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Polaroid and k-quasi-$*$-paranormal Operators

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Abstract. An operator T is said to be k-quasi-$*$-paranormal if $\|T^{k+2}x\|\|T^kx\| \geq \|T^*T^kx\|^2$ for all $x \in H$, where k is a natural number. In this paper, we give the inclusion relation of k-quasi-$*$-paranormal operators and k-quasi-$*$-A operators. And we prove that if T is a polynomially k-quasi-$*$-paranormal operator, then T is polaroid and has SVEP. We also show that if T is a polynomially k-quasi-$*$-paranormal operator, then Weyl type theorems hold for T.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space H. Recall [3, 8, 9, 15, 16, 18] that $T \in B(H)$ is hyponormal if $T^*T \geq TT^*$, T is class $*$-A if $|T^2| \geq |T^*|^2$, T is quasi-$*$-A if $T^*|T^2|T \geq T^*|T^*|^2T$, T is k-quasi-$*$-A, if $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$, T is $*$-paranormal, if $\|T^2x\|\|x\| \geq \|T^*x\|^2$ for all $x \in H$, T is k-quasi-$*$-paranormal, if $\|T^{k+2}x\|\|T^kx\| \geq \|T^*T^kx\|^2$ for all $x \in H$, and T is normaloid if $\|T^n\| = \|T\|^n$, for $n \in \mathbb{N}$ (equivalently, $\|T\| = r(T)$, the spectral radius of T). In general the following implications hold:
hyponormal $\Rightarrow$ class $*$-A $\Rightarrow$ $*$-paranormal $\Rightarrow$ normaloid.
hyponormal $\Rightarrow$ class $*$-A $\Rightarrow$ quasi-$*$-A $\Rightarrow$ k-quasi-$*$-A.

A 1-quasi-$*$-paranormal operator is a quasi-$*$-paranormal operator. We show that a k-quasi-$*$-A operator is a k-quasi-$*$-paranormal operator (see Theorem 2.3). Hence we have the following implications:
hyponormal $\Rightarrow$ class $*$-A $\Rightarrow$ $*$-paranormal $\Rightarrow$ k-quasi-$*$-paranormal.
hyponormal $\Rightarrow$ class $*$-A $\Rightarrow$ quasi-$*$-A $\Rightarrow$ k-quasi-$*$-A $\Rightarrow$ k-quasi-$*$-paranormal.

We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. If $T \in B(H)$, write $N(T)$ and $R(T)$ for the null space and range space of T; $\sigma(T)$, $\sigma_a(T)$ and iso $\sigma(T)$ for the spectrum, the approximate point spectrum and the isolated spectrum points of T, respectively.

In section 2, we give the inclusion relation of k-quasi-$*$-paranormal operators and k-quasi-$*$-A operators. Also, we obtain a sufficient condition for k-quasi-$*$-paranormal operators to be normaloid. In section 3, we prove that if T is a polynomially k-quasi-$*$-paranormal operator, then T is polaroid and has SVEP. Finally we show that Weyl's theorem holds for polynomially k-quasi-$*$-paranormal operators.

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2. k-quasi-$*$-paranormal Operators

Lemma 2.1. *[16] T is a k-quasi-$*$-paranormal operator* $\Leftrightarrow T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$ *for all* $\lambda > 0$.

Lemma 2.2. *[16] Let T be a k-quasi-$*$-paranormal operator, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \; on \;\; H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Then T_1 is a $*$-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Theorem 2.3. *Let T be a k-quasi-$*$-A operator. Then T is a k-quasi-$*$-paranormal operator.*

Proof. If T is a k-quasi-$*$-A operator, then

$$T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k,$$

which yields that

$$(T^{*k}|T^2|T^kx, x) \geq (T^{*k}|T^*|^2T^kx, x) \;\; \text{for all}\, x \in H,$$

and hence

$$||T^{k+2}x||||T^kx|| \geq ||T^*T^kx||^2.$$

Consequently, T is a k-quasi-$*$-paranormal operator. □

But the converse of Theorem 2.3 is not true. We shall give an operator which is a 2-quasi-$*$-paranormal operator but not a 2-quasi-$*$-A operator.

By straightforward computations, we have the following Lemma 2.4.

Lemma 2.4. *Let $K = \oplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. For given positive operators A and B on H, define the operator $T_{A,B}$ on K as follows:*

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & A & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then i) $T_{A,B}$ *belongs to* 2*-quasi-$*$-A if and only if*

$$A^2(B^2 - A^2)A^2 \geq 0.$$

ii) $T_{A,B}$ *belongs to* 2*-quasi-$*$-paranormal if and only if*

$$A^2(B^4 - 2\lambda A^2 + \lambda^2)A^2 \geq 0 \;\; \textit{for all} \;\; \lambda > 0.$$

Example 2.5. *A non-*2*-quasi-$*$-A and* 2*-quasi-$*$-paranormal operator.*

Proof. Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \;\; B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}^{\frac{1}{4}}.$$

Then

$$B^4 - 2\lambda A^2 + \lambda^2 = \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & \lambda^2 - 4\lambda + 8 \end{pmatrix} \geq 0 \text{ for all } \lambda > 0,$$

hence

$$A^2(B^4 - 2\lambda A^2 + \lambda^2)A^2 \geq 0 \text{ for all } \lambda > 0.$$

Thus $T_{A,B}$ is a 2-quasi-∗-paranormal operator.

On the other hand, by using the Maple program,

$$A^2(B^2 - A^2)A^2 = \begin{pmatrix} -0.2850\cdots & 0.0432\cdots \\ 0.0432\cdots & 1.1449\cdots \end{pmatrix} \ngeq 0.$$

Hence $T_{A,B}$ is not a 2-quasi-∗-A operator. □

Lemma 2.6. *[15] Let T be a quasi-∗-paranormal operator. Then T is normaloid.*

If $k > 1$, a nilpotent operator is a k-quasi-∗-paranormal operator, but it is not normaloid. However we have the following result.

Theorem 2.7. *Let T be a k-quasi-∗-paranormal operator and $\|T^k\| = \|T\|^k$. Then T is normaloid.*

Proof. Suppose that T is a k-quasi-∗-paranormal operator, i.e.,

$$\|T^{k+2}x\|\|T^k x\| \geq \|T^*T^k x\|^2 \quad \text{for every } x \in H,$$

which implies that

$$\|T^{k+2}\|\|T^k\| \geq \|T^*T^k\|^2.$$

Now assume that

$$\|T^k\| = \|T\|^k,$$

then, by the above inequality,

$$\begin{aligned}\|T\|^{3k-2}\|T^{k+2}\| = \|T\|^{2k-2}\|T\|^k\|T^{k+2}\| &\geq \|T^{*(k-1)}\|^2\|T^{k+2}\|\|T^k\| \\ &\geq \|T^{*(k-1)}\|^2\|T^*T^k\|^2 \\ &\geq \|T^{*k}T^k\|^2 \\ &= \|T^k\|^4 \\ &= \|T\|^{4k},\end{aligned}$$

and therefore

$$\|T^{k+2}\| = \|T\|^{k+2}.$$

Hence by induction,

$$\|T^{k+2j}\| = \|T\|^{k+2j} \text{ for every } j \geq 1.$$

Since $\{T^{k+2j}\}$ is a subsequence of $\{T^n\}$, and $\lim\limits_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$, we have

$$\lim \|T^{k+2j}\|^{\frac{1}{k+2j}} = \lim \|T^n\|^{\frac{1}{n}} = r(T),$$

i.e.,

$$r(T) = \lim_{j\to\infty} \|T^{k+2j}\|^{\frac{1}{k+2j}} = \lim_{j\to\infty} (\|T\|^{k+2j})^{\frac{1}{k+2j}} = \|T\|.$$

Thus T is normaloid. □

We say that $T \in B(H)$ has the single valued extension property (abbrev. SVEP), if for every open set U of $\mathbb{C}$, the only analytic solution $f : U \to H$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U.

The following theorem has been proved in [16], we give a new proof here.

Theorem 2.8. *[16] Let T be a k-quasi-∗-paranormal operator. Then T has SVEP.*

Proof. If the range of T^k is dense, then T is a ∗-paranormal operator, T has SVEP by [10]. Next we can assume that the range of T^k is not dense. By Lemma 2.2, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Suppose $(T-z)f(z) = 0$, $f(z) = f_1(z) \oplus f_2(z)$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2 f_2(z) \\ (T_3 - z)f_2(z) \end{pmatrix} = 0.$$

And T_3 is nilpotent, T_3 has SVEP, hence $f_2(z) = 0$, $(T_1 - z)f_1(z) = 0$. Since T_1 is a ∗-paranormal operator, T_1 has SVEP by [10], then $f_1(z) = 0$. Consequently, T has SVEP. □

3. Polynomially k-quasi-∗-paranormal Operators

An operator T is called Fredholm if $R(T)$ is closed and both $N(T)$ and $N(T^*)$ are finite dimensional. The index of a Fredholm operator T is given by $i(T) = \dim N(T) - \dim (H/R(T))$. An operator T is called Weyl if it is Fredholm of index zero. The Weyl spectrum $w(T)$ of T is defined by [12], $w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$.

We consider the sets

$\Phi_+(H) := \{T \in B(H) : R(T) \text{ is closed and } \dim N(T) < \infty\}$;
$\Phi_+^-(H) := \{T \in B(H) : T \in \Phi_+(H) \text{ and } i(T) \leq 0\}$.

We define

$$\sigma_{ea}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(H)\};$$

$$\pi_{00}(T) := \{\lambda \in \text{iso}\, \sigma(T) : 0 < \dim N(T-\lambda) < \infty\};$$

$$\pi_{00}^a(T) := \{\lambda \in \text{iso}\, \sigma_a(T) : 0 < \dim N(T-\lambda) < \infty\}.$$

Following [13], we say that Weyl's theorem holds for T if $\sigma(T) \backslash w(T) = \pi_{00}(T)$, and that a-Weyl's theorem holds for T if $\sigma_a(T) \backslash \sigma_{ea}(T) = \pi_{00}^a(T)$.

More generally, Berkani investigated generalized Weyl's theorem which extends Weyl's theorem. Berkani investigated B-Fredholm theory as follows (see [4–6]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) < \infty$ and $\dim N(T_{[n]}^*) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$.

The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{–Weyl}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues. Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [5]. Recently in [4] Berkani and Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T.

We define $T \in SBF_+^-(H)$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $i(T_{[n]}) \leq 0$ [6]. We define $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(H)\}$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim N(T - \lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T).$$

It's known from [5, 17] that if $T \in B(H)$ then we have
generalized a-Weyl's theorem ⇒ a-Weyl's theorem ⇒ Weyl's theorem;
generalized a-Weyl's theorem ⇒ generalized Weyl's theorem ⇒ Weyl's theorem.

We say that T is a polynomially k-quasi-∗-paranormal operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a k-quasi-∗-paranormal operator. From the above definition, T is a polynomially k-quasi-∗-paranormal operator, then so is $T - \lambda$ for each $\lambda \in \mathbb{C}$.

The following example provides an operator which is a polynomially 2-quasi-∗-paranormal operator but not a 2-quasi-∗-paranormal operator.

Example 3.1. *Let* $T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \in B(l_2 \oplus l_2)$. *Then T is a polynomially 2-quasi-∗-paranormal operator but not a 2-quasi-∗-paranormal operator.*

Proof. Since

$$T^* = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix},$$

we have

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 = \begin{pmatrix} (\lambda^2 - 2\lambda + 5)I & (-2\lambda + 2)I \\ (-2\lambda + 2)I & (\lambda^2 - 4\lambda + 1)I \end{pmatrix}.$$

Then

$$T^{*2}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^2 = \begin{pmatrix} (5\lambda^2 - 26\lambda + 17)I & (2\lambda^2 - 10\lambda + 4)I \\ (2\lambda^2 - 10\lambda + 4)I & (\lambda^2 - 4\lambda + 1)I \end{pmatrix}.$$

Since $(5\lambda^2 - 26\lambda + 17)I$ is not a positive operator for $\lambda = 1$,

$$T^{*2}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^2 \ngeq 0.$$

Therefore T is not a 2-quasi-∗-paranormal operator.

On the other hand, consider the complex polynomial $h(z) = (z - 1)^2$. Then $h(T) = 0$, and hence T is a polynomially k-quasi-∗-paranormal operator. □

We know that Weyl's theorem holds for hermitian operators [19], which has been extended from hermitian operators to hyponormal operators [7], to algebraically hyponormal operators by [11], to algebraically quasi-∗-A operators [21], and to polynomially ∗-paranormal operators [20]. In this section, we prove polynomially k-quasi-∗-paranormal operators satisfy generalized a-Weyl's theorem.

Theorem 3.2. *Let T be a quasinilpotent polynomially k-quasi-∗-paranormal operator. Then T is nilpotent.*

Proof. We first assume that T is a k-quasi-∗-paranormal operator. Consider two cases, Case I: If the range of T^k is dense, then T is a ∗-paranormal operator, which leads to that T is normaloid, hence $T = 0$. Case II: If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k})$$

where T_1 is a $*$-paranormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Lemma 2.2. Since $\sigma(T) = \{0\}$, we obtain $\sigma(T_1) = \{0\}$, then $T_1 = 0$. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

Now, suppose that T is a polynomially k-quasi-$*$-paranormal operator. Then there exists a nonconstant polynomial p such that $p(T)$ is a k-quasi-$*$-paranormal operator. If $(p(T))^k$ has dense range, then $p(T)$ is a $*$-paranormal operator. Thus T is a polynomially $*$-paranormal operator. It follows from [20] that it is nilpotent. If $(p(T))^k$ does not have a dense range, then, by Lemma 2.2 we can represent $p(T)$ as the upper triangular matrix

$$p(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } H = \overline{R((p(T))^k)} \oplus N((p(T))^{*k}),$$

where $A := p(T)|\overline{R((p(T))^k)}$ is a $*$-paranormal operator. Since $\sigma(T) = \{0\}$ and $\sigma(p(T)) = p(\sigma(T)) = \{p(0)\}$, the operator $p(T) - p(0)$ is quasinilpotent. But $\sigma(p(T)) = \sigma(A) \cup \{0\}$, thus $\sigma(A) \cup \{0\} = \{p(0)\}$. So $p(0) = 0$, and hence $p(T)$ is quasinilpotent. Since $p(T)$ is a k-quasi-$*$-paranormal operator, by the previous argument $p(T)$ is nilpotent. On the other hand, since $p(0) = 0$, $p(z) = cz^m(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$ for some natural number m. $p(T) = cT^m(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)$, then there exists $q \in \mathbb{N}$ such that

$$(p(T))^q = c^q T^{mq}(T - \lambda_1)^q (T - \lambda_2)^q \cdots (T - \lambda_n)^q = 0.$$

Since T is quasinilpotent, $(T - \lambda_1), (T - \lambda_2), \cdots, (T - \lambda_n)$ is invertible, we have $T^{mq} = 0$, i.e., T is nilpotent. □

Recall that an operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid then it is isoloid. However, the converse is not true. In [16] it is showed that every k-quasi-$*$-paranormal operator is isoloid, we can prove more.

Theorem 3.3. *Let T be a polynomially k-quasi-$*$-paranormal operator. Then T is polaroid.*

Proof. Suppose T is a polynomially k-quasi-$*$-paranormal operator. Then $p(T)$ is a k-quasi-$*$-paranormal operator for some nonconstant polynomial p. Let $\lambda \in \text{iso}\,\sigma(T)$ and E_λ be the Riesz idempotent associated to λ defined by $E_\lambda := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other point of $\sigma(T)$. We can represent T as the direct sum in the following form:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\lambda\}$, we have

$$p(T) = \begin{pmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{pmatrix},$$

since $p(T)$ is a k-quasi-$*$-paranormal operator, then $p(T_1)$ is a k-quasi-$*$-paranormal operator, i.e., T_1 is a polynomially k-quasi-$*$-paranormal operator, so is $T_1 - \lambda$. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Theorem 3.2 that $T_1 - \lambda$ is nilpotent, thus $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. $T - \lambda$ has finite ascent and descent, and hence λ is a pole of the resolvent of T, therefore T is polaroid. □

Corollary 3.4. *Let T be a polynomially k-quasi-$*$-paranormal operator. Then T is isoloid.*

Theorem 3.5. *Let T be a polynomially k-quasi-$*$-paranormal operator. Then T has SVEP.*

Proof. Suppose that T is a polynomially k-quasi-$*$-paranormal operator. Then $p(T)$ is a k-quasi-$*$-paranormal operator for some nonconstant complex polynomial p, and hence $p(T)$ has SVEP by Theorem 2.8. Therefore T has SVEP by [14, Theorem 3.3.9]. □

If $T \in B(H)$ has SVEP, then T and T^* satisfy Browder's (equivalently, generalized Browder's) theorem and a-Browder's (equivalently, generalized a-Browder's) theorem. A sufficient condition for an operator T satisfying Browder's (generalized Browder's) theorem to satisfy Weyl's (resp., generalized Weyl's) theorem is that T is polaroid. Then we have the following result:

Theorem 3.6. *Let $T \in B(H)$. If T is a polynomially k-quasi-$*$-paranormal operator, then generalized Weyl's theorem holds for T, so does Weyl's theorem.*

Proof. It is obvious from Theorem 3.3, Theorem 3.5 and the statements of the above. □

Theorem 3.7. *Let $T \in B(H)$.*
i) *If T^* is a polynomially k-quasi-$*$-paranormal operator, then generalized a-Weyl's theorem holds for T.*
ii) *If T is a polynomially k-quasi-$*$-paranormal operator, then generalized a-Weyl's theorem holds for T^*.*

Proof. i) It is well known that T is polaroid if and only if T^* is polaroid [2, Theorem 2.11]. Now since a polynomially k-quasi-$*$-paranormal operator is polaroid and has SVEP, [2, Theorem 3.10] gives us the result of the theorem. For ii) we can also apply [2, Theorem 3.10]. □

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