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Extending the Moore-Penrose Inverse

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Abstract. We show that it is possible to define generalized inverse similar to the Moore-Penrose inverse by slightly modified Penrose equations. Then we are investigating properties of this, so-called extended Moore-Penrose inverse.

1. Introduction

Let H and K be arbitrary Hilbert spaces, and let $\mathcal{L}(H,K)$ be the set of all bounded linear operators from H to K. If $H = K$, then we abbreviate $\mathcal{L}(H,H) = \mathcal{L}(H)$. For $A \in \mathcal{L}(H,K)$ by $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* we denote the range space, the null-space and adjoint, respectively.

Throughout the paper direct sum of the subspaces will be denoted by $\oplus$, and orthogonal direct sum by $\oplus^\perp$. An operator $P \in \mathcal{L}(H)$ is projection if $P^2 = P$, and orthogonal projection if $P^2 = P = P^*$. If $H = M \oplus N$, then $P_{M,N}$ denotes projection such that $\mathcal{R}(P_{M,N}) = M$, $\mathcal{N}(P_{M,N}) = N$. If $H = M \oplus^\perp N$, then we write P_M instead of $P_{M,N}$. Operator $A \in \mathcal{L}(H)$ is Hermitian (or selfadjoint) if $A = A^*$, normal if $AA^* = A^*A$, and unitary if $AA^* = A^*A = I$.

The Moore-Penrose inverse of $A \in \mathcal{L}(H,K)$, if it exists, is the unique operator $A^\dagger \in \mathcal{L}(K,H)$ satisfying the following, so-called Penrose equations:

$$(I)\, AA^\dagger A = A,\ (II)\, A^\dagger AA^\dagger = A^\dagger,\ (III)\, (AA^\dagger)^* = AA^\dagger,\ (IV)\, (A^\dagger A)^* = A^\dagger A.$$

It is well-known that $A^\dagger$ exists for given A if and only if $\mathcal{R}(A)$ is closed in K. For detailed introduction to the theory of generalized inverses, the reader is reffered, for example, to [1], [2], [4].

Closed-range operator $A \in \mathcal{L}(H)$ is EP ("equal-projection") if one of the following equivalent conditions holds: $AA^\dagger = A^\dagger A$, or $\mathcal{R}(A) = \mathcal{R}(A^*)$, or $\mathcal{N}(A) = \mathcal{N}(A^*)$.

In this paper we consider the following problem: for given closed-range operator $A \in \mathcal{L}(H,K)$ is there an operator $X \in \mathcal{L}(K,H)$ such that the following four Penrose-like equations are satisfied ($m, n \in \mathbb{N}$ are given):

(I_m) $\quad (AX)^m A = A,$

(II_n) $\quad X(AX)^n = X,$

(III) $\quad AX = (AX)^*,$

(IV) $\quad XA = (XA)^*.$

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It is obvious that case $m = n = 1$ reduces to well-known Moore-Penrose inverse.

Now we present some auxiliary results.

Lemma 1.1. *Let $A \in \mathcal{L}(X, Y)$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Lemma 1.2. *[3] Let $A \in \mathcal{L}(X, Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

Some properties of the Moore-Penrose inverse are collected in the following proposition.

Proposition 1.3. *Let $A \in \mathcal{L}(H, K)$ be closed-range operator. Then:*

1. $(\lambda A)^\dagger = \lambda^\dagger A^\dagger$, *where* $\lambda^\dagger = \lambda^{-1}$ *if* $\lambda \neq 0$ *and* $\lambda^\dagger = 0$ *if* $\lambda = 0$;
2. $(A^\dagger)^\dagger = A,\ (A^*)^\dagger = (A^\dagger)^*$;
3. $A^* = A^\dagger AA^* = A^*AA^\dagger,\ A = AA^*(A^*)^\dagger = (A^*)^\dagger A^*A$;
4. $A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^*,\ (AA^*)^\dagger = (A^*)^\dagger A^\dagger,\ (A^*A)^\dagger = A^\dagger(A^*)^\dagger$;
5. $\mathcal{R}(A) = \mathcal{R}(AA^\dagger) = \mathcal{R}(AA^*),\ \mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^*A)$;
6. $\mathcal{R}(I - A^\dagger A) = \mathcal{N}(A^\dagger A) = \mathcal{N}(A) = \mathcal{R}(A^*)^\perp$;
7. $\mathcal{R}(I - AA^\dagger) = \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$;
8. $(UAV)^\dagger = V^*A^\dagger U^*$, *when* $U \in \mathcal{L}(K)$ *and* $V \in \mathcal{L}(H)$ *are unitary operators (see for example [3] for some general reverse order law results).*

Theorem 1.4 ([5], Th. 12.29). *Suppose E is the spectral decomposition of a normal $T \in \mathcal{L}(H)$, $\lambda_0 \in \sigma(T)$, and $E_0 = E(\{\lambda_0\})$. Then*

(a) $\mathcal{N}(T - \lambda_0 I) = \mathcal{R}(E_0)$,

(b) *λ_0 is an eigenvalue of T if and only if $E_0 \neq 0$ and*

(c) *every isolated point of $\sigma(T)$ is an eigenvalue of T.*

(d) *Moreover, if $\sigma(T) = \{\lambda_1, \lambda_2, \lambda_3, ...\}$ is a countable set, then every $x \in H$ has a unique expansion of the form*

$$x = \sum_{i=1}^{\infty} x_i,$$

where $Tx_i = \lambda_i x_i$. Also, $x_i \perp x_j$ whenever $i \neq j$.

Theorem 1.5 ([6]). *Let M and N be closed subspaces of a Hilbert space H, and let P_M and P_N be the orthogonal projections onto M and N, respectively.*

(a) *We have $0 \leq P_M \leq I$.*

(b) *The following statements are equivalent:*

$$(i)\ P_M \leq P_N,\ (ii)\ P_N P_M = P_M,\ (iii)\ M \subset N,\ (iv)\ P_M P_N = P_M.$$

Theorem 1.6 ([6]). *Let M and N be closed subspaces of a Hilbert space H, and let P_M and P_N be the orthogonal projections onto M and N, respectively.*

(a) *$P = P_M P_N$ is an orthogonal projection if and only if $P_M P_N = P_N P_M$ holds; then we have $P = P_{M\cap N}$. We have $M \perp N$ if and only if $P_M P_N = 0$ (or $P_N P_M = 0$).*

(b) *$Q = P_M + P_N$ is an orthogonal projection if and only if $M \perp N$, then we have $Q = P_{M\oplus N}$.*

(c) *$R = P_M - P_N$ is an orthogonal projection if and only if $N \subset M$; then we have $R = P_{M\ominus N}$.*

Remark 1.7. *(See [6]) If H is a Hilbert space and T and T_1 are closed subspaces such that $T_1 \subset T$, then there exists exactly one closed subspace T_2 such that $T_2 \subset T$, $T_2 \perp T_1$ and $T = T_1 \oplus T_2$. For the uniquely defined subspace T_2, we write briefly $T_2 = T \ominus T_1$. The subspace T_2 is called the orthogonal complement of T_1 with respect to T. For $T = H$ we obtain that $H \ominus T_1 = T_1^{\perp}$.*

2. Main result

Lemma 2.1. *Let H be arbitrary Hilbert space and $T \in \mathcal{L}(H)$ closed-range operator such that $T^n = I$, $n \in \mathbb{N}$.*

i) There exists $T^{-1} \in \mathcal{L}(H)$ and $T^{-k} = T^{n-k}$, $k = \overline{0, n}$.

ii) If $T = T^*$, then $\sigma(T) = \{1\}$ for odd n and $\sigma(T) = \{-1; 1\}$. Moreover, $T = I$ for odd n, and for even n there exist nontrivial closed subspaces $Y_1, Y_2 \subset H$ such that $Y_1 \oplus^{\perp} Y_2 = H$ and

$$T = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} : \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \mapsto \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

Proof. i) We have $\{0\} = \mathcal{N}(I) = \mathcal{N}(T^n) \supseteq \mathcal{N}(T) \supseteq \mathcal{N}(I)$, so $\mathcal{N}(T) = \{0\}$ and operator T is injective. Also, $H = \mathcal{R}(I) = \mathcal{R}(T^n) \subseteq \mathcal{R}(T) \subseteq H$ implies $\mathcal{R}(T) = H$, so T is surjective. Therefore, there exists T^{-1}.

From $I = T^n = T^k T^{n-k} = T^{n-k} T^k$ it follows $T^{-k} = T^{n-k}$, $k = \overline{0, n}$.

ii) Operator T is Hermitian, so its spectrum is real. By the spectral mapping theorem for polynomials, we have

$$\{1\} = \sigma(I) = \sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\} \Rightarrow \sigma(T) = \{\lambda \in \mathbb{R} : \lambda^n = 1\}.$$

Therefore, for odd n we have $\sigma(T) = \{1\}$, while for even $n \in \mathbb{N}$ we have $\sigma(T) = \{-1; 1\}$.

It is clear that if $\sigma(T) = \{1\}$, then $T = I$. When the spectrum of the operator is a disjoint union of closed sets, then by Theorem 1.4.d there exist nontrivial closed subspaces $Y_1, Y_2 \subset H$ such that $Y_1 \oplus^{\perp} Y_2 = H$ (the sum is orthogonal because T is Hermitian!) and

$$T = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} : \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \mapsto \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.$$

□

Let H, K be arbitrary Hilbert spaces and $A \in \mathcal{L}(H, K)$ closed-range operator. Let us consider whether there is an operator $X \in \mathcal{L}(K, H)$ such that the following four Penrose-like equation are satisfied ($m, n \in \mathbb{N}$):

(I_m) $(AX)^m A = A,$

(II_n) $X(AX)^n = X,$

(III) $AX = (AX)^*,$

(IV) $XA = (XA)^*.$

Theorem 2.2. *Let H, K be arbitrary Hilbert spaces and $A \in \mathcal{L}(H, K)$ closed-range operator. Then we have*

$$\begin{cases} (I_m) & (AX)^m A = A \\ (II_n) & X(AX)^n = X \end{cases} \Leftrightarrow \begin{cases} (I_d) & (AX)^d A = A \\ (II_d) & X(AX)^d = X, \end{cases}$$

where $d = GCD(m, n)$ is the greatest common divisor of $m, n \in \mathbb{N}$.

Proof. $(\Leftarrow)$: Obvious.

$(\Rightarrow)$: Without the loss of generality, we may assume that $m > n$. By the Euclidean algorithm for the greatest common divisor, we have the finite sequence:

$$\begin{aligned} &m = q_0 n + r_0,\ 0 \le r_0 < n, \\ &n = q_1 r_0 + r_1,\ 0 \le r_1 < r_0, \\ &r_0 = q_2 r_1 + r_2,\ 0 \le r_2 < r_1, \\ &\qquad \dots \\ &r_{k-2} = q_k r_{k-1} + r_k,\ 0 \le r_k < r_{k-1}, \\ &r_{k-1} = q_{k+1} r_k + 0,\ d \equiv GCD(m, n) = r_k. \end{aligned}$$

So by (I_m) and (II_n) we have

$$\begin{aligned} A &= (AX)^m A = (AX)^{q_0 n + r_0} A = (AX)^{r_0 - 1} AX \underbrace{(AX)^n \dots (AX)^n}_{q_0 \text{ times}} A = \\ &= (AX)^{r_0 - 1} AX \underbrace{(AX)^n \dots (AX)^n}_{q_0 - 1 \text{ times}} A = \dots = (AX)^{r_0 - 1} AXA = (AX)^{r_0} A; \\ X &= X(AX)^n = X(AX)^{q_1 r_0 + r_1} = X \underbrace{(AX)^{r_0} \dots (AX)^{r_0}}_{q_1 \text{ times}} AX(AX)^{r_1 - 1} = \\ &= X \underbrace{(AX)^{r_0} \dots (AX)^{r_0}}_{q_1 - 1 \text{ times}} AX(AX)^{r_1 - 1} = \dots = XAX(AX)^{r_1 - 1} = X(AX)^{r_1}. \end{aligned}$$

By proceeding along the Euclidean algorithm, we have the proof. □

Therefore, it is enough to investigate the case $m = n$ in the sequel of the paper. Now we will consider the following four Penrose-like equations ($n \in \mathbb{N}$ given):

(I_n) $(AX)^n A = A$,

(II_n) $X(AX)^n = X$,

(III) $AX = (AX)^*$,

(IV) $XA = (XA)^*$.

By the Lemma 1.1, operator A has the following matrix form according to the space decompositions:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

We are looking for the operator X of the following form

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

By (III), the operator

$$AX = \begin{bmatrix} A_1X_1 & A_1X_2 \\ 0 & 0 \end{bmatrix}$$

is Hermitian, so by invertibility of A_1 it follows that $X_2 = 0$ and A_1X_1 is Hermitian. On the similar matter, from (IV) it follows $X_3 = 0$ and X_1A_1 is Hermitian. From (II_n) we have $X_4 = 0$ and $X_1(A_1X_1)^n = X_1$, and from (I_n) it follows $(A_1X_1)^n A_1 = A_1$. Therefore,

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}, \ AX = \begin{bmatrix} A_1X_1 & 0 \\ 0 & 0 \end{bmatrix}, \ XA = \begin{bmatrix} X_1A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

From $(A_1X_1)^n = I_{\mathcal{R}(A)}$, $(X_1A_1)^n = I_{\mathcal{R}(A^*)}$, $(A_1X_1)^* = A_1X_1$, $(X_1A_1)^* = X_1A_1$ by Lemma 2.1 we have **for odd** n:

$$A_1X_1 = I_{\mathcal{R}(A)}, \ X_1A_1 = I_{\mathcal{R}(A^*)} \Rightarrow X_1 = A_1^{-1} \Rightarrow X = A^\dagger.$$

By the same lemma, **for even** n we have:

$$A_1X_1 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \in \mathcal{L}\left(\begin{bmatrix} S \\ S^\perp_{\mathcal{R}(A)} \end{bmatrix}\right), \ X_1A_1 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \in \mathcal{L}\left(\begin{bmatrix} T \\ T^\perp_{\mathcal{R}(A^*)} \end{bmatrix}\right),$$

where $S \oplus^\perp S^\perp_{\mathcal{R}(A)} = \mathcal{R}(A)$, $T \oplus^\perp T^\perp_{\mathcal{R}(A^*)} = \mathcal{R}(A^*)$ (clearly, $S^\perp_{\mathcal{R}(A)} = \mathcal{R}(A) \ominus S$, $T^\perp_{\mathcal{R}(A^*)} = \mathcal{R}(A^*) \ominus T$). Therefore,

$$A_1X_1 = I_{\mathcal{R}(A)} - 2P_S, \ X_1A_1 = I_{\mathcal{R}(A^*)} - 2P_T,$$

so we have

$$X_1 = A_1^{-1}(I_{\mathcal{R}(A)} - 2P_S) = (I_{\mathcal{R}(A^*)} - 2P_T)A_1^{-1},$$

from where we see the relation between the subspaces T and S :

$$A_1^{-1}P_S = P_TA_1^{-1} \Leftrightarrow P_SA_1 = A_1P_T,$$

so those projections are similar. We can put

$$X_1 = A_1^{-1}(I_{\mathcal{R}(A)} - 2P_S) = (I_{\mathcal{R}(A^*)} - 2A_1^{-1}P_SA_1)A_1^{-1}$$

When we return to the operator X, we have

$$X = A^\dagger(P_{\mathcal{R}(A)} - 2P_S) = A^\dagger - 2A^\dagger P_S = A^\dagger(I - 2P_S),$$

and on similar way

$$X = (P_{\mathcal{R}(A^*)} - 2P_T)A^\dagger = A^\dagger - 2P_TA^\dagger = (I - 2P_T)A^\dagger.$$

Also, $A^\dagger P_S = P_TA^\dagger$, or, equivalently, $P_SA = AP_T$.

Remark 2.3. *If we suppose*

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} : \begin{bmatrix} T \\ T^{\perp}_{\mathcal{R}(A^*)} \end{bmatrix} \mapsto \begin{bmatrix} S \\ S^{\perp}_{\mathcal{R}(A)} \end{bmatrix},$$

from $P_S A_1 = A_1 P_T$ *we have* $A_{12} = 0,\ A_{13} = 0$, *so the operator* A_1 *must have the following form*

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{14} \end{bmatrix} : \begin{bmatrix} T \\ T^{\perp}_{\mathcal{R}(A^*)} \end{bmatrix} \mapsto \begin{bmatrix} S \\ S^{\perp}_{\mathcal{R}(A)} \end{bmatrix},$$

where A_{11} *and* A_{14} *are invertible operators.*

We have seen that odd n case reduces to $n = 1$, which coincides with the Moore-Penrose inverse. As an important result, because $(A_1X_1)^2 = I_{\mathcal{R}(A)}$ and $(X_1A_1)^2 = I_{\mathcal{R}(A^*)}$, we have that case $n = 2k$ actually reduces to $n = 2$. Therefore, we can define new generalized inverse which depends of some subspace(s).

Definition 2.4. *Let* H, K *be arbitrary Hilbert spaces and* $A \in \mathcal{L}(H, K)$ *be closed-range operator. For fixed subspace* $S \subset \mathcal{R}(A)$ *(or, equivalently,* $T \subset \mathcal{R}(A^*)$*), where* S *and* T *are related by*

$$AP_T = P_S A, \quad \text{or, equivalently, } A^\dagger P_S = P_T A^\dagger. \tag{1}$$

there exist unique operator denoted by $A^\ddagger \equiv A^\ddagger_{T,S}$ *such that the following four Penrose-like equations are satisfied:*

$$(AA^\ddagger)^2 A = A,\ A^\ddagger (AA^\ddagger)^2 = A^\ddagger,\ (AA^\ddagger)^* = AA^\ddagger,\ (A^\ddagger A)^* = A^\ddagger A. \tag{2}$$

Such inverse will be called **extended MP inverse**, *and can be explicitly given by*

$$A^\ddagger_{T,S} = A^\dagger (I - 2P_S) = (I - 2P_T) A^\dagger. \tag{3}$$

The existence and the uniqueness of extended Moore-Penrose inverse follows immediately by preceding construction. We use both subspaces in the index although they are uniquely related ($P_T = A_1^{-1} P_S A_1$, where $A_1 = A|_{\mathcal{R}(A^*)}$), because it is convenient in various identities. Note that for trivial closed subspaces $S = \{0\}$ and $S = \mathcal{R}(A)$ we also have $A^\ddagger_{\{0\},\{0\}} = A^\dagger (I - 2P_{\{0\}}) = A^\dagger$ and $A^\ddagger_{\mathcal{R}(A),\mathcal{R}(A^*)} = A^\dagger (I - 2P_{\mathcal{R}(A)}) = -A^\dagger$.

3. Properties of EMP

It is very likely that properties of extended Moore-Penrose inverse strongly resemble to those of Moore-Penrose inverse. Also, for given orthogonal projections P_S and P_T the operators $I - 2P_S$ and $I - 2P_T$ are unitary and they are square roots of unit operators $I_{\mathcal{R}(A)}$ and $I_{\mathcal{R}(A^*)}$ on appropriate Hilbert spaces.

Theorem 3.1. *Let* $A \in \mathcal{L}(H, K)$ *be closed-range operator, let* $S \subset \mathcal{R}(A)$ *and* $T \subset \mathcal{R}(A^*)$ *be nontrivial closed subsets.*

1. $A^\ddagger_{T,S} P_S = -A^\dagger P_S,\ P_T A^\ddagger_{T,S} = -P_T A^\dagger,\ P_T A^\ddagger_{T,S} P_S = -P_T A^\dagger P_S$;
2. $AA^\ddagger_{T,S} = P_{\mathcal{R}(A)} - 2P_S,\ A^\ddagger_{T,S} A = P_{\mathcal{R}(A^*)} - 2P_T$; *those operators are Hermitian, but they are not idempotents. Also we have:*
$$P_S = \frac{1}{2}(P_{\mathcal{R}(A)} - AA^\ddagger_{T,S}),\ P_T = \frac{1}{2}(P_{\mathcal{R}(A^*)} - A^\ddagger_{T,S} A);$$
3. $A^\ddagger_{T,S} = A^\dagger A A^\ddagger_{T,S} = A^\ddagger_{T,S} A A^\dagger = A^\dagger A A^\ddagger_{T,S} A A^\dagger$;
4. $A^\dagger - A^\ddagger_{T,S} = 2A^\dagger P_S = 2P_T A^\dagger$, *so* $\|A^\dagger - A^\ddagger_{T,S}\| \le 2\|A^\dagger\|$.

Proof. It follows from (3), with $S \subset \mathcal{R}(A) \Leftrightarrow P_{\mathcal{R}(A)} P_S = P_S$ and $T \subset \mathcal{R}(A^*) \Leftrightarrow P_T P_{\mathcal{R}(A^*)} = P_T$ (Th. 1.5.b). □

By the definition, for fixed $S \subset \mathcal{R}(A)$ and $T \subset \mathcal{R}(A^*)$, related by (1), there exists unique $A^{\ddagger}_{T,S}$. By the preceding theorem, part 2, for given $A^{\ddagger}_{T,S}$ one can reconstruct subspaces T and S, and the relation (1) holds.

Some properties of extended Moore-Penrose inverse, similar to those of the ordinary Moore-Penrose inverse, are presented in the next theorem (cf. Proposition 1.3).

Theorem 3.2. *Let $A \in \mathcal{L}(H,K)$ be closed-range operator, and $S \subset \mathcal{R}(A)$ and $T \subset \mathcal{R}(A^*)$ nontrivial closed subspaces. Then we have:*

1. $(\lambda A)^{\ddagger}_{T,S} = \lambda^{\dagger} A^{\ddagger}_{T,S}$, *where* $\lambda^{\dagger} = \lambda^{-1}$ *if* $\lambda \neq 0$ *and* $\lambda^{\dagger} = 0$ *if* $\lambda = 0$;
2. $(AA^{\ddagger}_{T,S})^2 = P_{\mathcal{R}(A)}$, $(A^{\ddagger}_{T,S}A)^2 = P_{\mathcal{R}(A^*)}$;
3. $A^*(AA^{\ddagger}_{T,S})^2 = A^* = (A^{\ddagger}_{T,S}A)^2A^*$, $A^* - A^*AA^{\ddagger}_{T,S} = 2A^*P_S$, $A^* - A^{\ddagger}_{T,S}AA^* = 2P_TA^*$;
4. $(A^{\ddagger}_{T,S})^* = (A^*)^{\ddagger}_{S,T}$;
5. $A^{\ddagger}_{T,S} = A^*(AA^*)^{\ddagger}_{S,S}$, $A^{\ddagger}_{T,S} = (A^*A)^{\ddagger}_{T,T}A^*$;
6. $(A^*)^{\ddagger}_{S,T}A^{\ddagger}_{T,S} = (AA^*)^{\dagger}$, $A^{\ddagger}_{T,S}(A^*)^{\ddagger}_{S,T} = (A^*A)^{\dagger}$;
7. $A - AA^{\ddagger}_{T,S}A = 2P_SA = 2AP_T \neq 0$;
8. $\mathcal{R}(A^{\ddagger}_{T,S}) = \mathcal{R}(A^*)$, $\mathcal{N}(A^{\ddagger}_{T,S}) = \mathcal{N}(A^*)$;
9. $(A^{\ddagger}_{T,S})^{\ddagger}_{S,T} = A$;
10. $(A^{\ddagger}_{T,S})^{\dagger} = (I - 2P_S)A = A(I - 2P_T) = (A^{\dagger})^{\ddagger}_{S,T}$.

Proof. Recall, $V \subset W \Leftrightarrow P_VP_W = P_WP_V = P_V$, by Theorem 1.5.b.

1. $(\lambda A)^{\ddagger}_{T,S} = (\lambda A)^{\dagger}(I - 2P_S) = \lambda^{\dagger}A^{\dagger}(I - 2P_S) = \lambda^{\dagger}A^{\ddagger}_{T,S}$;
2. We have $(AA^{\ddagger}_{T,S})^2 = (AA^{\dagger}(I - 2P_S))^2 = (P_{\mathcal{R}(A)} - 2P_S)^2 = P_{\mathcal{R}(A)}$ and $(A^{\ddagger}_{T,S}A)^2 = (P_{\mathcal{R}(A^*)} - 2P_T)^2 = P_{\mathcal{R}(A^*)}$.
3. By 2. and Proposition 1.3.3, we have $A^*P_{\mathcal{R}(A)} = A^* = P_{\mathcal{R}(A^*)}A$. The second part is due to $A^* - A^*AA^{\ddagger}_{T,S} = A^* - A^*(P_{\mathcal{R}(A)} - 2P_S) = 2A^*P_S$ and $A^* - A^{\ddagger}_{T,S}AA^* = A^* - (P_{\mathcal{R}(A^*)} - 2P_T)A^* = 2P_TA^*$.
4. $(A^{\ddagger}_{T,S})^* = (A^{\dagger}(I - 2P_S))^* = (I - 2P_S)(A^{\dagger})^* = (I - 2P_S)(A^*)^{\dagger} = (A^*)^{\ddagger}_{S,T}$; also $A^*P_S = (P_SA)^* = (AP_T)^* = P_TA^*$.
5. We have $A^*(AA^*)^{\ddagger}_{S,S} = A^*(AA^*)^{\dagger}(I - 2P_S) = A^{\dagger}(I - 2P_S) = A^{\ddagger}_{T,S}$, and $(A^*A)^{\ddagger}_{T,T}A^* = (I - 2P_T)(A^*A)^{\dagger}A^* = (I - 2P_T)A^{\dagger} = A^{\ddagger}_{T,S}$. Note that $S \subset \mathcal{R}(AA^*) = \mathcal{R}(A)$ and $T \subset \mathcal{R}(A^*A) = \mathcal{R}(A^*)$.
6. $(A^*)^{\ddagger}_{S,T}A^{\ddagger}_{T,S} = (A^{\ddagger}_{T,S})^*A^{\ddagger}_{T,S} = ((I - 2P_T)A^{\dagger})^*(I - 2P_T)A^{\dagger} = (A^{\dagger})^*(I - 2P_T)^2A^{\dagger} = (A^*)^{\dagger}A^{\dagger} = (AA^*)^{\dagger}$, and $A^{\ddagger}_{T,S}(A^*)^{\ddagger}_{S,T} = A^{\dagger}(I - 2P_S)(A^{\dagger}(I - 2P_S))^* = A^{\dagger}(I - 2P_S)^2(A^{\dagger})^* = A^{\dagger}(A^*)^{\dagger} = (A^*A)^{\dagger}$.
7. $S \subset \mathcal{R}(A) \Rightarrow P_SA \neq 0$, so this difference cannot be zero.
8. The operators $I - 2P_S$ and $I - 2P_T$ are unitary, hence invertible, so $\mathcal{R}(A^{\ddagger}_{T,S}) = \mathcal{R}(A^{\dagger}(I - 2P_S)) = A^{\dagger}((I - 2P_S)(K)) = A^{\dagger}(K) = \mathcal{R}(A^{\dagger}) = \mathcal{R}(A^*)$, $\mathcal{N}(A^{\ddagger}_{T,S}) = \mathcal{N}((I - 2P_T)A^{\dagger}) = \mathcal{N}(A^{\dagger}) = \mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$.
9. Let us note that the reverse order law $(A^{\dagger}(I - 2P_S))^{\dagger} = (I - 2P_S)A$ holds, because $I - 2P_S$ is unitary operator (hence Hermitian and invertible). Now we have
$$(A^{\ddagger}_{T,S})^{\ddagger}_{S,T} = (A^{\ddagger}_{T,S})^{\dagger}(I - 2P_T) = (A^{\dagger}(I - 2P_S))^{\dagger}(I - 2P_T) = (I - 2P_S)A(I - 2P_T) = A,$$
because $AP_T = P_SA \Rightarrow AP_T = P_SAP_T$. Also, $A^{\ddagger}_{T,S}P_S = P_TA^{\ddagger}_{T,S} \Leftrightarrow A^{\dagger}(I - 2P_S)P_S = P_T(I - 2P_T)A^{\dagger} \Leftrightarrow A^{\dagger}P_S = P_TA^{\dagger}$.
10. Because of $(A^{\ddagger}_{T,S})^{\dagger} = (A^{\dagger}(I - 2P_S))^{\dagger} = (I - 2P_S)A$, and $(A^{\dagger})^{\ddagger}_{S,T} = (A^{\dagger})^{\dagger}(I - 2P_T) = A(I - 2P_T)$, we have the proof. Note that 8. implies the existence of $(A^{\ddagger}_{T,S})^{\dagger}$.

□

Unlike the ordinary Moore-Penrose inverse, the extended Moore-Penrose inverse depends on some subspaces, and we present some related properties.

Theorem 3.3. *Let $A \in \mathcal{L}(H,K)$ be closed-range operator, and $S, S_1, S_2 \subset \mathcal{R}(A)$, $T, T_1, T_2 \subset \mathcal{R}(A^*)$ be nontrivial closed subsets. Then we have*

1. $A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = A^{\dagger}(I - 2P_{S_1} - 2P_{S_2} + 4P_{S_1}P_{S_2})$.
   - $S_1 \cap S_2 = \{0\} \Rightarrow A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = A^{\dagger}(I - 2(P_{S_1} + P_{S_2}))$;
   - $S_1 \oplus^{\perp} S_2 = \mathcal{R}(A) \Rightarrow A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = -A^{\dagger}$;
   - $S_1 \perp S_2 \Rightarrow A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = A^{\ddagger}_{T_1 \oplus T_2, S_1 \oplus S_2}$.
2. $A^{\ddagger}_{T_1,S_1} - A^{\ddagger}_{T_2,S_2} = 2A^{\dagger}(P_{S_2} - P_{S_1})$; *particularly, if $S_1 \subset S_2$ then* $A^{\ddagger}_{T_1,S_1} - A^{\ddagger}_{T_2,S_2} = A^{\dagger} - A^{\ddagger}_{T_2 \ominus T_1, S_2 \ominus S_1}$;
3. $AA^{\ddagger}_{T_1,S_1} - AA^{\ddagger}_{T_2,S_2} = 2(P_{S_2} - P_{S_1})$; *particularly, if $S_1 \subset S_2$ then* $AA^{\ddagger}_{T_1,S_1} - AA^{\ddagger}_{T_2,S_2} = 2P_{T_2 \ominus T_1, S_2 \ominus S_1}$;
4. $A^{\ddagger}_{T_1,S_1} A - A^{\ddagger}_{T_2,S_2} A = 2(P_{T_2} - P_{T_1})$; *particularly, if $T_1 \subset T_2$ then* $A^{\ddagger}_{T_1,S_1} A - A^{\ddagger}_{T_2,S_2} A = 2P_{T_2 \ominus T_1, S_2 \ominus S_1}$;
5. $A, B \in \mathcal{L}(H,K)$, $\mathcal{R}(A) = \mathcal{R}(B) \supsetneq S$, *then* $(A^{\ddagger}_{T,S} - B^{\ddagger}_{T,S})^{\dagger} = (A^{\dagger} - B^{\dagger})^{\ddagger}_{S,T}$.

Proof.

1. $A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = A^{\dagger}(I - 2P_{S_1}) A A^{\dagger}(I - 2P_{S_2}) = A^{\dagger}(I - 2P_{S_1}) P_{\mathcal{R}(A)} (I - 2P_{S_2}) = A^{\dagger}(I - 2P_{S_1} - 2P_{S_2} + 4P_{S_1}P_{S_2})$. If $S_1 \cap S_2 = \{0\}$, then $P_{S_1}P_{S_2} = 0$, so $A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = A^{\dagger}(I - 2(P_{S_1} + P_{S_2}))$. When $S_1 \oplus^{\perp} S_2 = \mathcal{R}(A)$, then $A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = -A^{\dagger}$. For $S_1 \perp S_2$, by Theorem 1.6 we have $P_{S_1}P_{S_2} = 0$ and $P_{S_1} + P_{S_2} = P_{S_1 \oplus S_2}$, therefore $A^{\ddagger}_{T_1,S_1} A A^{\ddagger}_{T_2,S_2} = A^{\dagger}(I - 2P_{S_1 \oplus S_2}) = A^{\ddagger}_{T_1 \oplus T_2, S_1 \oplus S_2}$.
2. $A^{\ddagger}_{T_1,S_1} - A^{\ddagger}_{T_2,S_2} = A^{\dagger}(I - 2P_{S_1}) - A^{\dagger}(I - 2P_{S_2}) = 2A^{\dagger}(P_{S_2} - P_{S_1})$. When $S_1 \subset S_2$, by Theorem 1.6 it follows $P_{S_2} - P_{S_1} = P_{S_2 \ominus S_1}$ is orthogonal projection, therefore

$$A^{\ddagger}_{T_1,S_1} - A^{\ddagger}_{T_2,S_2} = 2A^{\dagger} P_{S_2 \ominus S_1} = A^{\dagger}(I - (I - 2P_{S_2 \ominus S_1})) = A^{\dagger} - A^{\ddagger}_{T_2 \ominus T_1, S_2 \ominus S_1}.$$

   Note that $S_1 \subset S_2 \Leftrightarrow P_{S_1}P_{S_2} = P_{S_2}P_{S_1} = P_{S_1} \Leftrightarrow P_{S_1}P_{S_2}A = P_{S_2}P_{S_1}A = P_{S_1}A \Leftrightarrow P_{S_1}AP_{T_2} = P_{S_2}AP_{T_1} = AP_{T_1} \Leftrightarrow AP_{T_1}P_{T_2} = AP_{T_2}P_{T_1} = AP_{T_1} \Leftrightarrow A^{\dagger}AP_{T_1}P_{T_2} = A^{\dagger}AP_{T_2}P_{T_1} = A^{\dagger}AP_{T_1} \Leftrightarrow P_{T_1}P_{T_2} = P_{T_2}P_{T_1} = P_{T_1} \Leftrightarrow T_1 \subset T_2$.
3. $AA^{\ddagger}_{T_1,S_1} - AA^{\ddagger}_{T_2,S_2} = 2AA^{\dagger}(P_{S_2} - P_{S_1}) = 2(P_{S_2} - P_{S_1})$; the rest of the proof as in the second part.
4. Analogous to the proof of part 3.
5. By part 10 of Theorem 3.2 and part 8 of Proposition 1.1, we have

$$((A^{\dagger} - B^{\dagger})^{\ddagger}_{S,T})^{\dagger} = ((A^{\dagger} - B^{\dagger})^{\dagger}(I - 2P_T))^{\dagger} = (I - 2P_T)(A^{\dagger} - B^{\dagger}) = A^{\ddagger}_{T,S} - B^{\ddagger}_{T,S}.$$

□

Theorem 3.4. *Let $A \in \mathcal{L}(H,K)$ be closed-range operator, and S_i, $i = \overline{1,n}$, $n \geq 2$, be closed subspaces of $\mathcal{R}(A)$, such that $\mathcal{R}(A)$ is their orthogonal direct sum (i.e. $\mathcal{R}(A) = S_1 \oplus^{\perp} S_2 \oplus^{\perp} \ldots \oplus^{\perp} S_n$). Then:*

$$\sum_{k=1}^{n} A^{\ddagger}_{T_k,S_k} = (n-2)A^{\dagger}.$$

Here T_k, $k = \overline{1,n}$, are related to S_k, $k = \overline{1,n}$, by (1).

Proof. Because of

$$S_1 \oplus^{\perp} S_2 \oplus^{\perp} \ldots \oplus^{\perp} S_n = \mathcal{R}(A) \Leftrightarrow P_{S_1} + P_{S_2} + \ldots + P_{S_n} = I_{\mathcal{R}(A)},$$

we have

$$\sum_{k=1}^{n} A^{\ddagger}_{T_k,S_k} = A^{\dagger} \sum_{k=1}^{n} (I - 2P_{S_k}) = A^{\dagger}\left(\sum_{k=1}^{n-1} (I - 2P_{S_k}) + I - 2\Big(P_{\mathcal{R}(A)} - 2\sum_{k=1}^{n-1} P_{S_k}\Big)\right) = A^{\dagger}(nI - 2P_{\mathcal{R}(A)}) = (n-2)A^{\dagger}.$$

□

In the case when there are just two subspaces, the following corollary holds.

Corollary 3.5. *Let $A \in \mathcal{L}(H, K)$ be closed-range operator and $S \subset \mathcal{R}(A)$ nontrivial closed subspace. Then we have*

$$A^{\ddagger}_{T^{\perp},S^{\perp}} = -A^{\ddagger}_{T,S},$$

where $S^{\perp}_{\mathcal{R}(A)}$ is closed subspace such that $S \oplus^{\perp} S^{\perp}_{\mathcal{R}(A)} = \mathcal{R}(A)$. Here $T \subset \mathcal{R}(A^*)$ is related to S by (1).

Next result establishes the connection between extended Moore-Penrose equation and some other generalized inverses:

Theorem 3.6. *Let $A \in \mathcal{L}(H, K)$ be closed-range operator, and $S \subset \mathcal{R}(A)$ and $T \subset \mathcal{R}(A^*)$ nontrivial closed subspaces. Then we have:*

1. $A^{\dagger} = A^{\ddagger}_{T,S}(I - 2P_S) = (I - 2P_T)A^{\ddagger}_{T,S}$;
2. $A^{\ddagger}_{T,S} A A^{\ddagger}_{T,S} = A^{\dagger}$;
3. $A^{\ddagger}_{T,S}$ *is EP if and only if A is EP.*

Proof. Recall that $V \subset W \Leftrightarrow P_V P_W = P_W P_V = P_V$, by Theorem 1.5.b.

1. Operator $I - 2P_S$ is unitary, therefore $A^{\ddagger}_{T,S} = A^{\dagger}(I - 2P_S) \Leftrightarrow A^{\dagger} = A^{\ddagger}_{T,S}(I - 2P_S)$.
2. $A^{\ddagger}_{T,S} A A^{\ddagger}_{T,S} = A^{\ddagger}_{T,S}(P_{\mathcal{R}(A)} - 2P_S) = A^{\dagger}(I - 2P_S)(P_{\mathcal{R}(A)} - 2P_S) = A^{\dagger}P_{\mathcal{R}(A)} = A^{\dagger}$;
3. The proof follows from the following equivalence chain:

$$A^{\ddagger}_{T,S}(A^{\ddagger}_{T,S})^{\dagger} = (A^{\ddagger}_{T,S})^{\dagger}A^{\ddagger}_{T,S} \Leftrightarrow A^{\dagger}(I - 2P_S)(A^{\dagger}(I - 2P_S))^{\dagger} = ((I - 2P_T)A^{\dagger})^{\dagger}(I - 2P_T)A^{\dagger} \Leftrightarrow$$
$$\Leftrightarrow A^{\dagger}(I - 2P_S)^2 A = A(I - 2P_T)^2 A^{\dagger} \Leftrightarrow A^{\dagger}A = AA^{\dagger}.$$

□

Theorem 3.7. *Let $A \in \mathcal{L}(H, K)$ be closed range operators and $S \subset \mathcal{R}(A)$, $T \subset \mathcal{R}(A^*)$ nontrivial closed subsets. Then we have the following norm estimates:*

1. $\|A^{\ddagger}_{T,S}\| = \|A^{\dagger}\|$;
2. $\|A - AA^{\ddagger}_{T,S}A\| \le 2\|A\|$;

Proof. 1. From (3) we have

$$\|A^{\ddagger}_{T,S}\| = \|A^{\dagger}(I - 2P_S)\| \le \|A^{\dagger}\|,$$

while from Theorem 3.6, part 1, it follows

$$\|A^{\dagger}\| = \|A^{\ddagger}_{T,S}(I - 2P_S)\| \le \|A^{\ddagger}_{T,S}\|.$$

2. It follows from $A - AA^{\ddagger}_{T,S}A = 2P_S A$, because $\|P_S\| = 1$.

□

Proposition 3.8. *Consider the operator equation $Ax = b$. We have the following possibilities:*

- $b \notin \mathcal{R}(A) : AA^{\ddagger}_{T,S}b = (P_{\mathcal{R}(A)} - 2P_S)b = 0,$
- $b \in \mathcal{R}(A) \setminus S : AA^{\ddagger}_{T,S}b = (P_{\mathcal{R}(A)} - 2P_S)b = b,$
- $b \in S : AA^{\ddagger}_{T,S}b = (P_{\mathcal{R}(A)} - 2P_S)b = -b.$

Therefore, $x = A^{\ddagger}_{T,S}b$ is a solution when $b \in \mathcal{R}(A) \setminus S$, and $x = -A^{\ddagger}_{T,S}b$ is solution for $b \in S$.

4. Some examples

- It is obvious that $A = 0 \Leftrightarrow A^{\ddagger} = 0$.
- For $A = I \in \mathcal{L}(H)$ and given subspace $S \subset H$ we have $X^* = X$ and $X^2 = I$, so $I^{\ddagger}_{T,S} = I - 2P_S = I - 2P_T$.
- Suppose $A \in \mathcal{L}(H)$ is invertible, and $S, T \subset H$ are given. By the equations, we have

  $$(AX)^2 = I = (XA)^2, \ (AX)^* = AX, \ (XA)^* = XA.$$

  The reasonings similar to those preceding the definition gives us $AX = I - 2P_S, \ XA = I - 2P_T$. Therefore,

  $$A^{\ddagger}_{T,S} = A^{-1}(I - 2P_S) = (I - 2P_T)A^{-1}, \text{ where } A^{-1}P_S = P_T A^{-1}.$$

  So, the subspaces S, T are similar $P_T = A^{-1}P_S A$. Also in this case we have

  $$P_S = \frac{1}{2}(I - AA^{\ddagger}_S), \ P_T = \frac{1}{2}(I - A^{\ddagger}_S A).$$

- Let R and L be the right shift and left shift operator, respectively, defined on separable Hilbert space ℓ^2 with canonical basis $(\{e_1, e_2, ...\})$ on usual way

  $$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots), \ L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

  It is not hard to see that $R^{\dagger} = R^* = L$ and $\mathcal{R}(R) = lin\{e_2, e_3, \ldots\}$.

  Let $S_1 = lin\{e_3, e_5, \ldots\}$ and $S_2 = \{e_2, e_4, \ldots\}$ be given subspaces of $\mathcal{R}(R)$ such that $S_1 \oplus^{\perp} S_2 = \mathcal{R}(R)$. Then we have for any $x \in \ell^2$:

  $$\begin{aligned} R^{\ddagger}_{T_1,S_1}x &= R^{\dagger}(I - 2P_{S_1})x = R^{\dagger}(x - 2(0, 0, x_3, 0, x_5, 0, \ldots)) = L(x_1, x_2, -x_3, x_4, -x_5, \ldots) = (x_2, -x_3, x_4, -x_5, \ldots), \\ R^{\ddagger}_{T_2,S_2}x &= R^{\dagger}(I - 2P_{S_2})x = R^{\dagger}(x - 2(0, x_2, 0, x_4, 0, \ldots)) = L(x_1, -x_2, x_3, -x_4, x_5, \ldots) = (-x_2, x_3, -x_4, x_5, \ldots). \end{aligned}$$

  It is obvious that $R_{T_1,S_1}x + R_{T_2,S_2}x = 0$, therefore $R_{T_1,S_1} + R_{T_2,S_2} = 0$.

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