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A New Perspective to Convergence Types in Classical Real Analysis Using Double Sequences

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Abstract. In this work, following their counterparts for single sequences in classical real analysis we will introduce and examine convergence types and relationship between them for double sequences of functions defined on a subset E with finite measure in real numbers.

1. Introduction

In this work we examine the concepts of convergence almost everywere, convergence in measure and uniformly convergence, also relationships between these concepts, for double function sequences. Let us clearly note that we utilize from methods, given for single function sequences, in [6].

Let E be a measurable subset with finite Lebesgue measure E in real numbers. We receive a single sequence (f_n) of real-valued measurable functions and a measurable function f defined on E. If for an element x in E and any $\varepsilon > 0$ there is a natural number N such that $\left|f_n(x) - f(x)\right| < \varepsilon$ for all $n \geq N$, then this sequence (f_n) converges to f in the point x. If for any $\varepsilon > 0$ there is a natural number N such that $\left|f_n(x) - f(x)\right| < \varepsilon$ for all $n \geq N$ and all $x \in E$, then this sequence (f_n) is said to be uniformly convergence on E to f.

Let a single sequence of measurable functions (f_n) be defined and finite almost everywhere on E. Let f be a measurable function which is finite almost everywhere on E. If

$$\lim_{n\to\infty}\left[\mu\left\{x \in E : \left|f_n(x) - f(x)\right| \geq \sigma\right\}\right] = 0$$

for all $\sigma > 0$, then this sequence (f_n) is said to be convergence in measure to f [6].

Throughout this work, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers respectively. A real double sequence is a function x from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and briefly denoted by $(x_{k,l})$. If for all $\varepsilon > 0$ there is $n_\varepsilon \in N$ such that $\left|x_{k,l} - a\right| < \varepsilon$ where $k > n_\varepsilon$ and $l > n_\varepsilon$, then a double sequence $(x_{k,l})$ is said to be converges in Pringsheim's sense to $a \in \mathbb{R}$ and to indicate it is briefly written

$$P - \lim x_{k,l} = a$$

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A double function sequence $(f_{k,l})$ of real-valued functions defined on E corresponds to bring a real double sequence $(f_{k,l}(x))$ to each $x \in E$. If, for each $x \in X$, the sequence $(f_{k,l}(x))$ converges in Pringsheim's sense to $f(x)$, then this sequence $(f_{k,l})$ is said to be pointwise convergence in Pringsheim's sense to f. If, for any $\varepsilon > 0$, there is a natural number N such that $\left|f_{kl}(x) - f(x)\right| < \varepsilon$ for all $k,l \geq N$ and all $x \in E$, then this sequence $(f_{k,l})$ is said to be uniformly convergence in Pringsheim's sense on E to f. These said things can be seen from ([1], [2], [3], [4], [5]).

Let a double sequence of measurable functions $(f_{k,l})$ be defined and finite almost everywhere on E. Let f be a measurable function which is finite almost everywhere on E. If

$$\lim_{n\to\infty}\left[\mu\left\{x \in E : \left|f_{k,l}(x) - f(x)\right| \geq \sigma\right\}\right] = 0$$

in Pringsheim's sense for all $\sigma > 0$, then this sequence $(f_{k,l})$ is said to be converges in measure in Pringsheim's sense to f.

2. Main Results

Theorem 2.1. *Let $(f_{m,n})$ be a double sequence of measurable functions which is finite almost everywere and a function f be finite in almost everywere on E. If $(f_{m,n})$ converges almost everywhere in Pringsheim's sense on E to f, then*

$$P - \lim \mu\left\{x \in E : \left|f_{m,n}(x) - f(x)\right| \geq \sigma\right\} = 0$$

for all $\sigma > 0$.

Proof. Firstly we define the sets A, $A_{m,n}$ for $m, n \in \mathbb{N}$, B and Q as follows:

$$\begin{aligned} A &= \left\{x \in E : \left|f(x)\right| = +\infty\right\},\ A_{m.n} = \left\{x \in E : \left|f_{m,n}(x)\right| = +\infty\right\} \ \ (m, n \in \mathbb{N}), \\ B &= \left\{x \in E : P - \lim f_{m,n}(x) \neq f(x)\right\},\ \ Q = A \cup \left(\cup_{m,n=1}^{\infty} A_{m.n}\right) \cup B. \end{aligned}$$

By the hypothesis, we have $\mu(A) = \mu(B) = \mu\left(\cup_{m,n=1}^{\infty} A_{m.n}\right) = 0$. Since $\mu(Q) \leq \mu(A) + \mu(B) + \sum_{m,n=1}^{\infty} \mu(A_{m.n})$, we get $\mu(Q) = 0$. Further, let

$$E_{m,n}(\sigma) = \left\{x \in E : \left|f_{m,n}(x) - f(x)\right| \geq \sigma\right\} \text{ for all } m,\ n \in \mathbb{N},$$

and

$$R_{m,n}(\sigma) = \cup_{i=0}^{\infty} E_{i+m,i+n}(\sigma) \text{ for all } m, n \in \mathbb{N}.$$

Using these sets, we define the sets

$$S_{m,n}(\sigma) = \cup_{k=m}^{\infty} \cup_{l=n}^{\infty} R_{k,l}(\sigma) \text{ for all } m, n \in \mathbb{N}.$$

Then clearly

$$S_{1,1}(\sigma) \supset S_{2,2}(\sigma) \supset S_{3,3}(\sigma) \supset ...$$

Now we take

$$M = \cap_{n=1}^{\infty} S_{n,n}(\sigma),$$

and show the inclusion $M \subset Q$. If we accept $x_0 \notin Q$, then $P - \lim f_{k,l}(x_0) = f(x_0)$, each $f_{k,l}(x_0)$ is finite and $f(x_0)$ is finite. Thus there is $n_0 \in \mathbb{N}$ such that

$$\left|f_{k,l}(x_0) - f(x_0)\right| < \sigma$$

for all $k, l \geq n_0$. In other words, $x_0 \notin E_{k,l}(\sigma)$ for all $k, l \geq n_0$. Hence $x_0 \notin S_{n,n}(\sigma)$ for each $n \geq n_0$. From here, we have $x_0 \notin M$ and so $M \subset Q$. Hence, since $\mu(Q) = 0$ we find $\mu(M) = 0$ and so

$$\lim_{n\to\infty} \mu(S_{n,n}(\sigma)) = \mu(M) = 0.$$

Considering the inclusions

$$E_{m,n}(\sigma) \subset R_{m,n}(\sigma) \subset S_{m,n} \subset S_{m,m} \cup S_{n,n}$$

for $m \geq n_0$ and $n \geq n_0$, we have

$$\mu(E_{m,n}(\sigma)) \leq \mu(S_{m,m} \cup S_{n,n}) \leq \mu(S_{m,m}) + \mu(S_{n,n})$$

and hence

$$P - \lim \mu(E_{m,n}(\sigma)) = 0.$$

This completes the proof. □

Corollary 2.2. *If a double sequence of functions converges almost everywhere, then it converges in measure to same function.*

The opposite of the above theorem is not usually true. We explain this with an example:

Example 2.3. *We define $k - funtions$ on* $[0.1)$ *for all* $k \in \mathbb{N}$ *as follows:*

$$f_i^{(k)}(x) = \begin{cases} 1, & for\ x \in \left[\frac{i-1}{k}, \frac{i}{k}\right) \\ 0, & for\ x \notin \left[\frac{i-1}{k}, \frac{i}{k}\right) \end{cases}, \quad i = 1, 2, 3, ..., k$$

Using these functions, we define a double sequence as $\varphi_{m,n} \equiv 0$ *on* $[0.1)$ *for* $m \neq n$ *and*

$$\varphi_{1,1}(x) = f_1^{(1)}(x), \ \varphi_{2,2}(x) = f_1^{(2)}(x), \ \varphi_{3,3}(x) = f_2^{(2)}(x), \ \varphi_{4,4}(x) = f_1^{(3)}(x), ...$$

We can easily see that this double sequence $(\varphi_{m,n})$ *converges in measure to zero. In fact, for each* σ *with* $0 < \sigma \leq 1$ *if* $m \neq n$*, then* $\left\{x \in E : \left|\varphi_{m,n}(x)\right| \geq \sigma\right\} = \phi$ *where* ϕ *denotes to empty set and if* $m = n$*, then taking* $\varphi_{n,n}(x) = f_i^{(k)}(x)$ *we have*

$$\left\{x \in E : \left|\varphi_{n,n}(x)\right| \geq \sigma\right\} = \left[\frac{i-1}{k}, \frac{i}{k}\right),$$

and the measure of this set is $\frac{1}{k}$ *and converges to zero as* $n \to \infty$*. Thus*

$$P - \lim \left\{x \in E : \left|\varphi_{n,n}(x)\right| \geq \sigma\right\} = 0.$$

Also the relation

$$P - \lim \varphi_{m,n}(x) = 0$$

does not occur for any point of the interval $[0, 1)$*. In fact, given any* $x_0 \in [0, 1)$ *we can find* $i \in \mathbb{N}$ *such that*

$$x_0 \in \left[\frac{i-1}{k}, \frac{i}{k}\right),$$

and so $f_i^{(k)}(x_0) = 1$*. In other words,* $\varphi_{n,n}(x_0) = f_i^{(k)}(x_0) = 1$ *for large enough* n *and hence*

$$P - \lim \varphi_{m,n}(x_0) \neq 0.$$

Theorem 2.4. *If double sequence of functions $(f_{m,n})$ converges in measure to two different f and g, then this limit functions are equivalent.*

Proof. It is easy to verify that

$$\left\{x \in E : \left|f(x) - g(x)\right| \geq \sigma\right\} \subset \left\{x \in E : \left|f_{m,n}(x) - f(x)\right| \geq \frac{\sigma}{2}\right\} \cup \left\{x \in E : \left|f_{m,n}(x) - g(x)\right| \geq \frac{\sigma}{2}\right\}.$$

for all $\sigma > 0$. The measure of set in right side of this inclusion converges to zero in Pringsheim's sense and hence we have

$$\mu\left(\left\{x \in E : \left|f(x) - g(x)\right| \geq \sigma\right\}\right) = 0.$$

This gives the desired. □

Theorem 2.5. *Let $(f_{m,n})$ be a double sequence of measurable functions which converges in measure to a function f on E. Then there exists a subsequence*

$$f_{m_1,n_1},\ f_{m_2,n_2},\ f_{m_3,n_3}, \ldots \qquad (m_1 < m_2 < m_3 < \ldots;\ n_1 < n_2 < n_3 < \ldots)$$

of $(f_{m,n})$ which converges almost everywhere to f.

Proof. We consider a sequence of positive numbers such that

$$\lim_{k\to\infty} \sigma_k = 0, \quad \sigma_1 > \sigma_2 > \sigma_3 > \ldots.$$

Further, let

$$\eta_1 + \eta_2 + \eta_3 + \ldots$$

be a convergence series with positive terms. According to hypotesis, there exist two natural numbers m_1 and n_1 such that

$$\mu\left(\left\{x \in E : \left|f_{m_1,n_1}(x) - f(x)\right| \geq \sigma_1\right\}\right) < \eta_1.$$

Again there two natural numbers m_2 and n_2 such that

$$\mu\left(\left\{x \in E : \left|f_{m_2,n_2}(x) - f(x)\right| \geq \sigma_2\right\}\right) < \eta_2 \qquad (m_2 > m_1,\ n_2 > n_1).$$

In general we can choose two natural numbers m_k and n_k such that

$$\mu\left(\left\{x \in E : \left|f_{m_k,n_k}(x) - f(x)\right| \geq \sigma_k\right\}\right) < \eta_k \qquad (m_k > m_{k-1},\ n_k > n_{k-1}).$$

We thus have defined a subsequence (f_{m_k,n_k}) of $(f_{m,n})$. Next we denote that

$$P - \lim f_{m_k,n_k}(x) = f(x)$$

almost everywhere. Firstly, we define the sets

$$R_i = \cup_{k=i}^{\infty} \left\{x \in E : \left|f_{m_k,n_k}(x) - f(x)\right| \geq \sigma_k\right\}$$

for each $i \in \mathbb{N}$ and a set

$$Q = \cap_{k=i}^{\infty} R_i.$$

Since

$$R_1 \supset R_2 \supset R_3 \supset ...,$$

we have

$$\mu(R_i) \to \mu(Q)$$

as $i \to \infty$. On the other hand, the inequality

$$\mu(R_i) < \sum_{k=i}^{\infty} \eta_k$$

holds and hence it is clear that $\mu(R_i) \to 0$ as $i \to \infty$. Therefore we obtain

$$\mu(Q) = 0.$$

Finally we denote that $P - \lim f_{m_k,n_k}(x) = f(x)$ on $E - Q$. Let $x_0 \in E - Q$. Then $x_0 \notin R_{i_0}$ for at least one $i_0 \in \mathbb{N}$. Namely

$$x_0 \notin \left\{ x \in E : \left| f_{m_k,n_k}(x) - f(x) \right| \geq \sigma_k \right\}$$

for all $k \geq i_0$ and hence we have

$$\left| f_{m_k,n_k}(x) - f(x) \right| < \sigma_k$$

for all $k \geq i_0$. Since $\lim_{k\to\infty} \sigma_k = 0$, it is clear that $P - \lim f_{m_k,n_k}(x) = f(x)$. This completes the proof. □

Theorem 2.6. *Let $(f_{m,n})$ be a double sequence of measurable functions which is defined on E and finite almost everywere. We accept that $(f_{m,n})$ converges almost everywhere in Pringsheim's sense to a measurable function f, which is finite almost everywere. Then, for all $\delta > 0$ there is a measurable set $E_\delta \subset E$ such that*

(1) $\mu(E_\delta) > \mu(E) - \delta$,
(2) $(f_{m,n})$ is uniformly convergence to f on E_δ.

Proof. We consider the sets $(S_{m,n}(\sigma))$ in Theorem 1. We consider a sequence of positive numbers such that

$$\lim_{k\to\infty} \sigma_k = 0, \quad \sigma_1 > \sigma_2 > \sigma_3 > ....$$

and a convergence series

$$\eta_1 + \eta_2 + \eta_3 + ...$$

with positive terms. Since $\mu(S_{n,n}(\sigma)) \to 0$ as $n \to \infty$, there exists $n_i \in \mathbb{N}$ such that

$$\mu(S_{n_i,n_i}(\sigma_i)) < \eta_i$$

for all $i \in \mathbb{N}$. We now select $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0}^{\infty} \eta_i < \delta$$

also receive

$$e = \cup_{i=i_0}^{\infty} S_{n_i,n_i}(\sigma_i).$$

Then clearly

$$\mu(e) \leq \sum_{i=i_0}^{\infty} S_{n_i,n_i}(\sigma_i) < \sum_{i=i_0}^{\infty} \eta_i < \delta.$$

We now say $E_\delta = E - e$. We obviously have $\mu(E_\delta) > \mu(E) - \delta$. So the condition (1) is proved. Finally we prove the condition (2). For this we select arbitrary $\varepsilon > 0$.

If $y \in E_\delta$, then $x \notin e$ and therefore $y \notin S_{n_i,n_i}(\sigma)$ for all $i \geq i_0$. This implies that

$$y \notin \left\{x \in E : \left|f_{k,l}(x) - f(x)\right| \geq \sigma_i\right\}$$

$k \geq n_i$, $l \geq n_i$ and so

$$\left|f_{k,l}(y) - f(y)\right| < \sigma_i.$$

Hence

$$\left|f_{k,l}(y) - f(y)\right| < \varepsilon$$

$k \geq n_i$, $l \geq n_i$. This completes the proof. □

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