U_R = U_R^*U_s = O$, $U_s^*U_s = U_R^*U_R = I$. Similarly, one can verify identities $V_s^*V_R = V_R^*V_s = O$, $V_s^*V_s = V_R^*V_R = I$.

In this case,

$$W = U_s(CV_s^*) = Q_sCV_s^* \tag{3.8}$$

is a full–rank factorization of W which is based on its URV decomposition (3.7).

3.1. Computing {2,4} and {2,3}-inverses by SVD Factorization

Lemma 3.1. *Assume that the matrix $A \in \mathbb{C}_r^{m\times n}$ is given and $0 < s \le r$ is a selected integer. Assume that T is a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and S is a subspace of $\mathbb{C}^m$ of dimensions $m - s$. Let $G \in \mathbb{C}_s^{s\times m}$ be an arbitrary matrix satisfying $\mathcal{N}(G) = S$, $\mathcal{N}(GA) = T^\perp$ and $\mathrm{rank}(GA) = \mathrm{rank}(G) = s$. Let*

$$GA = U_s\Sigma_sV_s^*, \;\; \Sigma_s = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_s) \tag{3.9}$$

be the compact SVD of GA. Then

$$A_{T,S}^{(2,4)} \;=\; V_s\,\mathrm{diag}\left(\sigma_1^{-1}, \ldots, \sigma_s^{-1}\right)U_s^*\,G. \tag{3.10}$$

Proof. Using known SVD representation of the Moore-Penrose inverse we obtain

$$V_s\,\mathrm{diag}\left(\sigma_1^{-1}, \ldots, \sigma_s^{-1}\right)U_s^*\,G = (GA)^\dagger G.$$

According to Lemma 2.1 we get

$$(GA)^\dagger G = A^{(2,4)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} = A^{(2,4)}_{T,S},$$

which is a verification of the identity (3.10). □

Lemma 3.2. *Assume that the matrix $A \in \mathbb{C}_r^{m\times n}$ is given and $0 < s \le r$ is a selected integer. Assume that T is a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and S is a subspace of $\mathbb{C}^m$ of dimensions $m - s$. Let $F \in \mathbb{C}_s^{n\times s}$ be an arbitrary matrix satisfying $\mathcal{R}(F) = T$, $\mathcal{R}(AF) = S^\perp$ and $\mathrm{rank}(AF) = \mathrm{rank}(F) = s$. Let*

$$AF = U_s\Sigma_sV_s^*, \;\; \Sigma_s = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_s) \tag{3.11}$$

be the compact SVD of AF. Then

$$A_{T,S}^{(2,3)} \;=\; F\,V_s\mathrm{diag}\left(\sigma_1^{-1}, \ldots, \sigma_s^{-1}\right)U_s^*. \tag{3.12}$$

In Theorem 3.1 we show that $\{2,4\}$ and $\{2,3\}$-inverses of prescribed range and null space can be computed without computation of the *SVD* of the matrix products GA or AF, respectively.

Theorem 3.1. *Let $A \in \mathbb{C}_r^{m\times n}$ be the given matrix and $s \leq r$ be a given integer.*

(a) *Let G be $s\times m$ matrix of rank s satisfying* $\mathrm{rank}((GA)^*G) = s$ *and* (3.4) *be the full-rank factorization of $W = (GA)^*G \in \mathbb{C}_s^{n\times m}$ implied by the compact SVD of W. Then the following statements hold:*

$$\begin{aligned} A^{(2,4)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} &= A^{(2,4)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)} \\ &= U_s(\Sigma_s V_s^* A U_s)^{-1}\Sigma_s V_s^* \\ &= (WA)^\dagger W \\ &= (GA)^\dagger G \in A\{2,4\}_s. \end{aligned} \tag{3.13}$$

(b) *Let F be $n\times s$ matrix of rank s satisfying* $\mathrm{rank}(F(AF)^*) = s$. *If* (3.4) *is a full–rank factorization of $W = F(AF)^* \in \mathbb{C}_s^{n\times m}$ the following statements are valid:*

$$\begin{aligned} A^{(2,3)}_{\mathcal{R}(F),\mathcal{N}(AF)^\perp} &= A^{(2,3)}_{\mathcal{R}(U_s),\mathcal{N}(V_s^*)} \\ &= U_s(\Sigma_s V_s^* A U_s)^{-1}\Sigma_s V_s^* \\ &= W(AW)^\dagger \\ &= F(AF)^\dagger \in A\{2,3\}_s. \end{aligned} \tag{3.14}$$

Proof. (a) In this case we have

$$W = U_s(\Sigma_s V_s^*) = (GA)^*G,$$

so that

$$\begin{aligned} \mathcal{R}(W) &= \mathcal{R}(U_s) = \mathcal{R}((GA)^*) = \mathcal{N}(GA)^\perp, \\ \mathcal{N}(W) &= \mathcal{N}(\Sigma_s V_s^*) = \mathcal{N}(V_s^*) = \mathcal{N}(G). \end{aligned}$$

Later, the identities $\mathrm{rank}(\Sigma_s V_s^*) = \mathrm{rank}(\Sigma_s V_s^* A) = s$ are satisfied. Therefore, conditions from the part (3) of Lemma 3.3 from [3] are satisfied, which implies

$$A^{(2)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} = U_s(\Sigma_s V_s^* A U_s)^{-1}\Sigma_s V_s^*.$$

Since $\Sigma_s V_s^* A U_s$ is a full-rank factorization of the invertible matrix $\Sigma_s V_s^* A U_s$ the reverse order low of the Moore-Penrose inverse is applicable, which implies

$$A^{(2)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} = (\Sigma_s V_s^* A)^\dagger \Sigma_s V_s^* \in A\{2,4\}_s.$$

This part of the proof can be completed using

$$\begin{aligned} (\Sigma_s V_s^* A)^\dagger \Sigma_s V_s^* &= (U_s\Sigma_s V_s^* A)^\dagger U_s\Sigma_s V_s^* \\ &= (WA)^\dagger W \\ &= ((GA)^*GA)^\dagger\,(GA)^*G \\ &= (GA)^\dagger \left(GA(GA)^\dagger\right)^* G \\ &= (GA)^\dagger GA(GA)^\dagger G \\ &= (GA)^\dagger G. \end{aligned}$$

(b) This part of the proof can be verified in a similar way, using the part (2) of Lemma 3.3 from [3]. □

A combined SVD of two matrices of identical dimensions has been introduced with the motivation to avoid explicit computation of matrix products and quotients in the computation of the SVD of some matrix products and quotients. It is possible to use the combined SVD to compute $\{2,3\}$-inverses of $A \in \mathbb{C}_r^{m\times n}$ of the form $F(AF)^\dagger$, where $F \in \mathbb{C}^{n\times s}$, $AF \in \mathbb{C}_s^{m\times s}$. Instead of computation of the matrix product AF it seems more appropriate to use a proper combination of truncated factorization of A of the order s

$$A_s = U_s^A \Sigma_s^A V_s^* = U_s^A \operatorname{diag}\left(\sigma_1^A, \dots, \sigma_s^A\right) V_s^* \tag{3.15}$$

and compact SVD of F^*:

$$F^* = U_s^F \Sigma_s^F V_s^* = U_s^F \operatorname{diag}\left(\sigma_1^F, \dots, \sigma_s^F\right) V_s^*. \tag{3.16}$$

In Theorem 3.2 we investigate further properties of the thin Moore-Penrose inverse $\left(A^\dagger\right)_s$, which is introduced in [7].

Theorem 3.2. *Under the assumptions of Theorem 3.1 the following statements are valid:*

$$A_s^\dagger = \left(A^\dagger\right)_s = A^{(2,3)}_{\mathcal{R}(V_s),\mathcal{R}\left(U_s^F\right)^\perp} \in A_s\{2,3\}_s.$$

Proof. Using (3.15) and (3.16) we have

$$A_s F = U_s^A \Sigma_s^A \left(\Sigma_s^F\right)^* \left(U_s^F\right)^* = U_s^A \operatorname{diag}\left(\overline{\sigma_1^G}\sigma_1^A, \dots, \overline{\sigma_1^G}\sigma_s^A\right)\left(U_s^F\right)^*.$$

This further implies

$$\begin{aligned} A_s^\dagger &= V_s \operatorname{diag}\left(\frac{1}{\sigma_1^A}, \dots, \frac{1}{\sigma_s^A}\right)\left(U_s^A\right)^* \\ &= V_s \operatorname{diag}\left(\overline{\sigma_1^F}, \dots, \overline{\sigma_s^F}\right)\left(U_s^F\right)^* U_s^F \operatorname{diag}\left(\frac{1}{\overline{\sigma_1^F}\sigma_1^A}, \dots, \frac{1}{\overline{\sigma_s^F}\sigma_s^A}\right)\left(U_s^A\right)^* \\ &= F(A_s F)^\dagger. \end{aligned}$$

The proof can be completed using $\mathcal{R}(F) = \mathcal{R}(V_s)$, $\mathcal{N}(F) = \mathcal{N}\left(\left(U_s^F\right)^*\right) = \mathcal{R}\left(U_s^F\right)^\perp$. □

3.2. Computing {2,4} and {2,3}-inverses by QR Factorization

Proposition 3.1. [19] *The following two statements are valid:*

$$\begin{aligned} Q_s(R_sP^*AQ_s)^{-1}R_sP^* &= A^{(2)}_{\mathcal{R}(W),\mathcal{N}(W)} = A^{(2)}_{\mathcal{R}(Q_s),\mathcal{N}(R_sP^*)} && (3.17)\\ &= Q_s(Q_s^*WAQ_s)^{-1}Q_s^*W. && (3.18) \end{aligned}$$

Theorem 3.3. *Let $A \in \mathbb{C}_r^{m\times n}$ be the given matrix, $s \le r$ be a given integer and the matrices F, G are chosen as in Proposition 1.2.*

(a) *If* (3.5) *is a full–rank factorization of $W = (GA)^*G \in \mathbb{C}_s^{n\times m}$ the following is satisfied:*

$$Q_s(R_sP^*AQ_s)^{-1}R_sP^* = (GA)^\dagger G = A^{(2,4)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} \in A\{2,4\}_s. \tag{3.19}$$

(b) *If* (3.5) *is a full–rank factorization of $W = F(AF)^* \in \mathbb{C}_s^{n\times m}$ the following holds:*

$$Q_s(R_sP^*AQ_s)^{-1}R_sP^* = F(AF)^\dagger = A^{(2,3)}_{\mathcal{R}(F),\mathcal{R}(AF)^\perp} \in A\{2,3\}_s. \tag{3.20}$$

Proof. (a) In this case we have

$$W = Q_s(R_sP^*) = (GA)^*G. \tag{3.21}$$

Therefore, conditions from the part (3) of Lemma 3.3 from [3] are satisfied, which implies

$$Q_s(R_sP^*AQ_s)^{-1}R_sP^* = (R_sP^*A)^{\dagger}R_sP^* \in A\{2,4\}_s.$$

This part of the proof can be completed using

$$\begin{aligned}(R_sP^*A)^{\dagger}R_sP^* &= (Q_sR_sP^*A)^{\dagger}Q_sR_sP^*\\ &= ((GA)^*GA)^{\dagger}\,(GA)^*G\\ &= (GA)^{\dagger}G\\ &= A^{(2)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)}.\end{aligned}$$

(b) This dual statement can be verified in a similar way, using the part (2) of Lemma 3.3 from [3]. □

3.3. **Computing {2,4} and {2,3}-inverses by URV Factorization**

Theorem 3.4. *Let $A \in \mathbb{C}_r^{m\times n}$ be the given matrix, $s \le r$ be a given integer and the matrices F, G are chosen as in Proposition 1.2.*

(a) *If* (3.8) *is a full–rank factorization of $W = (GA)^*G \in \mathbb{C}_s^{n\times m}$ the following is satisfied:*

$$\begin{aligned}A^{(2,4)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} &= Q_s(CV_s^*AQ_s)^{-1}CV_s^*\\ &= Q_s\,(Q_s^*WAQ_s)^{-1}\,Q_s^*W\\ &= (GA)^{\dagger}G \in A\{2,4\}_s.\end{aligned} \tag{3.22}$$

(b) *If* (3.8) *is a full–rank factorization of $W = F(AF)^* \in \mathbb{C}_s^{n\times m}$ the following holds:*

$$A^{(2,3)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} = Q_s(CV_s^*AQ_s)^{-1}CV_s^* = F(AF)^{\dagger} \in A\{2,3\}_s. \tag{3.23}$$

Proof. (a) In this case we have

$$W = U_s(CV_s^*) = Q_sCV_s^* = (GA)^*G.$$

Therefore, conditions from the part (3) of Lemma 3.3 from [3] are satisfied, which implies

$$Q_s(CV_s^*AQ_s)^{-1}CV_s^* = (U_sCV_s^*A)^{\dagger}U_sCV_s^* \in A\{2,4\}_s.$$

From (3.8) we obtain

$$CV_s^* = Q_s^*W$$

and later

$$Q_s(CV_s^*AQ_s)^{-1}CV_s^* = Q_s\,(Q_s^*WAQ_s)^{-1}\,Q_s^*W,$$

which confirms the second identity in (3.22).

The second identity in (3.22) can be verified using

$$\begin{aligned}Q_s(CV_s^*AQ_s)^{-1}CV_s^* &= (U_sCV_s^*A)^{\dagger}U_sCV_s^*\\ &= ((GA)^*GA)^{\dagger}\,(GA)^*G\\ &= (GA)^{\dagger}G\\ &= A^{(2)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)}.\end{aligned}$$

(b) This part of the proof can be verified in a similar way, using the part (2) of Lemma 3.3 from [3]. □

4. Algorithm

In the sequel we state the next unique algorithm for generating $A^{(2,4)}_{T,S}$ and $A^{(2,3)}_{T,S}$ inverses of a given matrix A by means of three matrix decompositions.

Algorithm 4.1 Computing the $A^{(2,4)}_{T,S}$ inverse of the matrix A using the complete orthogonal factorization of $W = (GA)^*G$. **(Algorithm COFATS24)**
or
$W = F(AF)^*$. **(Algorithm COFATS23)**

Require: The matrix A of dimensions $m \times n$ and of rank r.
Require: The matrix G of dimensions $s \times m$ and of rank s or the matrix F of dimensions $n \times s$ and of rank s.
1: Compute the matrix $W = (GA)^*G$ or the matrix $W = F(AF)^*$.
2: Compute the SVD, QR or URV decomposition of the matrix W in the form (3.1), (3.5) or (3.7), respectively.
3: Generate the full–rank decomposition of the matrix W as in (3.4), (3.21) or (3.8), respectively.
4: Solve one of the following matrix equations:
- In the case of SVD decomposition solve
$\Sigma_s V_s^* A U_s X = \Sigma_s V_s^*$.
- In the case of QR decomposition solve
$R_s P^* A Q_s X = R_s P^*$
or
$Q_s^* W A Q_s X = Q_s^* W$.
- In the case of URV decomposition solve
$C V_s^* A Q_s X = C V_s^*$.

5: Compute the output:
- In the case of SVD decomposition return the matrix $U_s X$.
- In the case of QR or URV decomposition return the matrix $Q_s X$.

We present a comparison of Algorithm 4.1 with the modified Successive Matrix Squaring (SMS) iterative algorithm for computing $\{2,3\}$ and $\{2,4\}$-inverses from [16]. The main difference between the modified SMS algorithm and Algorithm 4.1 are contained in the following:

1. The modified SMS algorithm is iterative, while Algorithm 4.1 is a direct method for computing generalized inverses.

2. Moreover, QR variant of Algorithm 4.1 is applicable in exact computation of generalized inverses of matrices whose entries are rational numbers. More precisely, the programming package *Mathematica* yields the QR decomposition for a numerical matrix in the form of a matrix with rational entries. Later, using the exact computation possibilities in *Mathematica*, it is possible to produce the exact output of Algorithm 4.1. About the package *Mathematica* see, for example [23]. Let us mention that it is possible to force *Mathematica* to give an approximate numerical output by ending the user's input with //N.

3. Roundoff errors accumulate during the iterative steps of the SMS method. Sometimes, these roundoff errors can endanger its convergence. In these cases, direct methods are appropriate choices.

4. Finally, in our numerical experiments we observed an additional drawback of the modified SMS algorithm. Namely, the optimal value β^C_{opt} of the parameter β (from [16]) is very small, which causes a very large number of iterative steps (approximately ∞). This choice causes the collapse of the SMS algorithm.

5. Numerical Examples

Numerical examples are performed in the programming package *Mathematica*.

In both Example 5.1, part (b) and Example 5.2, part (b), we derive exact generalized inverses.

Example 5.1. *The starting matrix in this example is*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 3 & 4 & 6 & 2 \\ 2 & 3 & 4 & 5 & 3 \\ 3 & 4 & 5 & 6 & 4 \\ 4 & 5 & 6 & 7 & 6 \\ 6 & 6 & 7 & 7 & 8 \end{bmatrix}$$

and the matrix G is equal to

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) *The SVD decomposition of the matrix $W = (GA)^*G$ associated with the $s = 2$ largest singular values of W is defined by $W = U_2\Sigma_2V_2^*$, where*

$$\{U_2, \Sigma_2, V_2\} = \left\{ \begin{bmatrix} 0.298885 & 0.533279 \\ 0.391543 & 0.0126312 \\ 0.483791 & -0.103401 \\ 0.576449 & -0.624049 \\ 0.437053 & 0.561538 \end{bmatrix}, \begin{bmatrix} 137.513 & 0. \\ 0. & 2.46756 \end{bmatrix}, \begin{bmatrix} 0.315726 & 0.0178024 \\ 0.0562962 & -0.998414 \\ 0. & 0. \\ 0. & 0. \\ 0.947179 & 0.0534072 \\ 0. & 0. \end{bmatrix} \right\}.$$

The representation (3.13) *gives the following* $\{2,4\}$*–inverse of A:*

$$\begin{aligned} X_1 = A^{(2,3)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} &= U_2(\Sigma_2V_2^*AU_2)^{-1}\Sigma_2V_2^* \\ &= \begin{bmatrix} 0.0453361 & -0.215651 & 0. & 0. & 0.136008 & 0. \\ 0.00990099 & -0.0049505 & 0. & 0. & 0.029703 & 0. \\ 0.00364773 & 0.0420358 & 0. & 0. & 0.0109432 & 0. \\ -0.0317874 & 0.252736 & 0. & 0. & -0.0953622 & 0. \\ 0.0505472 & -0.227028 & 0. & 0. & 0.151641 & 0. \end{bmatrix} \\ &= (GA)^\dagger G. \end{aligned}$$

b) *The QR decomposition of $W = (GA)^*G$ is determined by*

$$\{Q, R, P\} = \left\{ \begin{bmatrix} \sqrt{\frac{13}{145}} & -191\sqrt{\frac{13}{1669530}} \\ \frac{17}{\sqrt{1885}} & -\sqrt{\frac{57}{380770}} \\ \frac{21}{\sqrt{1885}} & 242\sqrt{\frac{2}{10851945}} \\ 5\sqrt{\frac{5}{377}} & 97\sqrt{\frac{30}{723463}} \\ \frac{19}{\sqrt{1885}} & -1307\sqrt{\frac{2}{10851945}} \end{bmatrix}, \begin{bmatrix} \sqrt{1885} & \frac{336}{\sqrt{1885}} & 0 & 0 & 3\sqrt{1885} & 0 \\ 0 & \sqrt{\frac{11514}{1885}} & 0 & 0 & 0 & 0 \end{bmatrix}, I_6 \right\}.$$

The representation (3.17) *produces the following* $\{2,4\}$*–inverse of A:*

$$A^{(2,4)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)} = Q_s(R_sP^*AQ_s)^{-1}R_sP^* = \begin{bmatrix} \frac{87}{1919} & -\frac{2483}{11514} & 0 & 0 & \frac{261}{1919} & 0 \\ \frac{1}{101} & -\frac{1}{202} & 0 & 0 & \frac{3}{101} & 0 \\ \frac{7}{1919} & \frac{242}{5757} & 0 & 0 & \frac{21}{1919} & 0 \\ -\frac{61}{1919} & \frac{485}{1919} & 0 & 0 & -\frac{183}{1919} & 0 \\ \frac{97}{1919} & -\frac{1307}{5757} & 0 & 0 & \frac{291}{1919} & 0 \end{bmatrix}.$$

Real approximation of the rational matrix $A^{(2,4)}_{\mathcal{N}(GA)^\perp,\mathcal{N}(G)}$ is identical to X_1.

Also, simple verification shows that

$$(WA)^{\dagger}W = (GA)^{\dagger}G = X_1.$$

c) *The modification of the SMS method from* [16] *produces the following approximate outer inverse after 40 iterations:*

$$X_{SMS} = \begin{bmatrix} 0.0453336 & -0.215638 & 0. & 0. & 0.136001 & 0. \\ 0.00990136 & -0.00495131 & 0. & 0. & 0.0297041 & 0. \\ 0.00364769 & 0.0420319 & 0. & 0. & 0.0109431 & 0. \\ -0.0317878 & 0.252739 & 0. & 0. & -0.0953633 & 0. \\ 0.0505491 & -0.227036 & 0. & 0. & 0.151647 & 0. \end{bmatrix}.$$

But, after 50 iterative steps we get the wrong result:

$$\begin{bmatrix} 0.0428389 & -0.203391 & 0. & 0. & 0.128517 & 0. \\ 0.0102711 & -0.00575505 & 0. & 0. & 0.0308135 & 0. \\ 0.00360076 & 0.0382315 & 0. & 0. & 0.0108022 & 0. \\ -0.0321686 & 0.256387 & 0. & 0. & -0.0965057 & 0. \\ 0.052478 & -0.235296 & 0. & 0. & 0.157434 & 0. \end{bmatrix}.$$

After 60 iterative steps we observed beginning of the divergence:

$$\begin{bmatrix} -27.3097 & 119.577 & 0. & 0. & -81.9255 & 0. \\ -460.605 & 2016.8 & 0. & 0. & -1381.75 & 0. \\ -9.6306 & 42.161 & 0. & 0. & -28.8905 & 0. \\ 167.628 & -733.808 & 0. & 0. & 502.86 & 0. \\ 220.94 & -967.379 & 0. & 0. & 662.789 & 0. \end{bmatrix}.$$

Let us observe that the approximation of the exact outer inverse with 10 decimals is equal to

$$\begin{bmatrix} 0.04533611256 & -0.2156505124 & 0 & 0 & 0.1360083377 & 0 \\ 0.009900990099 & -0.004950495050 & 0 & 0 & 0.02970297030 & 0 \\ 0.003647733194 & 0.04203578253 & 0 & 0 & 0.01094319958 & 0 \\ -0.03178738927 & 0.2527357999 & 0 & 0 & -0.09536216780 & 0 \\ 0.05054715998 & -0.2270279660 & 0 & 0 & 0.1516414799 & 0 \end{bmatrix}.$$

It is easy to observe that the best possible result produced by the modified SMS method, denoted by X_{SMS}, is a much worse approximation than the approximation X_1 produced by the SVD factorization. To verify this statement, we compute corresponding matrix norms:

$$\|A^{(2,4)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} - X_1\| = 9.726792725073938 \times 10^{-15},\ \|A^{(2,4)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} - X_{SMS}\| = 0.00001827337135490787.$$

Example 5.2. *Let us choose the same matrix A as in Example 5.1 and the matrix F equal to*

$$F = \begin{bmatrix} -3 & -2 \\ -7 & 6 \\ 0 & 0 \\ 7 & 5 \\ -4 & 0 \end{bmatrix}.$$

The matrix W, appropriate to A, is now given by $W = F(AF)^*$.

a) *The SVD decomposition of $W = F(AF)^*$ is*

$$\{U_2, \Sigma_2, V_2\} = \left\{ \begin{bmatrix} -0.107919 & 0.35296 \\ 0.938716 & 0.246259 \\ 0. & 0. \\ 0.289015 & -0.841614 \\ 0.15374 & 0.326285 \end{bmatrix}, \begin{bmatrix} 1025.55 & 0. \\ 0. & 400.128 \end{bmatrix}, \begin{bmatrix} 0.180315 & -0.432229 \\ 0.279943 & -0.644462 \\ 0.296222 & -0.235186 \\ 0.393255 & -0.137902 \\ 0.509162 & 0.0591417 \\ 0.622475 & 0.565702 \end{bmatrix} \right\}.$$

The representation (3.14) *gives the following* $\{2,3\}$*–inverse of A:*

$$X_2 = A^{(2,3)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} = U_s(\Sigma_s V_s^* A U_s)^{-1}\Sigma_s V_s^*$$
$$= \begin{bmatrix} -0.0397756 & -0.059307 & -0.0216562 & -0.012713 & 0.00540651 & 0.0520006 \\ -0.0395358 & -0.0581092 & -0.00651577 & 0.0128002 & 0.0458202 & 0.11673 \\ 0. & 0. & 0. & 0. & 0. & 0. \\ 0.0944746 & 0.140892 & 0.0519066 & 0.0309905 & -0.0115775 & -0.121479 \\ -0.0397157 & -0.0590075 & -0.0178711 & -0.00633466 & 0.0155099 & 0.0681828 \end{bmatrix}.$$

Let us mention that $F(AF)^{\dagger}$ *is the rational matrix*

$$\begin{bmatrix} -\frac{837851}{21064424} & -\frac{312317}{5266106} & -\frac{456175}{21064424} & -\frac{267791}{21064424} & \frac{113885}{21064424} & \frac{547681}{10532212} \\ -\frac{832799}{21064424} & -\frac{306009}{5266106} & -\frac{137251}{21064424} & \frac{269629}{21064424} & \frac{965177}{21064424} & \frac{1229421}{10532212} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{995027}{10532212} & \frac{370975}{2633053} & \frac{546691}{10532212} & \frac{326399}{10532212} & -\frac{121937}{10532212} & -\frac{639719}{5266106} \\ -\frac{209147}{5266106} & -\frac{155370}{2633053} & -\frac{94111}{5266106} & -\frac{33359}{5266106} & \frac{81677}{5266106} & \frac{179529}{2633053} \end{bmatrix}$$

whose approximation is just the matrix X_2.

b) *The QR decomposition of* $W = F(AF)^*$ *is given by*

$$\{Q_2, R_2, P\} = \left\{ \begin{bmatrix} -\frac{27}{\sqrt{7123}} & \frac{1389}{\sqrt{56941262}} \\ \frac{131}{3\sqrt{7123}} & \frac{18587}{3\sqrt{56941262}} \\ 0 & 0 \\ \frac{199}{3\sqrt{7123}} & -\frac{4723\sqrt{\frac{2}{28470631}}}{3} \\ -\frac{28}{3\sqrt{7123}} & \frac{3886\sqrt{\frac{2}{28470631}}}{3} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 3\sqrt{7123} & \frac{97688}{3\sqrt{7123}} & \frac{24151}{\sqrt{7123}} & \frac{28041}{\sqrt{7123}} & \frac{30823}{\sqrt{7123}} & \frac{78904}{3\sqrt{7123}} \\ 0 & \frac{22\sqrt{\frac{7994}{7123}}}{3} & 131\sqrt{\frac{7994}{7123}} & 222\sqrt{\frac{7994}{7123}} & 353\sqrt{\frac{7994}{7123}} & \frac{1703\sqrt{\frac{7994}{7123}}}{3} \end{bmatrix}, I_6 \right\}.$$

The representation (3.17) *gives the following exact* $\{2,3\}$*–inverse of A:*

$$A^{(2,3)}_{\mathcal{N}(GA)^{\perp},\mathcal{N}(G)} = Q_2(R_2P^*AQ_2)^{-1}R_2P^* = \begin{bmatrix} -\frac{837851}{21064424} & -\frac{312317}{5266106} & -\frac{456175}{21064424} & -\frac{267791}{21064424} & \frac{113885}{21064424} & \frac{547681}{10532212} \\ -\frac{832799}{21064424} & -\frac{306009}{5266106} & -\frac{137251}{21064424} & \frac{269629}{21064424} & \frac{965177}{21064424} & \frac{1229421}{10532212} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{995027}{10532212} & \frac{370975}{2633053} & \frac{546691}{10532212} & \frac{326399}{10532212} & -\frac{121937}{10532212} & -\frac{639719}{5266106} \\ -\frac{209147}{5266106} & -\frac{155370}{2633053} & -\frac{94111}{5266106} & -\frac{33359}{5266106} & \frac{81677}{5266106} & \frac{179529}{2633053} \end{bmatrix}.$$

Again, simple verification shows that X_2 *is a real approximation of* $Q_2(R_2P^*AQ_2)^{-1}R_2P^*$.

Example 5.3. *In this example we consider outer inverses of the invertible* 8×8 *matrix, generated applying* $a = 1$ *and* $n = 8$ *on the test matrix* S_n *from* [27].

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

initiated by the randomly generated 2×8 *matrix*

$$G = \begin{bmatrix} 0.840945 & 0.538775 & 0.305695 & 0.225754 & 0.997242 & 0.695873 & 0.409878 & 0.0989846 \\ 0.183869 & 0.538072 & 0.721692 & 0.109286 & 0.728451 & 0.627618 & 0.247737 & 0.799947 \end{bmatrix}.$$

The matrix $W = (GA)^*G$ *should be used in construction of corresponding* $\{2,4\}$*-inverse of* A.

a) *SVD decomposition* $W = U_2\Sigma_2V_2^*$ *of* W *is defined by*

$$\{U_2, \Sigma_2, V_2\} = \left\{ \begin{bmatrix} -0.415962 & -0.168585 \\ -0.291191 & -0.063356 \\ -0.378075 & -0.410255 \\ -0.321869 & -0.163915 \\ -0.408221 & 0.0581344 \\ -0.280822 & -0.105142 \\ -0.363613 & 0.0090359 \\ -0.343399 & 0.870558 \end{bmatrix}, \begin{bmatrix} 36.7405 & 0. \\ 0. & 0.771884 \end{bmatrix}, \begin{bmatrix} -0.342289 & 0.563382 \\ -0.351771 & -0.0370266 \\ -0.330844 & -0.415312 \\ -0.110778 & 0.0944943 \\ -0.566792 & 0.184763 \\ -0.433113 & 0.0159507 \\ -0.216673 & 0.124863 \\ -0.28562 & -0.670697 \end{bmatrix} \right\}.$$

The representation (3.17) *produces the following* $\{2,4\}$*–inverse of the invertible matrix* A*:*

$$X_3 = \begin{bmatrix} -0.0561094 & 0.0230657 & 0.0715567 & -0.00664794 & 0.0051328 & 0.0203345 & -0.00514498 & 0.102684 \\ -0.0151947 & 0.0147109 & 0.0325671 & -0.000593331 & 0.01167 & 0.015083 & 0.00179136 & 0.0434828 \\ -0.164248 & 0.0277175 & 0.147448 & -0.0251358 & -0.0333136 & 0.0144951 & -0.0300357 & 0.226874 \\ -0.0581621 & 0.0187273 & 0.0660998 & -0.00763009 & -0.000973076 & 0.0152156 & -0.00728292 & 0.0968433 \\ 0.0434484 & 0.0167623 & -0.00146126 & 0.0100848 & 0.0380744 & 0.0234245 & 0.0170104 & -0.0153941 \\ -0.0340591 & 0.0153441 & 0.0455284 & -0.0038439 & 0.00474666 & 0.0138626 & -0.00261781 & 0.0648173 \\ 0.0198662 & 0.0160537 & 0.0124039 & 0.00580735 & 0.0275976 & 0.0202017 & 0.0109341 & 0.00849762 \\ 0.398583 & -0.00748643 & -0.264674 & 0.0695208 & 0.153439 & 0.0324514 & 0.0953776 & -0.439858 \end{bmatrix}.$$

b) *On the other hand, QR decomposition of* W *is defined by the triple of matrices*

$$\{Q, R, P\} = \left\{ \begin{bmatrix} -0.414798 & -0.29075 & -0.375257 & -0.320739 & -0.408609 & -0.280095 & -0.363667 & -0.349353 \\ -0.17143 & -0.0653488 & -0.412835 & -0.166115 & 0.0553373 & -0.107063 & 0.00654552 & 0.868186 \\ 0.766079 & 0.212335 & -0.149937 & -0.456384 & -0.354179 & -0.0180863 & -0.103023 & 0.0297519 \\ 0.241165 & -0.178279 & 0.203815 & -0.0282679 & 0.517042 & -0.661777 & -0.402783 & 0.0141814 \\ -0.362132 & 0.353185 & 0.540532 & -0.660523 & 0.0670047 & 0.0452746 & -0.0385762 & 0.0873309 \\ -0.0899308 & 0.395851 & -0.0761581 & 0.0897057 & -0.182705 & -0.665358 & 0.581887 & -0.0818038 \\ 0.0980838 & -0.664458 & 0.491088 & -0.0469233 & -0.398228 & -0.122557 & 0.301896 & 0.201888 \\ -0.0683294 & 0.33374 & 0.29255 & 0.4625 & -0.490657 & -0.109714 & -0.513453 & 0.260848 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} -20.8247 & -10.49 & -15.9125 & -12.5785 & -12.9238 & -7.96114 & -12.1529 & -4.07043 \\ 0. & -0.5896 & -0.0966653 & 0.3487 & -0.117 & 0.04186 & -0.4038 & 0.04506 \\ 0. & 0. & 2.2527\times10^{-15} & 1.3215\times10^{-16} & 7.7206\times^{-16} & 4.1133^{-16} & 4.3718\times10^{-16} & 8.8256\times10^{-17} \\ 0. & 0. & 0. & -1.4092\times10^{-15} & 2.542\times10^{-16} & -4.156\times10^{-16} & -3.111\times10^{-16} & -8.971\times10^{-17} \\ 0. & 0. & 0. & 0. & -8.834\times10^{-16} & -1.875\times10^{-16} & 2.296\times10^{-17} & -1.717\times10^{-16} \\ 0. & 0. & 0. & 0. & 0. & 6.966\times10^{-16} & -4.533\times10^{-17} & 7.938\times10^{-17} \\ 0. & 0. & 0. & 0. & 0. & 0. & 6.346\times10^{-17} & 4.927\times10^{-17} \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & -6.214\times10^{-18} \end{bmatrix}, I_8 \right\}.$$

The matrix $R_2P^*AQ_2$, equal to

$$\begin{bmatrix} 4.62211 & -3.23702 & -60.8101 & 134.471 & 132.254 & 178.293 & 68.296 & -86.2181 \\ -0.308102 & 0.23398 & -0.0152944 & 1.55327 & 0.651791 & 1.40038 & 0.145 & -0.376974 \\ 6.198\times10^{-16} & -6.0623\times10^{-16} & 2.701\times10^{-15} & -4.884\times10^{-15} & -5.0062\times10^{-15} & -6.327\times10^{-15} & -3.492\times10^{-15} & 3.818\times10^{-15} \\ 4.695\times10^{-16} & -3.965\times10^{-17} & -9.306\times10^{-16} & 2.223\times10^{-15} & 3.931\times10^{-15} & 4.358\times10^{-15} & 2.152\times10^{-15} & -2.220\times10^{-15} \\ -1.092\times10^{-16} & -7.632\times10^{-17} & -1.052\times10^{-15} & 1.221\times10^{-15} & 1.662\times10^{-15} & 2.155\times10^{-15} & 5.255\times10^{-16} & -5.253\times10^{-16} \\ 1.351\times10^{-17} & -3.8797\times10^{-16} & 7.081\times10^{-16} & -8.330\times10^{-16} & -1.175\times10^{-15} & -1.434\times10^{-15} & -1.412\times10^{-16} & 8.914\times10^{-16} \\ 3.659\times10^{-17} & 3.311\times10^{-17} & 3.836\times10^{-17} & -1.546\times10^{-16} & -1.827\times10^{-16} & -1.849\times10^{-16} & -4.657\times10^{-17} & 1.174\times10^{-16} \\ 1.507\times10^{-18} & -1.706\times10^{-18} & -1.926\times10^{-18} & 6.826\times10^{-18} & 1.064\times10^{-17} & 7.815\times10^{-18} & 7.978\times10^{-19} & -6.319\times10^{-18} \end{bmatrix},$$

is almost singular and its inversion produces computational errors which cause that the generated approximation (3.19) *of outer inverse does not satisfy the matrix equation* (2).

Example 5.4. *In this example we verify the results of Theorem 3.2. Consider the matrix*

$$As = \begin{bmatrix} 0.572082 & 2.04881 & 2.98051 & 4.45724 & 0.939071 \\ 1.3795 & 2.92615 & 4.01624 & 5.56289 & 2.05963 \\ 2.09302 & 3.11513 & 4.00852 & 5.03063 & 2.99024 \\ 2.81009 & 3.93239 & 4.98831 & 6.11061 & 3.98929 \\ 4.33103 & 4.99871 & 6.01632 & 6.684 & 6.04046 \\ 5.76162 & 6.00495 & 6.98839 & 7.23172 & 7.97013 \end{bmatrix}$$

defined by $As = U_2^A \Sigma_2^A V_2^*$, where

$$\left\{ \begin{bmatrix} -0.222859 & -0.573902 \\ -0.3093 & -0.513042 \\ -0.319292 & -0.205769 \\ -0.400374 & -0.168183 \\ -0.496807 & 0.19995 \\ -0.58788 & 0.54481 \end{bmatrix}, \begin{bmatrix} 1. & 0. \\ 0. & 1. \end{bmatrix}, \begin{bmatrix} -7.88638 & 2.06563 \\ -9.94431 & 0.291642 \\ -12.2808 & -0.424493 \\ -14.3388 & -2.19848 \\ -11.0847 & 2.66816 \end{bmatrix} \right\}$$

and the matrix F defined by $F = V_2 \left(\Sigma_2^F\right)^* \left(U_2^F\right)^*$, where

$$\left\{ \begin{bmatrix} 0. & 0. \\ 0. & 0. \\ -1. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 1. \end{bmatrix}, \begin{bmatrix} 0.999233 & 0. \\ 0. & 1. \end{bmatrix}, \begin{bmatrix} -7.88638 & 2.06563 \\ -9.94431 & 0.291642 \\ -12.2808 & -0.424493 \\ -14.3388 & -2.19848 \\ -11.0847 & 2.66816 \end{bmatrix} \right\}.$$

After a verification we get

$$\begin{aligned} (A_s)^\dagger &= \begin{bmatrix} -0.0654337 & -0.057383 & -0.021043 & -0.0157914 & 0.0285993 & 0.0701909 \\ -0.000266749 & 0.00145754 & 0.00359279 & 0.00508443 & 0.00894397 & 0.0125709 \\ 0.0275701 & 0.0268407 & 0.0146572 & 0.0147588 & 0.00184593 & -0.0102359 \\ 0.092737 & 0.0856812 & 0.039293 & 0.0356346 & -0.0178094 & -0.0678559 \\ -0.0836078 & -0.0731491 & -0.0265137 & -0.0196538 & 0.0374403 & 0.0909355 \end{bmatrix} \\ &= F(AF)^\dagger. \end{aligned}$$

Example 5.5. *In this example we compare Algorithm 4.1 and the modified SMS algorithm with 10 iterative steps. Matrices A of the order $m \times n$, where $m = 30 * k, n = 10 * k$, and of rank $r = n/2$. Matrices A of the order $n \times m$ and of rank $sr = n/3$. By X_{SVD}, X_{QR}, X_{SMS} we denote outer inverses produced by the SVD decomposition, QR decomposition and the modified SMS iterative process. The strike $-$ in Table 1 means that the modified SMS method produces the incorrect zero matrix in the output $X_{SMS} = 0$.*

Dimensions	$\|X_{SVD}AX_{SVD} - X_{SVD}\|$	$\|X_{QR}AX_{QR} - X_{QR}\|$	$\|X_{SMS}AX_{SMS} - X_{SMS}\|$
20×10	$5.867783065059582 \times 10^{-15}$	$1.54799100168391 \times 10^{-14}$	$2.113921195787184 \times 10^{-9}$
40×20	$7.062781516703317 \times 10^{-15}$	$4.082564210023391 \times 10^{-14}$	$4.31108324776876 \times 10^{-12}$
80×40	$1.315758645995428 \times 10^{-14}$	$1.04151672512349 \times 10^{-13}$	–
160×80	$8.332624649753079 \times 10^{-15}$	$1.100200054051695 \times 10^{-13}$	–
320×160	$1.320514308782867 \times 10^{-14}$	$2.996279369089772 \times 10^{-13}$	–
640×320	$1.352323356362689 \times 10^{-14}$	$4.304300513903187 \times 10^{-13}$	–
1280×640	$1.011325773008253 \times 10^{-14}$	$7.864989586776998 \times 10^{-13}$	–

Table 1: CPU times on the test block matrix V_n

6. Conclusion

In the current paper we further investigate conditions for the existence, representations and algorithms for computing $\{2,4\}$ and $\{2,3\}$-inverses with prescribed range and null space. We also derive corresponding representations arising from SVD, QR and URV matrix decomposition. An unique algorithm for computing $\{2,4\}$ and $\{2,3\}$-inverses is defined.

Our numerical experience shows that direct methods defined in 4.1 produce results with significantly better precision than the iterative method proposed in [16].

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