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Structure of Weighted Hardy Spaces in the Plane

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Abstract. We characterize certain weighted Hardy spaces on the unit disk and completely describe their dual spaces.

1. Introduction and Preliminaries

By a recent paper of Poletsky and Stessin [5] to each subharmonic function on a bounded regular domain G which is continuous near the boundary corresponds a space H^p_u of analytic functions in G with a certain growth condition. These are namely Poletsky-Stessin Hardy spaces. They include and generalize the well-known classical Hardy spaces. This new theory unifies the standpoints of various analytic function spaces into one.

The first generalizations in this direction of the theory of Hardy spaces on hyperconvex domains in $\mathbb{C}^n$ was suggested and studied in [1]. More recently the theory is extended to hyperconvex domains in [5]. Boundedness and compactness of the composition operators on these new Poletsky-Stessin Hardy and Bergman type spaces were investigated there. After this motivating work more investigation [2], [12], [10] revealed the structure and first examples of these Hardy type spaces in the plane.

In [2] to understand the scale of weighted Hardy spaces $u \to H^p_u$ Alan and the author completely characterized H^p_u spaces in the plane domains by their boundary values or by possessing a harmonic majorant with a certain growth (see also [12], [10]). Basically the version of the Beurling's theorem proved in [2] states that when G is the unit disk in the plane, to each subharmonic exhaustion G corresponds an outer function φ which belongs to the class H^p_u so that H^p_u isometrically equals to $\mathcal{M}_{\varphi,p}$ for $p > 0$, where $\mathcal{M}_{\varphi,p}$ is the space $\varphi^{2/p}H^p$ endowed with the norm

$$\|f\|_{\mathcal{M}_{\varphi,p}} := \|f/\varphi^{2/p}\|_p, \quad f \in \mathcal{M}_{\varphi,p}.$$

This result is especially useful to construct examples of analytic function spaces enjoying certain desired properties. The space $\mathcal{M}_{\varphi,2}$, when $\|\varphi\|_\infty \leq 1$, was studied as a tool to understand certain sub-Hardy Hilbert spaces in the unit disk in [11]. Two problems were not answered in [2]:

1. Can we go back? That is, given analytic φ can one find a subharmonic exhaustion u so that $H^p_u = \mathcal{M}_{\varphi,p}$?

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2. For the space H^p_u, consider the class of all representatives, i.e., subharmonic exhaustions v so that $H^p_u = H^p_v$. What kind of "good" representatives are there?

In this note we give answers for both questions. We show under certain growth conditions on the analytic function φ on the disk that it is possible to construct a subharmonic exhaustion u on the disk so that H^p_u equals to $\mathcal{M}_{\varphi,p}$. Moreover, by the construction, u is a decreasing limit of subharmonic exhaustions which are smooth on the closure of the unit disk. In specific cases u is real analytic and satisfies the bi-Laplacian in the unit disk. This is a new information related to the second question.

In addition, using the boundary value characterization from [2] we completely characterize the dual space of H^p_u and discuss the corresponding extremal and dual extremal problems.

After several months of submission of this paper there appeared a preprint [7]. Theorem 3.3 in this paper is similar to Theorem 2.1 below. In Theorem 2.1 we do not require any integrability condition on the subharmonic exhaustion u, however the authors in [7] require u to be intagrable.

Let us start to recall basic definitions. A function $u \leq 0$ on a bounded open set $G \subset \mathbb{C}$ is called an exhaustion on G if the set

$$B_{c,u} := \{z \in G : u(z) < c\}$$

is relatively compact in G for any $c < 0$. When u is an exhaustion and $c < 0$, we set

$$u_c := \max\{u, c\}, \quad S_{c,u} := \{z \in G : u(z) = c\}.$$

Let $u \in sh(G)$ be an exhaustion function which is continuous with values in $\mathbb{R} \cup \{-\infty\}$. Following Demailly [3] we define

$$\mu_{c,u} := \Delta u_c - \chi_{G\setminus B_{c,u}} \Delta u,$$

where χ_ω is the characteristic function of a set $\omega \subset G$. We denote the class of negative subharmonic exhaustion functions on G by $\mathcal{E}(G)$. The class of all functions $u \in \mathcal{E}(G)$ for which $\int \Delta u < \infty$ is denoted by $\mathcal{E}_0(G)$.

If $u \in \mathcal{E}(G)$, then the Demailly-Lelong-Jensen formula ([3]) takes the form

$$\int_{S_{c,u}} v\, d\mu_{c,u} = \int_{B_{c,u}} (v\Delta u - u\Delta v) + c \int_{B_{c,u}} \Delta v, \tag{1}$$

where $\mu_{c,u}$ is the Demailly measure which is supported in the level sets $S_{c,u}$ of u and $v \in sh(G)$. Let us recall that by [3] if $\int_G \Delta u < \infty$, then the measures $\mu_{c,u}$ converge as $c \to 0$ weak-* in $C^*(\overline{G})$ to a measure μ_u supported in the boundary ∂G.

Following [5] we set

$$sh_u(G) := sh_u := \left\{ v \in sh(G) : v \geq 0, \sup_{c<0} \int_{S_{c,u}} v\, d\mu_{c,u} < \infty \right\},$$

and

$$H^p_u(G) := H^p_u := \{f \in hol(G) : |f|^p \in sh_u\}$$

for every $p > 0$. We write

$$\|v\|_u := \sup_{c<0} \int_{S_{c,u}} v\, d\mu_{c,u} = \int_G (v\Delta u - u\Delta v) \tag{2}$$

for a nonnegative function $v \in sh(G)$ and set

$$\|f\|_{u,p} := \sup_{c<0} \left(\int_{S_{c,u}} |f|^p\, d\mu_{c,u} \right)^{1/p} \tag{3}$$

for a holomorphic function f on G. We will use $\|f\|_u = \|f\|_{u,p}$ when $p = 1$. By Theorem 4.1 of [5], H^p_u is a Banach space when $p \geq 1$. It is clear that the function $f \equiv 1$ belongs to H^p_u if and only if the Demailly measure μ_u has finite mass. If G is a regular bounded domain in $\mathbb{C}$ and $w \in G$, then the Green function $v(z) = g_G(z, w)$ is a subharmonic exhaustion function for G. For example, when G is the unit disk and $v(z) = \log|z|$, then μ_v is the normalized arclength measure on the unit circle. We denote by $P_G(z, w)$ the Poisson kernel for the domain G.

The following Theorems are recollections from [2].

Theorem 1.1. *[2, Theorem 2.3] Let G be a bounded domain, $v \geq 0$ be a function on G, $p > 0$, and $u \in \mathcal{E}(G)$. The following statements are equivalent:*

i. $v \in sh_u(G)$.

ii. The least harmonic majorant $h = P_G(v)$ of φ in G belongs to the class sh_u.

Furthermore,

$$\|v\|_u = \int_G h\Delta u = \|h\|_u.$$

We will denote by $H^p(G)$ the space of analytic functions f in G for which $|f|^p$ has a harmonic majorant in G (see for example [4]). We always have $H^p_u \subset H^p$ by [5]. We will denote by ν the usual arclength measure on ∂G normalized so that $\nu(\partial G) = 1$.

Theorem 1.2. *[2, Theorem 2.10] Let G be a Jordan domain with rectifiable boundary or a bounded domain with C^2 boundary, $p > 1$, and $u \in \mathcal{E}(G)$. The following statements are equivalent:*

i. $f \in H^p_u(G)$.

ii. $f \in H^p(G)$ and $|f^*| \in L^p(V_u\nu)$, where

$$V_u(\zeta) := \int_G P_G(z, \zeta)\Delta u(z), \quad \zeta \in \partial G. \tag{4}$$

iii. $f \in H^p(G)$ and there exists a positive measure $\widetilde{\mu_u}$ on ∂G such that $|f^*| \in L^p(\widetilde{\mu_u})$. Moreover, if E is any Borel subset of ∂G with measure $\nu(E) = 0$, then $\widetilde{\mu_u}(E) = 0$ and we have the equality

$$\int_{\partial G} \gamma\, d\widetilde{\mu_u} = \int_G P_G(\gamma)\Delta u \tag{5}$$

for every $\gamma \in L^1(\nu)$.

In addition, if $f \in H^p_u(G)$, then $\|f\|_{u,p} = \|f^*\|_{L^p(\widetilde{\mu_u})}$ and $d\widetilde{\mu_u} = V_u d\nu$.

Remark 1.3. *i. Theorem 1.2 is valid when $p > 0$ and G is the unit disk or more generally a Jordan domain with rectifiable boundary. In this case the Poisson integral of an $L^p(d\nu)$ function u has non-tangential limits equal to u on ν-almost every boundary point. This is indeed what is needed in the proof of Theorem 1.2.*

ii. By replacing the function u by a suitable positive multiple tu, $t > 0$, we may assume that $V_u \geq 1$ on ∂G. To do this it is enough to take a compact set $K \subset G$ so that $\Delta u(K) > r > 0$. Let $m := \min_{\zeta\in\partial G} \min_{z\in K} P_G(z, \zeta)$. Then take $t := 1/(rm)$. We will use the assumption that $V_u \geq 1$ when convenient.

iii. The weight function V_u is lower semicontinuous. To see this, suppose $\zeta_j \in \partial G$, $\zeta_j \to \zeta$. By Fatou's lemma

$$\liminf_j V_u(\zeta_j) = \liminf_j \int_G P_G(z, \zeta_j)\Delta u(z) \geq \int_G P_G(z, \zeta)\Delta u(z) = V_u(\zeta).$$

Note that V_u is the balayage *of the measure Δu on ∂G.*

iv. Suppose G is a bounded domain with C^2 boundary or a Jordan domain with rectifiable boundary and $u \in \mathcal{E}_0(G)$. Then

$$u(z) = \int_G g_G(z, w)\Delta u(w), \quad z \in G.$$

Since

$$\frac{\partial g_G(\zeta, w)}{\partial n} = P_G(w, \zeta)$$

when $\zeta \in \partial G$ and $w \in G$, $\frac{\partial u}{\partial n}(\zeta)$ exists for every $\zeta \in \partial G$, where $\frac{\partial}{\partial n}$ denotes the normal derivative in the outward direction on ∂G and

$$\frac{\partial u(\zeta)}{\partial n} = V_u(\zeta) = \int_G P_G(w, \zeta)\Delta u(w), \quad \zeta \in \partial G.$$

By property (5) in Theorem 1.1

$$\int_{\partial G} V_u(\zeta)d\nu(\zeta) = \int_G \Delta u = \int_{\partial G} \frac{\partial u}{\partial n}(\zeta)d\nu(\zeta).$$

To obtain Fatou's type results we would like to compute the Radon-Nikodym derivative of the Demailly measures with respect to the usual arclength measure on the level sets. In the next result we provide this. Let ν_c denote the arclength measure on $S_{c,u}$. Define

$$V_{c,u}(\zeta) := \int_{B_{c,u}} P_{B_{c,u}}(z, \zeta)\Delta u(z), \quad \zeta \in S_{c,u},$$

where $P_{B_{c,u}}(z, \zeta)$ denotes the Poisson kernel for $B_{c,u}$.

Proposition 1.4. *Let $u \in \mathcal{E}(G)$, where G is a bounded regular domain. Suppose that u is Lipschitz in every compact subset of G. Then the measures ν_c and $\mu_{c,u}$ are mutually absolutely continuous and $\mu_{c,u} = V_{c,u}\nu_c$ with $V_{c,u} \in L^1(\nu_c)$. Moreover, for each $c < 0$ there is a constant $k_c > 0$ so that $V_{c,u} \geq k_c$ on $S_{c,u}$.*

Proof. Let φ be a continuous function on $S_{c,u}$ and let $h(z)$ be the harmonic function in $B_{c,u}$ with boundary values equal to φ. By equality (1) we have

$$\begin{aligned}\int_{S_{c,u}} \varphi(\zeta)d\mu_{c,u}(\zeta) &= \int_{B_{c,u}} h(z)\Delta u(z) \\ &= \int_{S_{c,u}} \left(\int_{B_{c,u}} P_{B_{c,u}}(z, \zeta)\Delta u(z)\right)\varphi(\zeta)\, d\nu_c(\zeta) \\ &= \int_{S_{c,u}} \varphi(\zeta)V_{c,u}(\zeta)\, d\nu_c(\zeta).\end{aligned}$$

Hence $\mu_{c,u} = V_{c,u}\nu_c$. Another observation using Fubini's theorem gives

$$\int_{S_{c,u}} V_{c,u}(\zeta)d\nu_c(\zeta) = \int_{B_{c,u}} \Delta u(z) = \|\mu_{c,u}\| < \infty.$$

Thus $V_{c,u} \in L^1(\nu_c)$. Note that $\nu_c \leq k'_c\mu_{c,u}$ for some positive constant k'_c by [3]. Hence $V_{c,u} \geq k_c$ on $S_{c,u}$ for some $k_c > 0$. This completes the proof. □

Remark 1.5. *The requirement that u is Lipschitz is only needed to write the harmonic measure on $B_{c,u}$ of the form $P_{B_{c,u}}d\nu_c$. There are much weaker conditions on domains for which the harmonic measure is absolutely continuous.*

The next auxiliary result allows one to compare the Demailly measures on $S_{c,u}$ with a measure on an arbitrary level set.

Proposition 1.6. *Let u be a subharmonic exhaustion function on a bounded regular domain G in $\mathbb{C}$. Let G_j be relatively compact regular open sets in G so that $\overline{G_j} \subset G_{j+1}$ and $\cup G_j = G$. Then for each j there is a $u_j \in \mathcal{E}(G_j)$ and for each $c < 0$ there is a number s with $c < s < 0$ so that for any nonnegative function $v \in sh(G)$, the integrals $\mu_{u_j}(v)$ are increasing and*

$$\mu_{c,u}(v) \le \mu_{u_j}(v) = \|v\|_{u_j} \le \mu_{s,u}(v).$$

This means $\|v\|_u = \lim_j \mu_{u_j}(v)$ for every nonnegative subharmonic function v on G.

Proof. Set $u_j := u - P_{G_j}u$. Clearly $u_j \in \mathcal{E}(G_j)$. Take an integer $j_0 \ge 1$ and a number $s < 0$ with $c < s$ so that $B_{c,u} \subset G_{j_0} \subset B_{s,u}$. The comparison follows from (1) and (2) if we note that $c \le P_{G_j}u$ on $B_{c,u}$ and $P_{G_j}u \le s$ on $B_{s,u}$. □

If φ is a nonzero analytic function on $\mathbb{D}$, let $\mathcal{M}_{\varphi,p}$ denote the space $\varphi^{2/p}H^p$ endowed with the norm

$$\|f\|_{\mathcal{M}_{\varphi,p}} := \|f/\varphi^{2/p}\|_p, \quad f \in \mathcal{M}_{\varphi,p}.$$

We will call a function $\varphi \in H^2_u$ a u-inner function if $|\varphi^*(\zeta)|^2 V_u(\zeta)$ equals 1 for almost every $\zeta \in \partial\mathbb{D}$. If, moreover, $\varphi(z)$ is zero-free, we will say that φ is a singular u-inner function. The next result is Theorem 3.2 and Corollary 3.3 from [2].

Theorem 1.7. *Let $Y \neq \{0\}$ be a closed M_z-invariant subspace of $H^2_u(\mathbb{D})$. Then there exists a function $\varphi \in H^2_u$ so that $|\varphi^*(\zeta)|^2 V_u(\zeta) = 1$ for almost every $\zeta \in \partial\mathbb{D}$ and $Y = \mathcal{M}_{\varphi,2}$. In particular, there exists a u-inner and an outer function $\varphi \in H^2_u$ so that $H^2_u = \mathcal{M}_{\varphi,2}$ and these spaces are isometric.*

This function φ is determined uniquely up to a unit constant. Note that

$$V_u(e^{i\theta}) = \frac{1}{|\varphi(e^{i\theta})|^2} = \frac{1}{\varphi^2(e^{i\theta})}\,\mathrm{sgn}\,\frac{1}{\varphi^2(e^{i\theta})}, \tag{6}$$

where we set $sgn\alpha := |\alpha|/\alpha$ for any complex number $\alpha \neq 0$ and $sgn0 := 0$. If $V \ge 1$ on $\partial\mathbb{D}$, then $|\varphi(\zeta)| \le 1$ for almost every ζ. Suppose that $\int \Delta u < \infty$. Then the function 1 belongs to H^p_u. Hence φ^{-1} belongs to H^2. Then it is an easy exercise to show that φ is an outer function.

Theorem 1.8. *The set $L^p(V_u d\theta)$ coincides with $\varphi^{2/p}L^p(d\theta)$ and the map $f \mapsto \varphi^{-2/p}f$ is an isometric isomorphism from the space $L^p(V_u d\theta)$ onto $L^p(d\theta)$.*

Theorem 1.9. *[2, Theorem 3.4] Suppose $0 < p < \infty$, $f \in H^p_u(\mathbb{D})$, $f \not\equiv 0$, and B is the Blaschke product formed with the zeros of f. Then there are zero-free $\varphi \in H^2_u \cap H^\infty$, $S \in H^\infty$ and $F \in H^p$ so that φ is outer and singular u-inner, S is singular inner, F is outer, and*

$$f = BS\varphi^{2/p}F. \tag{7}$$

Moreover, $\|f\|_{p,u} = \|F\|_p$ and $H^p_u(\mathbb{D}) = \mathcal{M}_{\varphi,p}$.

Corollary 1.10. *The map $f \mapsto \varphi^{-2/p}f$ is an isometric isomorphism from the space H^p_u onto H^p.*

The following Lemma will be useful in the next section. Its proof is a simple calculation and we outline it here.

Lemma 1.11. *Let c be a number with $-1 < c < 0$. Then there exists a function $\kappa = \kappa_c$ defined on $(-\infty, 0]$ with the following properties:*

i. $\kappa : (-\infty, 0] \to (-\infty, 0]$ *is non-decreasing, convex and* C^∞*,*

ii. κ *is real-analytic in* $(c, 0]$*,*

iii. $\kappa(t) \equiv c$ *when* $t \le c$*,* $\kappa(0) = 0$*, and* $\kappa'(0) = 1$.

Proof. Let $a := -\frac{\ln(-c)}{e}$, $b := \frac{-1}{\ln(-c)}$, and

$$\kappa(t) := \begin{cases} c + e^{\frac{-a}{(t-c)^b}}, & t > c, \\ c, & t \le c. \end{cases}$$

Then

$$\kappa'(t) = \frac{1}{e(t-c)^{b+1}} e^{\frac{-a}{(t-c)^b}}$$

and

$$\kappa''(t) = \frac{1}{e(t-c)^{2b+2}} (1/e - (b+1)(t-c)^{b+1}) e^{\frac{-a}{(t-c)^b}}$$

for $t > c$. For $t \le c$, $\kappa'(t) = \kappa''(t) = 0$. It can be checked that $\kappa''(t) > 0$ for $c < t \le 0$, and κ satisfies all properties in i., ii. and iii. □

2. Finding Subharmonic Exhaustion

Theorem 1.2 describes the weight function V_u corresponding to the Hardy space H^p_u when the Laplacian of u is known. In Theorem 1.9 we obtain a canonical factorization for functions in H^p_u and we see that this space is a certain multiple $\varphi^{2/p} H^p$ of H^p. The singular u-inner function φ appearing in this factorization is related to the weight V_u by

$$V_u(e^{i\theta}) = \frac{1}{|\varphi(e^{i\theta})|^2}, \quad a.e.\ \theta. \tag{8}$$

In this section we seek a converse to these results.

Let G be a Jordan domain with rectifiable boundary and ψ be a given analytic function in $H^1(G)$. The problem is to find a subharmonic exhaustion u on G so that $V_u(\zeta) = |\psi(\zeta)|$ when $\zeta \in \partial G$. Taking a conformal map of G onto $\mathbb{D}$ we can always suppose that $G = \mathbb{D}$. This is a type of *inverse balayage* problem. We solve this next.

Theorem 2.1. *Let* ψ *be a lower semicontinuous function on* $\partial\mathbb{D}$ *so that* $\psi \ge c$ *for some constant* $c > 0$*. Then there exists a function* $u \in \mathcal{E}$ *so that* $\psi = V_u$*. Moreover we have the following properties:*

a. u *is the decreasing limit of functions in* $\mathcal{E}_0 \cap C^\infty(\overline{\mathbb{D}})$ *converging uniformly to* u *on* $\overline{\mathbb{D}}$*.*

b. $u \in \mathcal{E}_0(\mathbb{D})$ *if and only if* $\psi \in L^1(d\nu)$*.*

c. If ψ *is* C^k*,* $0 \le k \le \infty$*, on* $\partial\mathbb{D}$*, then* u *is* C^k *on* $\overline{\mathbb{D}}$*. If* ψ *is real-analytic, then there exists a compact* K *so that* u *is real-analytic on* $\overline{\mathbb{D}}\backslash K$*.*

Proof. Suppose first that ψ is C^2 on $\partial\mathbb{D}$ and let $\rho(re^{i\theta}) := \frac{1}{2}(r^2-1)\psi(e^{i\theta})$ for $re^{i\theta} \in \mathbb{D}$. Computing the Laplacian of ρ we get

$$\Delta\rho(re^{i\theta}) = 2\psi(e^{i\theta}) + \frac{r^2-1}{2r^2} \frac{d^2\psi(e^{i\theta})}{d\theta^2}.$$

By assumption $\Delta\rho(e^{i\theta}) = 2\psi(e^{i\theta}) \geq 2c > 0$. Hence there exists a compact $B \subset \mathbb{D}$ so that $\Delta\rho(z) > 0$ on the open set $\Omega := \mathbb{D}\backslash B$. Hence ρ is a non-positive subharmonÅc function on Ω and $\rho|_{\partial\mathbb{D}} \equiv 0$. Since ρ is continuous on $\overline{\mathbb{D}}$, there exists a constant $c < 0$ so that the set $B_{c,\rho}$ is relatively compact in $\mathbb{D}$ and $S_{c,\rho} \subset \Omega$. Let $\kappa = \kappa_c$ be the function proivided in Lemma 1.11. Define $u(z) := \kappa(\rho(z))$ for $z \in \mathbb{D}$. Now $u = \kappa(\rho)$ is subharmonic in Ω, $u \equiv c$ on $B_{c,\rho}$ and $u \geq c$ on $\mathbb{D}\backslash B_{c,\rho} \subset \Omega$. Hence $u \in \mathcal{E}$ and $V_u = \frac{\partial u}{\partial r} = \kappa'(0)\frac{\partial \rho}{\partial r} = \psi$ on $\partial\mathbb{D}$.

Now let ψ be lower semicontinuous. There exists ψ_n, all C^∞ on $\partial\mathbb{D}$ so that $c \leq \psi_n(\zeta) \leq \psi_{n+1}(\zeta)$, and $\psi(\zeta) = \lim_n \psi_n(\zeta)$ for every $\zeta \in \partial\mathbb{D}$. We let $\psi_0 \equiv 0$. Replacing ψ_n by $\psi_n - 2^{-n}$ we may assume that $d_n := \psi_{n+1} - \psi_n \geq 2^{-n-1}$. As in the first part of the proof we let $\rho_n(z) := \frac{1}{2}(r^2 - 1)d_n(e^{i\theta})$. There exists a compact $B_n \subset \mathbb{D}$ so that $\Delta\rho_n(z) > 0$ on the open set $\Omega_n := \mathbb{D}\backslash B_n$. This time we choose constants $-2^{-n} \leq c_n < 0$ so that B_{c_n,ρ_n} is relatively compact in $\mathbb{D}$, $S_{c_n,\rho_n} \subset \Omega_n$, and $B_{c_n,\rho_n} \subset B_{c_{n+1},\rho_{n+1}}$. Let $u_n(z) := \kappa_{c_n}(\rho_n(z))$ so that as proved in the first part, $V_{u_n} = d_n$ and $u_n \in C^\infty(\overline{\mathbb{D}})$.

Let

$$u(z) := \sum_{n=0}^{\infty} u_n(z).$$

Since $|u_n| \leq |c_n| \leq 2^{-n}$ for all n, the sum converges uniformly on $\overline{\mathbb{D}}$. This shows that $u \in \mathcal{E}$ and properties in a. and c. are satisfied. Using (4) in Theorem 1.2,

$$V_u(\zeta) = \int_{\mathbb{D}} P(z,\zeta)\Delta u(z) = \sum_{n=0}^{\infty} \int_{\mathbb{D}} P(z,\zeta)\Delta u_n(z) = \sum_{n=0}^{\infty} d_n(\zeta) = \psi(\zeta).$$

Due to an equality in Remark 1.3,

$$\int_{\mathbb{D}} \Delta u(z) = \int_{\partial\mathbb{D}} V_u d\nu = \int_{\partial\mathbb{D}} \psi d\nu.$$

Hence $u \in \mathcal{E}_0$ if and only if $\psi \in L^1(d\nu)$. The proof is completed. □

We have now the following converse to Theorem 1.9 to answer the first question in the introduction.

Theorem 2.2. *Let φ be a zero free analytic function on $\mathbb{D}$ so that $|\varphi^*|$ equals ν-almost everywhere to an upper semicontinuous function on $\partial\mathbb{D}$. Then there exists a $u \in \mathcal{E}(\mathbb{D})$ so that $H^p_u = \mathcal{M}_{\varphi,p}$ and we have isometric isomorphism of two spaces.*

Proof. It is enough to prove the theorem when $p = 2$. Since $|\varphi^*|$ is upper semicontinuous on $\partial\mathbb{D}$, there exists a constant m so that $|\varphi^*| \leq m$. Hence the function $\psi := 1/|\varphi^*|^2$ is lower semicontinuous and $\psi \geq 1/m$. Let $u \in \mathcal{E}(\mathbb{D})$ be the exhaustion provided by Theorem 2.3 for the function ψ so that $V_u = 1/|\varphi^*|^2$. If $f \in H^2_u$, we write $f = \varphi f_0$, where $f_0 = f/\varphi$. Then

$$\|f\|^2_{2,u} = \int_0^{2\pi} |f(e^{i\theta})|^2 V_u(e^{i\theta})\, d\theta = \|f\|^2_{\mathcal{M}_{\varphi,2}} = \|f_0\|^2_2 < \infty.$$

Thus $f_0 \in H^2$ and we have shown that $H^2_u \subset \mathcal{M}_{\varphi,2}$. Conversely, if $f \in \mathcal{M}_{\varphi,2}$, then clearly $f \in H^2_u$ from the same equality above. The mapping $f \mapsto \varphi f_0$ is clearly an isomorphism of H^2_u onto $\mathcal{M}_{\varphi,2}$ which is an isometry. □

When the weight function V_u is smooth enough, there is a connection with the corresponding subharmonic exhaustions and the bi-Laplacian equation $\Delta^2 u = 0$. This is explained in the next result.

Theorem 2.3. *Let $\psi \in C^1(\partial\mathbb{D})$ be a nonnegative function. Then there exists a function u and a constant M with the following properties:*

a. $u \in \mathcal{E}_0(\mathbb{D})$ and u is real analytic on $\mathbb{D}$.

b. $V_u(\zeta) = \frac{\partial u}{\partial n}(\zeta) = \psi(\zeta) + M$ *for every* $\zeta \in \partial\mathbb{D}$.

c. u satisfies the bi-Laplacian equation $\Delta^2 u = 0$ *on* $\mathbb{D}$.

Proof. Let $u(z) := \frac{1}{2}(|z|^2 - 1)[P\psi(z) + M]$, where $P\psi(z)$ is the harmonic extension of ψ on $\mathbb{D}$. Then using polar coordinates $\Delta u(z) = 2[P\psi(z) + M] + 2|z|\frac{\partial P\psi(z)}{\partial r}$. Note that $P\psi \in C^1(\overline{\mathbb{D}})$. Now take M large enough so that $\Delta u(z) \geq 0$ on $\mathbb{D}$. Again taking the Laplacian it can be checked that $\Delta^2 u(z) = 0$. Hence Δu is harmonic on $\mathbb{D}$ and since

$$\int_{\mathbb{D}} \Delta u = c(P\psi(0) + M) \leq c||\psi||_\infty + cM < \infty,$$

$u \in \mathcal{E}_0$. Clearly u is real analytic on $\mathbb{D}$. On $\partial\mathbb{D}$ we have

$$V_u(\zeta) = \frac{\partial u}{\partial r}(\zeta) = \psi(\zeta) + M$$

for every $\zeta \in \partial\mathbb{D}$. □

Remark 2.4. *Equality in (5) shows also that*

$$2\int_0^{2\pi} \log|z - e^{i\theta}| \left[\psi(e^{i\theta}) + \frac{\partial P\psi(e^{i\theta})}{\partial r} + M\right] d\theta = \int_{\mathbb{D}} \log|1 - \overline{w}z|\Delta u(w)$$

for every $z \in \overline{\mathbb{D}}$. *Therefore, in fact, u can be written as the difference of two potentials*

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{D}} \log|z - w|\Delta u(w)dw - \frac{1}{\pi}\int_0^{2\pi} \log|z - e^{i\theta}| \left[\psi(e^{i\theta}) + \frac{\partial P\widetilde{\psi}(e^{i\theta})}{\partial r} + M\right] d\theta$$

for every $z \in \overline{\mathbb{D}}$. *Here* Δu *is harmonic.*

When $v \in \mathcal{E}(\mathbb{D})$, let $R(v)$ denote the class of all functions $u \in \mathcal{E}(\mathbb{D})$ which generates the same space $H^p_v = H^p_u$. We know a "good" representative in $R(v)$ for certain cases as a consequence of Theorem 2.3.

Theorem 2.5. *Let* $v \in \mathcal{E}_0(\mathbb{D})$ *so that* V_v *is bounded and* $PV_v + |z|\frac{\partial PV_v}{\partial r} \geq 0$ *on* $\mathbb{D}$. *Then* $R(v)$ *contains a function* $u \in \mathcal{E}_0$ *which is real analytic and satisfies the bi-Laplacian equation* $\Delta^2 u = 0$ *on* $\mathbb{D}$. *Moreover,* $V_u = V_v$ *and the weight function* V_u *can be found by using the equation*

$$V_u(e^{i\theta}) = \frac{1}{2}\int_0^1 \Delta u(se^{i\theta})ds.$$

Proof. Let $u(z) := \frac{1}{2}(|z|^2 - 1)PV_v(z)$. Then $\Delta u(z) = 2PV_v(z) + 2|z|\frac{\partial PV_v(z)}{\partial r} \geq 0$ by assumption. Hence $u \in \mathcal{E}_0$, u is real analytic and satisfies the bi-Laplacian equation $\Delta^2 u = 0$ on $\mathbb{D}$. Let $h(z) := \Delta u(z)$ and $h_s(z) := h(sz)$ for $0 < s < 1$. By (4) of Theorem 1.2

$$\begin{aligned}
V_u(e^{i\theta}) &= \int_{\mathbb{D}} P(z, e^{i\theta})h(z)dz = \lim_{s\to 1}\int_{\mathbb{D}} P(z, e^{i\theta})h_s(z)dz \\
&= \lim_{s\to 1}\int_0^1 r\int_0^{2\pi} P(re^{it}, e^{i\theta})\left[\frac{1}{2\pi}\int_0^{2\pi} h_s(e^{i\eta})P(re^{it}, e^{i\eta})d\eta\right] dtdr \\
&= \lim_{s\to 1}\int_0^1 r\int_0^{2\pi} h_s(e^{i\eta})\left[\frac{1}{2\pi}\int_0^{2\pi} P(re^{it}, e^{i\eta})P(re^{i\theta}, e^{it})dt\right] d\eta dr \\
&= \lim_{s\to 1}\int_0^1 r\int_0^{2\pi} h_s(e^{i\eta})P(r^2e^{i\theta}, e^{i\eta})d\eta dr \\
&= \lim_{s\to 1}\int_0^1 rh_s(r^2e^{i\theta})dr = \frac{1}{2}\int_0^1 \Delta u(re^{i\theta})dr.
\end{aligned}$$

□

3. Representation of Linear Functionals

First we describe the the space of annihilators of H^p_u in $L^q(Vd\theta)$, where $1/p + 1/q = 1$. Let $G \in L^q(d\theta)$ and $f = \varphi^{2/p}F \in H^p_u$. Define

$$L_G(f) = L_G(\varphi^{2/p}F) := \int_0^{2\pi} F(e^{i\theta})G(e^{i\theta})d\theta.$$

Then L_G belongs to $(H^p_u)^*$ since $|L_G(f)| \le \|F\|_p\|G\|_q = \|f\|_{u,p}\|G\|_q$. We denote by $H^q_{u,0}$ the class of functions g in H^q_u with $g(0) = 0$. Then $H^q_{u,0}$ is isometrically isomorphic to H^q_0 which is the space of functions $g \in H^q$ with $g(0) = 0$.

Theorem 3.1. *For $1 \le p < \infty$, $(H^p_u)^\perp$ is isometrically isomorphic to H^q_0 which is isometrically isomorphic to $H^q_{u,0}$ or H^q_u.*

Proof. Suppose $g \in L^q(Vd\theta)$ is an annihilator of H^p_u. Then

$$\int_0^{2\pi} \varphi^{2/p}(e^{i\theta})g(e^{i\theta})V(e^{i\theta})e^{in\theta}d\theta = 0$$

for every $n = 0, 1, 2, \ldots$. Therefore $\varphi^{2/p}gV$ is the boundary function of some $G \in H^1$ with $G(0) = 0$. In fact G is determined uniquely by g. From the equality

$$|g|^q V = |\varphi|^2|G|^q V = |G|^q$$

we see that $G \in H^q$ and $\|g\|_{u,q} = \|G\|_q$. Take any $f = \varphi^{2/p}F \in H^p_u$. Then

$$\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})V(e^{i\theta})d\theta = \int_0^{2\pi} F(e^{i\theta})G(e^{i\theta})d\theta = 0.$$

Conversely, take any $G \in H^q$. Now from [4, Sec. 7.2] if $G \in H^q_0$, then $L_G \in (H^p_u)^\perp$. Hence the map $G \mapsto L_G$ from H^q_0 onto $(H^p_u)^\perp$ is an isometric isomorphism. □

Theorem 3.1 gives a canonical representation of $(H^p_u)^*$ as in the next statement which can be compared to the classical case (see [4, Theorem 7.3] for example).

Theorem 3.2. *For $1 \le p < \infty$, $(H^p_u)^*$ is isometrically isomorphic to $L^q(Vd\theta)/H^q_u$. Furthermore, if $1 < p < \infty$, for each $L \in (H^p_u)^*$ there exists a unique $G \in H^q_u$ so that $L(f) = L_G(f)$ for every $f \in H^q_u$. For each $L \in (H^1_u)^*$ there exists a function $G \in H^\infty_u$ so that $L(f) = L_G(f)$ for every $f \in H^1_u$.*

The next theorem describes the preduals of H^p_u.

Theorem 3.3. *Let u be a subharmonic exhaustion function on $\mathbb{D}$. If $1 < p \le \infty$ and $1/p + 1/q = 1$, then:*

i. $H^p_u = \left(L^q_u/H^q_{u,0}\right)^*$.

ii. $H^p_{u,0} = \left(L^q_u/H^q_u\right)^*$.

Proof. Let Γ be a bounded linear functional on $L^q_u/H^q_{u,0}$. Then, by composing Γ with the canonical projection of L^q_u onto $L^q_u/H^q_{u,0}$, Γ gives a linear functional on L^q_u with the same norm as on $L^q_u/H^q_{u,0}$. So, for any $f \in L^q_u$, using equation (6), Theorem 1.8 and Corollary 1.10 we have

$$\Gamma(f) = \Gamma(f + H^q_{u,0}) = \int_0^{2\pi} [f(e^{i\theta})\varphi^{-2/q}(e^{i\theta})][G(e^{i\theta})\varphi^{-2/p}(e^{i\theta})]d\theta, \tag{9}$$

where $G \in L^p_u$ with $\|G\|_{p,u} = \|\Gamma\|$. We have $\Gamma(e^{in\theta}\varphi^{2/q}) = 0$ for every integer $n \ge 1$. Hence $G \in H^p_u$. Conversely, any function $G \in H^p_u$ gives rise to a linear functional Γ on $L^q_u/H^q_{u,0}$ by formula (9). This proves the first assertion. The second part is proved by a similar argument. □

4. Extremal Problems

We are now ready to discuss the related extremal problems. For fixed $g \in L^q(Vd\theta)$ the extremal problem is to find

$$\Lambda(g) := \sup\left\{|\lambda(f)| : f \in H^p_u, \|f\|_{p,u} \le 1\right\}, \tag{10}$$

where

$$\lambda(f) := \frac{1}{2\pi i}\int_{|z|=1} F(z)G(z)dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})V(e^{i\theta})e^{i\theta}d\theta \tag{11}$$

and we use the correspondence $f = \varphi^{2/p}F$, $g = \varphi^{2/q}sgn(\varphi^2)G$ provided by Theorem 1.8 and Corollary 1.10. The related dual extremal problem is to find the function $g_0 \in H^q_u$ so that

$$\Gamma(g) := \inf\left\{\|g-h\|_{q,u} : h \in H^q_u\right\} = \|g - g_0\|_{q,u}. \tag{12}$$

The proof of the following existence and uniqueness theorem for the extremal problems follows in view of Theorem 1.8, Corollary 1.10 and [4, Theorem 8.1].

Theorem 4.1. *Let* $1 \le p \le \infty$, $1/p + 1/q = 1$ *and* $g \in L^q(Vd\theta)$.

i. The duality relation $\Lambda(g) = \Gamma(g)$ *holds.*

ii. If $p > 1$, *there is a unique extremal function* $f \in H^p_u$ *for which* $\lambda(f) > 0$. *The dual extremal problem has a unique solution.*

iii. If $p = 1$ *and* $G(e^{i\theta})$ *is continuous, at least one solution to the extremal problem exists. If* $p = 1$, *the dual extremal problem has at least one solution; it is unique if the extremal problem has a solution.*

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