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Approximation by Operators Including Generalized Appell Polynomials

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Abstract. In this work, the problem of the approximation by certain polynomials is addressed. A new type operators sequence including generalized Appell polynomials are defined, qualitative and quantitative approximation theorems are proved. Some explicit examples of our operators involving Hermite polynomials of ν variance, Gould-Hopper polynomials and Miller-Lee polynomials are given. Also, we present some numerical examples to confirm our theoretical results.

1. Introduction

In mathematics, approximation theory is related to how functions can be approximated with uncomplicated functions such as polynomials, wavelets or special functions and to quantitatively characterizing the errors. The essence of the theory of approximation of functions is a theorem proved by Weierstrass which is great significance in the advancement of the whole of mathematical analysis.

In 1950, Szasz [16] introduced and investigated the following operators known as Szasz-Mirakjan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1}$$

where $n \in \mathbb{N}$, $x \geq 0$ and $f \in C[0,\infty)$ whenever the above sum converges. These operators are generalizations of Bernstein polynomials to the infinite interval. Some approximation properties of the operators (1) are discussed by many authors.

Cheney and Sharma [3] presented the operators including orthogonal polynomials as follows

$$P_n(f;x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t)\, x^k$$

where $t \leq 0$ and $L_k^{(n)}(t)$ denotes the Laguerre polynomials. After this construction, the notion of orthogonal polynomials has appeared in the positive approximation processes.

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Later, Jakimovski and Leviatan [12] constructed a generalization of Szasz operators with Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function in the disc $|z| < R$ $(R > 1)$ and assume that $g(1) \neq 0$. The Appell polynomials $p_k(x)$ have the generating functions of the form

$$g(u) e^{ux} = \sum_{k=0}^{\infty} p_k(x) u^k . \tag{2}$$

Under the assumption $p_k(x) \geq 0$ for $x \in [0, \infty)$, Jakimovski and Leviatan introduced the linear positive operators $P_n(f; x)$ via

$$P_n(f; x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) , \quad \text{for } n \in \mathbb{N} \tag{3}$$

and gave the approximation properties of these operators with the help of Szasz's method.

Then, Ismail [11] obtained another generalization of Szasz operators (1) and also Jakimovski and Leviatan operators (3) through the instrument of Sheffer polynomials. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ and $H(z) = \sum_{k=1}^{\infty} h_k z^k$ $(h_1 \neq 0)$ be analytic functions in the disc $|z| < R$ $(R > 1)$ where a_k and h_k are real. The Sheffer polynomials $p_k(x)$ have the generating functions of the type

$$A(t) e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x) t^k, \qquad |t| < R .$$

By the help of following assumptions

(i) for $x \in [0, \infty)$, $p_k(x) \geq 0$,
(ii) $A(1) \neq 0$ and $H'(1) = 1$,

Ismail investigated the convergence properties of linear positive operators given by

$$T_n(f; x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) , \quad \text{for } n \in \mathbb{N} .$$

Brenke type polynomials [4] have the generating functions of the form

$$A(t) B(xt) = \sum_{k=0}^{\infty} p_k(x) t^k \tag{4}$$

where A and B are analytic functions

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \tag{5}$$

$$B(t) = \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \ (r \geq 0) \tag{6}$$

and have the following explicit expression

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \quad k = 0, 1, 2, \ldots . \tag{7}$$

With the help of following assumptions

$$\begin{array}{ll} (i) & A(1) \neq 0,\ \frac{a_{k-r}b_r}{A(1)} \geq 0,\ 0 \leq r \leq k,\ k = 0,1,2,...,\\ (ii) & B : [0,\infty) \longrightarrow (0,\infty),\\ (iii) & (4) \text{ and the power series } (5) \text{ and } (6) \text{ converge for } |t| < R\ (R > 1), \end{array} \tag{8}$$

Varma et al. [18] introduced the following linear positive operators involving the Brenke type polynomials

$$L_n(f;x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{9}$$

where $x \geq 0$ and $n \in \mathbb{N}$. There have also been many remarkable contributions to Szasz type operators ([1],[9],[10]).

Motivated by the above mentioned works, we consider the linear operators as follows

$$\mathcal{M}_n(f;x) := \frac{1}{A(g(1))B(nxg(1))} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{10}$$

where $p_k(x)$ are generalized Appell polynomials [15] having the generating functions of the following form

$$A(g(t))B(xg(t)) = \sum_{k=0}^{\infty} p_k(x) t^k \tag{11}$$

and A, B and g are analytic functions such that

$$A(t) = \sum_{k=0}^{\infty} a_k t^k\ (a_0 \neq 0),\ B(t) = \sum_{k=0}^{\infty} b_k t^k\ (b_k \neq 0),\ g(t) = \sum_{k=1}^{\infty} g_k t^k\ (g_1 \neq 0). \tag{12}$$

We shall restrict ourselves to the generalized Appell polynomials (11) satisfying

$$\begin{array}{ll} (i) & A(g(1)) \neq 0,\ g'(1) = 1,\ p_k(x) \geq 0\ \ k = 0,1,2,...,\\ (ii) & B : \mathbb{R} \longrightarrow (0,\infty),\\ (iii) & (11) \text{ and the power series } (12) \text{ converge for } |t| < R\ (R > 1). \end{array} \tag{13}$$

By virtue of the above restrictions, $\mathcal{M}_n$ linear operators defining by (10) are positive.

Remark 1.1. *Let $g(t) = t$. The operators* (10) (*resp.* (11)) *reduce to the operators given by* (9) (*resp.* (4)).

Remark 1.2. *Let $g(t) = t$ and $B(t) = e^t$. It is obvious that one can get the operators* (3) *from the operators* (10). *In addition, if we choose $A(t) = 1$, we meet again well-known Szasz operators* (1).

The outline of this paper is as follows. Section 2 contains qualitative and quantitative results obtained by classical modulus of continuity and Peetre's K functional for the operators (10). In section 3, some examples are provided to illustrate the main ideas given in Section 2 and also, we give some numerical examples to confirm our theoretical results.

2. Approximation to Functions Using $\mathcal{M}_n$ Operators

In this section, we get qualitative convergence result by means of $\mathcal{M}_n$ operators with the help universal Korovkin-type property with respect to positive linear operators. Next, we state quantitative results for estimating the error of approximation using the classical approach, the second modulus of continuity and Peetre's K functional in the continuous functions space and the Lipschitz class.

Let us define the class of E as follows

$$E := \left\{ f : x \in [0, \infty), \ \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}.$$

For the proof of our theorems, the following lemmas are required.

Lemma 2.1. *For the operators $\mathcal{M}_n$, we have*

$$\begin{aligned}
\mathcal{M}_n(1;x) &= 1 \\
\mathcal{M}_n(s;x) &= \frac{B'(nxg(1))}{B(nxg(1))}x + \frac{A'(g(1))}{nA(g(1))} \\
\mathcal{M}_n\left(s^2;x\right) &= \frac{B''(nxg(1))}{B(nxg(1))}x^2 + \frac{\left[2A'(g(1)) + A(g(1))\left(g''(1)+1\right)\right]B'(nxg(1))}{nA(g(1))B(nxg(1))}x \\
&\quad + \frac{A'(g(1))\left(1+g''(1)\right) + A''(g(1))}{n^2A(g(1))},
\end{aligned}$$

for any $x \in [0, \infty)$.

Proof. *Using the generating functions of the generalized Appell polynomials given by* (11), *we have*

$$\begin{aligned}
\sum_{k=0}^{\infty} p_k(nx) &= A(g(1))B(nxg(1)) \\
\sum_{k=0}^{\infty} kp_k(nx) &= A'(g(1))B(nxg(1)) + nxA(g(1))B'(nxg(1)) \\
\sum_{k=0}^{\infty} k^2p_k(nx) &= \left(A'(g(1))\left(g''(1)+1\right) + A''(g(1))\right)B(nxg(1)) \\
&\quad + nxB'(nxg(1))\left(A(g(1))\left(g''(1)+1\right) + 2A'(g(1))\right) \\
&\quad + (nx)^2A(g(1))B''(nxg(1)).
\end{aligned}$$

In accordance with the above equalities, the proof of lemma follows. □

Theorem 2.2. *Let f be in $C[0,\infty) \cap E$ and assume that the following conditions*

$$\lim_{y\to\infty} \frac{B'(y)}{B(y)} = 1 \quad \text{and} \quad \lim_{y\to\infty} \frac{B''(y)}{B(y)} = 1 \tag{14}$$

are satisfied. Then,

$$\lim_{n\to\infty} \mathcal{M}_n(f;x) = f(x)$$

uniformly on each compact subset of $[0, \infty)$.

Proof. Considering the assumptions (14) in Lemma 2.1, we find

$$\lim_{n\to\infty} \mathcal{M}_n\left(s^i;x\right) = x^i, \quad i = 0, 1, 2$$

uniformly on each compact subset of $[0, \infty)$. Applying the universal Korovkin-type property (vi) of Theorem 4.1.4 from [2], the proof is completed. □

In order to estimate the order of approximation, we will give some definitions and lemmas.

Definition 2.3. *Let $f \in \tilde{C}[0,\infty)$ and $\delta > 0$. The modulus of continuity $\omega(f;\delta)$ of the function f is defined by*

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty) \\ |x-y|\le\delta}} \left|f(x) - f(y)\right|$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Let us denote the $C_B[0,\infty)$ space of all bounded and continuous functions on $[0,\infty)$.

Definition 2.4. *The second modulus of continuity of the function $f \in C_B[0,\infty)$ is defined by*

$$\omega_2(f;\delta) := \sup_{0<t\le\delta} \left\|f(.+2t) - 2f(.+t) + f(.)\right\|_{C_B[0,\infty)}$$

where $\left\|f\right\|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} \left|f(x)\right|$.

Definition 2.5. *([6]) The Peetre's K functional of function $f \in C_B[0,\infty)$ is defined by*

$$\mathcal{K}(f;\delta) = \inf\left\{\left\|f-h\right\|_{C_B[0,\infty)} + \delta\left\|h\right\|_{C_B^2[0,\infty)}\right\}$$

where $C_B^2[0,\infty) = \left\{h \in C_B[0,\infty) :\ h',\ h'' \in C_B[0,\infty)\right\}$ with the norm

$$\|h\|_{C_B^2[0,\infty)} = \|h\|_{C_B[0,\infty)} + \left\|h'\right\|_{C_B[0,\infty)} + \left\|h''\right\|_{C_B[0,\infty)}.$$

It is known that there exists a constant $C > 0$ such that

$$\mathcal{K}(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right). \tag{15}$$

Lemma 2.6. *For $x \in [0,\infty)$, the following identities*

$$\begin{aligned}
\mathcal{M}_n(s-x;x) &= \frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x \\
\mathcal{M}_n\left((s-x)^2;x\right) &= \frac{\left(1+g''(1)\right)A'(g(1)) + A''(g(1))}{n^2A(g(1))} \\
&\quad + \frac{2A'(g(1))\left(B'(nxg(1)) - B(nxg(1))\right)}{nA(g(1))B(nxg(1))}x \\
&\quad + \frac{B'(nxg(1))\left(1+g''(1)\right)}{nB(nxg(1))}x \\
&\quad + \frac{B''(nxg(1)) - 2B'(nxg(1)) + B(nxg(1))}{B(nxg(1))}x^2
\end{aligned}$$

are satisfied.

Proof. Using the linearity property of $\mathcal{M}_n$ operators and applying Lemma 2.1, we obtain the equalities stated in the lemma. □

The error of approximation can be obtained as the estimation of the difference of $\left|\mathcal{M}_n(f;x) - f(x)\right|$. In order to get the quantitative results, we investigate this difference in some function spaces.

Theorem 2.7. *Let* $f \in \tilde{C}[0,\infty) \cap E$*, then*

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \leq 2\omega(f;\delta_n(x))$$

where $\delta_n(x) := \sqrt{\mathcal{M}_n\left((s-x)^2;x\right)}$.

Proof. With the help of Lemma 2.1, monotonicity properties of operators $\mathcal{M}_n$ and property of modulus of continuity, one can write

$$\begin{aligned} \left|\mathcal{M}_n(f;x) - f(x)\right| &\leq \mathcal{M}_n\left(\left|f(s) - f(x)\right|;x\right) \\ &\leq \omega(f;\delta)\left(1 + \frac{1}{\delta}\mathcal{M}_n(|s-x|;x)\right). \end{aligned}$$

According to the Cauchy-Schwarz inequality, we get

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \leq \omega(f;\delta)\left(1 + \frac{1}{\delta}\sqrt{\mathcal{M}_n\left((s-x)^2;x\right)}\right).$$

By taking $\delta := \delta_n(x) = \sqrt{\mathcal{M}_n\left((s-x)^2;x\right)}$, we obtain the desired result. □

Lipschitz class of order α, $Lip_M(\alpha)$ $(0 < \alpha \leq 1,\ M > 0)$, is defined as follows

$$Lip_M(\alpha) := \left\{f \in C_B[0,\infty) : \left|f(t) - f(x)\right| \leq M|t-x|^\alpha,\ \ t,x \in [0,\infty)\right\}.$$

The following result provides an estimate for the approximation error of $\mathcal{M}_n$ operators to function $f \in Lip_M(\alpha)$.

Theorem 2.8. *Let* f *be in* $Lip_M(\alpha)$*. For* $x \geq 0$ *we have*

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \leq M\delta_n^\alpha(x)$$

where $\delta_n(x) := \sqrt{\mathcal{M}_n\left((s-x)^2;x\right)}$.

Proof. From the monotonicity properties of operators $\mathcal{M}_n$, we get

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \leq M\mathcal{M}_n(|s-x|^\alpha;x). \tag{16}$$

Applying the Hölder inequality, one can deduce from (16)

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \leq M\left(\mathcal{M}_n\left((s-x)^2;x\right)\right)^{\alpha/2}.$$

Thus, we complete the proof of theorem. □

The Peetre's K functional turned out to be a very instrumental tool in approximation theory for estimating the error. For this purpose, we are going to evaluate the degree of approximation with Peetre's K-functional in the following theorem.

Theorem 2.9. *For every* $f \in C_B[0,\infty)$ *and* $x \in [0,\infty)$*, the following statement holds*

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \leq 2\mathcal{K}(f;\lambda_n(x))$$

where

$$\begin{aligned}\lambda_n(x) &= \frac{B''(nxg(1)) - 2B'(nxg(1)) + B(nxg(1))}{4B(nxg(1))}x^2 \\ &+ \frac{\left(A'(g(1)) + nA(g(1))\right)\left(B'(nxg(1)) - B(nxg(1))\right)}{2nA(g(1))B(nxg(1))}x \\ &+ \frac{B'(nxg(1))\left(1 + g''(1)\right)}{4nB(nxg(1))}x + \frac{\left(2n + 1 + g''(1)\right)A'(g(1)) + A''(g(1))}{4n^2A(g(1))}.\end{aligned}$$

Proof. Let $h \in C_B^2[0,\infty)$. By the Taylor's expansion and linearity property of $\mathcal{M}_n$ operators, we have

$$\mathcal{M}_n(h;x) - h(x) = h'(x)\mathcal{M}_n(s - x;x) + \frac{h''(\eta)}{2}\mathcal{M}_n\left((s-x)^2;x\right), \quad \eta \in (x,s).$$

From the above equality, one can write

$$\begin{aligned}|\mathcal{M}_n(h;x) - h(x)| &\le \left\{\frac{B''(nxg(1)) - 2B'(nxg(1)) + B(nxg(1))}{2B(nxg(1))}x^2\right. \\ &+ \frac{\left(A'(g(1)) + nA(g(1))\right)\left(B'(nxg(1)) - B(nxg(1))\right)}{nA(g(1))B(nxg(1))}x \\ &+ \frac{B'(nxg(1))\left(1 + g''(1)\right)}{2nB(nxg(1))}x \\ &\left.+ \frac{\left(2n + 1 + g''(1)\right)A'(g(1)) + A''(g(1))}{2n^2A(g(1))}\right\}\|h\|_{C_B^2[0,\infty)}.\end{aligned} \tag{17}$$

On the other hand, using Lemma 2.1 and expression (17), we get

$$\begin{aligned}\left|\mathcal{M}_n(f;x) - f(x)\right| &\le \left|\mathcal{M}_n(f - h;x)\right| + |\mathcal{M}_n(h;x) - h(x)| + \left|f(x) - h(x)\right| \\ &\le 2\left\|f - h\right\|_{C_B[0,\infty)} + |\mathcal{M}_n(h;x) - h(x)| \\ &\le 2\left(\left\|f - h\right\|_{C_B[0,\infty)} + \lambda_n(x)\|h\|_{C_B^2[0,\infty)}\right).\end{aligned} \tag{18}$$

If we take the infimum on the right-hand side of (18) over all $h \in C_B^2[0,\infty)$, we obtain the following desired result

$$\left|\mathcal{M}_n(f;x) - f(x)\right| \le 2\mathcal{K}(f;\lambda_n(x)).$$

□

Theorem 2.10. *For the operators* (10), *if* $f \in C_B[0,\infty)$, *then we have*

$$\begin{aligned}\left|\mathcal{M}_n(f;x) - f(x)\right| &\le C\omega_2\left(f;\sqrt{\nu_n(x)}\right) \\ &+ \omega\left(f;\frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x\right)\end{aligned} \tag{19}$$

where C is constant and

$$\nu_n(x) = \frac{1}{4}\left\{\mathcal{M}_n\left((s-x)^2;x\right) + \left(\frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x\right)^2\right\}.$$

Proof. Let us define an operator $\mathcal{F}_n$ by

$$\mathcal{F}_n(f;x) = \mathcal{M}_n(f;x) - f\left(\frac{A'(g(1))}{nA(g(1))} + \frac{B'(nxg(1))}{B(nxg(1))}x\right) + f(x).$$

We deduce from Lemma 2.6

$$\mathcal{F}_n(s-x;x) = 0. \tag{20}$$

By the Taylor formula with integral reminder term for $h \in C_B^2[0,\infty)$, we can write

$$h(s) = h(x) + (s-x)h'(x) + \int_x^s (s-u)h''(u)\,du.$$

Through the instrumentality of the above equality and (20), one gets

$$\begin{aligned}
|\mathcal{F}_n(h;x) - h(x)| &= \left|\mathcal{F}_n\left(\int_x^s (s-u)h''(u)\,du;x\right)\right| \\
&\leq \left|\mathcal{M}_n\left(\int_x^s (s-u)h''(u)\,du;x\right)\right| \\
&\quad + \left|\int_x^{\frac{A'(g(1))}{nA(g(1))}+\frac{B'(nxg(1))}{B(nxg(1))}x} \left(\frac{A'(g(1))}{nA(g(1))} + \frac{B'(nxg(1))}{B(nxg(1))}x - u\right)h''(u)\,du\right| \\
&\leq \left\{\mathcal{M}_n\left((s-x)^2;x\right)\right. \\
&\quad \left. + \left(\frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x\right)^2\right\}\left\|h''\right\|_{C_B[0,\infty)} \\
&\leq 4\nu_n(x)\|h\|_{C_B^2[0,\infty)}.
\end{aligned} \tag{21}$$

Taking into account of the definition of $\mathcal{F}_n$ operator, Lemma 2.1 and (21), we deduce that

$$\begin{aligned}
\left|\mathcal{M}_n(f;x) - f(x)\right| &\leq \left|\mathcal{F}_n(f-h;x) - (f-h)(x)\right| \\
&\quad + |\mathcal{F}_n(h;x) - h(x)| + \left|f\left(\frac{A'(g(1))}{nA(g(1))} + \frac{B'(nxg(1))}{B(nxg(1))}x\right) - f(x)\right| \\
&\leq 4\left\|f-h\right\|_{C_B[0,\infty)} + 4\nu_n(x)\|h\|_{C_B^2[0,\infty)} \\
&\quad + \omega\left(f;\frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x\right).
\end{aligned}$$

Using the above inequality and considering (15), we conclude that

$$\begin{aligned}
\left|\mathcal{M}_n(f;x) - f(x)\right| &\leq 4\mathcal{K}(f;\nu_n(x)) + \omega\left(f;\frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x\right) \\
&\leq C\omega_2\left(f;\sqrt{\nu_n(x)}\right) + \omega\left(f;\frac{A'(g(1))}{nA(g(1))} + \left(\frac{B'(nxg(1))}{B(nxg(1))} - 1\right)x\right).
\end{aligned}$$

This completes the proof. □

3. Examples

In this section, we will give some explicit examples of operators (10) including generalized Appell polynomials satisfying all restrictions (13) and assumptions (14).

Example 3.1. *The Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν [14] have the following generating functions of the form*

$$e^{-\frac{\nu t^2}{2}+xt} = \sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(x)}{k!} t^k \tag{22}$$

and the explicit representations

$$H_k^{(\nu)}(x) = \sum_{r=0}^{\left[\frac{k}{2}\right]} \left(-\frac{\nu}{2}\right)^r \frac{k!}{r!\,(k-2r)!} x^{k-2r}$$

where, as usual, $[.]$ *denotes the integer part. It is obvious that the Hermite polynomials of variance ν are the generalized Appell polynomials for*

$$A(t) = e^{-\frac{\nu t^2}{2}} \quad , \quad B(t) = e^t \ \text{ and } \ g(t) = t.$$

Under the assumption $\nu \leq 0$; the restrictions (13) *and assumptions* (14) *for the operators $\mathcal{M}_n$ given by* (10) *are satisfied. With the help of the generating functions* (22)*, we get the explicit form of $\mathcal{M}_n$ operators involving the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν by*

$$\mathcal{M}_n^*(f;x) = e^{-nx+\frac{\nu}{2}} \sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(nx)}{k!} f\left(\frac{k}{n}\right) \tag{23}$$

where $x \in [0,\infty)$.

The estimates found by Algorithm 3.5 are given in Table 1. In the following Table 1, we establish error estimates for the approximation with $\mathcal{M}_n^*$ operators including Hermite polynomials of variance ν. If we pay attention to Table 1, we see that the approximation of $\mathcal{M}_n^*$ operators to function $f(x) = \frac{x^2}{\sqrt{1+x^2}}$ depends on ν parameter.

n	Estimation for $\nu = -0.001$	Estimation for $\nu = -3$	Estimation for $\nu = -5$
10	0.6423802906	0.9669999676	1.2083613350
10^2	0.2100791236	0.2250970464	0.2436322218
10^3	0.0668967314	0.0673950602	0.0680538580
10^4	0.0211952608	0.0212111354	0.0212322864
10^5	0.0067064286	0.0067069314	0.0067076008
10^6	0.0021211434	0.0021211592	0.0021211806
10^7	0.0006708028	0.0006708034	0.0006708044

Table 1. The error estimation of function f by using modulus of continuity

Example 3.2. *Gould-Hopper polynomials [8] have the generating functions of the type*

$$e^{ht^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x,h) \frac{t^k}{k!} \tag{24}$$

and their explicit representations are

$$g_k^{d+1}(x,h) = \sum_{s=0}^{\left[\frac{k}{d+1}\right]} \frac{k!}{s!\,(k-(d+1)\,s)!} h^s x^{k-(d+1)s} \ .$$

Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are d-orthogonal polynomial set of Hermite type [7]. Van Iseghem [17] and Maroni [13] discovered the notion of d-orthogonality. Gould-Hopper polynomials are the generalized Appell polynomials with

$$A(t) = e^{ht^{d+1}} \ , \quad B(t) = e^t \ \text{and} \quad g(t) = t \ .$$

Under the assumption $h \geq 0$; the restrictions (13) *and assumptions* (14) *for the operators $\mathcal{M}_n$ given by* (10) *are satisfied. With the help of the generating functions* (24)*, we obtain the explicit form of $\mathcal{M}_n$ operators including Gould-Hopper polynomials by*

$$\mathcal{M}_n^{**}(f;x) = e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} f\left(\frac{k}{n}\right) \tag{25}$$

where $x \in [0,\infty)$.

The following table gives a bound the error of the approximation of function $f(x) = \frac{x^2}{\sqrt{1+x^2}}$ by $\mathcal{M}_n^{**}$ operators involving Gould-Hopper polynomials. Each of the estimates depending on h and d parameters list in the following table as follows:

n	Estimation for $h = 0.00001$	Estimation for $h = 1$	Estimation for $h = 1.8$
10	0.6423250428	1.1805569940	1.4096205150
10^2	0.2100772120	0.2409468484	0.2812002468
10^3	0.0668966700	0.0679554484	0.0695336528
10^4	0.0211952592	0.0212291148	0.0212804728
10^5	0.0067064284	0.0067075012	0.0067091310
10^6	0.0021211432	0.0021211772	0.0021212294
10^7	0.0006708028	0.0006708042	0.0006708056

Table 2. The error estimation of function f by using modulus of continuity

Remark 3.3. *It is worthy to note that for $h = 0$ and $\nu = 0$, respectively, we obtain that*

$$g_k^{d+1}(nx,0) = (nx)^k \quad \text{and} \quad H_k^{(0)}(nx) = (nx)^k \, .$$

Substituting $H_k^{(0)}(nx) = (nx)^k$ for $\nu = 0$ in the operators (23) *and similarly $g_k^{d+1}(nx,0) = (nx)^k$ for $h = 0$ in the operators* (25)*, we get the well-known Szasz operators given by* (1).

Example 3.4. *Miller-Lee polynomials $G_k^{(m)}(x)$ [5] have the generating functions of the form*

$$\frac{1}{(1-t)^{m+1}} \exp(xt) = \sum_{k=0}^{\infty} G_k^{(m)}(x)\, t^k \ , \qquad |t| < 1 \, . \tag{26}$$

From the generating relation (26)*, the Miller-Lee polynomials are the generalized Appell polynomials for*

$$A(t) = \frac{1}{(1-t)^{m+1}} \ , \ B(t) = e^t \ \text{and} \quad g(t) = t$$

and have the explicit expression

$$G_k^{(m)}(x) = \sum_{r=0}^{k} \frac{(m+1)_r}{r!\,(k-r)!} x^{k-r}$$

where $(\alpha)_k$ *is the Pochhammer's symbol given by*

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1) \qquad k = 1, 2, \ldots\ .$$

For ensuring the restrictions (13) *, we have to modify the generating function* (26) *by* $t \to \frac{t}{2}$ *and* $x \to 2x$

$$\frac{1}{\left(1-\frac{t}{2}\right)^{m+1}} \exp(xt) = \sum_{k=0}^{\infty} \frac{G_k^{(m)}(2x)}{2^k} t^k \ , \qquad |t| < 2\,.$$

By the help of the above generating function, we can construct the following linear and positive operator as an example of the operator $\mathcal{M}_n$ *when* $m > -1$ *and* $x \in [0, \infty)$

$$\mathcal{M}_n^{***}(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{G_k^{(m)}(2nx)}{2^{m+k+1}} f\left(\frac{k}{n}\right).$$

The results obtained by a similar algorithm with Algorithm 3.5 are shown in Table 3. We derive error estimates depending on m for the convergence to the function $f(x) = \frac{x^2}{\sqrt{1+x^2}}$ with $\mathcal{M}_n^{***}$ operators including Miller-Lee polynomials as follows:

n	Estimation for $m = -0.01$	Estimation for $m = 1$	Estimation for $m = 2.1$
10	0.7242273912	0.8396862662	0.9798716350
10^2	0.2131282788	0.2182214998	0.2258785894
10^3	0.0669953140	0.0671629400	0.0674218672
10^4	0.0211983934	0.0212037278	0.0212119932
10^5	0.0067065276	0.0067066968	0.0067069578
10^6	0.0021211464	0.0021211524	0.0021211606
10^7	0.0006708028	0.0006708032	0.0006708034

Table 3. The error estimation of function f by using modulus of continuity

Algorithm 3.5.
```
restart;
f:=x->(x^2)*(1/sqrt(x^2+1));
n:=1:
v1:=-0.001: v2:=-3: v3:=-5:
for i from 1 to 7 do
n:=10*n;
delta1:=evalf(sqrt((1/n)+(v1^2-2*v1)/(n^2))):
delta2:=evalf(sqrt((1/n)+(v2^2-2*v2)/(n^2))):
delta3:=evalf(sqrt((1/n)+(v3^2-2*v3)/(n^2))):
omega1(f,delta1):=evalf(maximize(expand(f(x+h)-f(x)),x=0..1-delta1,h=0..delta1)):
omega2(f,delta2):=evalf(maximize(expand(f(x+h)-f(x)),x=0..1-delta2,h=0..delta2)):
omega3(f,delta3):=evalf(maximize(expand(f(x+h)-f(x)),x=0..1-delta3,h=0..delta3)):
error1:=evalf(2*omega1(f,delta1));
error2:=evalf(2*omega2(f,delta2));
error3:=evalf(2*omega3(f,delta3));
end do;
```

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