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Ideal Convergent Function Sequences in Random 2-Normed Spaces

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Abstract. In the present paper we are concerned with $\mathcal{I}$-convergence of sequences of functions in random 2-normed spaces. Particularly, following the line of recent work of Karakaya et al. [23], we introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and give some basic properties of these concepts.

1. Introduction and Preliminaries

The theory of probabilistic normed (PN) spaces is important area of research in functional analysis. Much work has been done in this theory and it has many important applications in real world problems. PN spaces are the vector spaces in which the norms of the vectors are uncertain due to randomness. A PN space is a generalization of an ordinary normed linear space. In a PN space, the norms of the vectors are represented by probability distribution functions instead of nonnegative real numbers. If x is an element of a PN space, then its norm is denoted by F_x, and the value $F_x(t)$ is interpreted as the probability that the norm of x is smaller than t. PN spaces were first introduced by Sherstnev in [42] by means of a definition that was closely modelled on the theory of normed spaces. In 1993, Alsina et al. [1] presented a new definition of a PN space which includes the definition of Sherstnev [43] as a special case. This new definition has naturally led to the definition of the principal class of PN spaces, the Menger spaces, and is compatible with various possible definitions of a probabilistic inner product space. It is based on the probabilistic generalization of a characterization of ordinary normed spaces by means of a betweenness relation and relies on the tools of the theory of probabilistic metric (PM) spaces (see [39, 40]). This new definition quickly became the standard one and it has been adopted by many authors (for instance, [2, 17, 21, 25, 32–35, 38, 50]), who have investigated several properties of PN spaces. A detailed history and the development of the subject up to 2006 can be found in [41].

In [15], Gähler introduced an attractive theory of 2-normed spaces in the 1960's. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George White [51]. Siddiqi [44] delivered a series of lectures on this theme in various conferences. His joint paper with Gähler and Gupta [16] also provided valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by Siddiqi [44]. Since then, many researchers have studied these subjects and obtained various results [9, 17–22, 30, 46, 50].

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The concepts of statistical convergence for sequences of real numbers was introduced (independently) by Steinhaus [45] and Fast [12]. The concept of statistical convergence was further discussed and developed by many authors including [45]. There has been an effort to introduce several generalizations and variants of statistical convergence in different spaces [5, 14, 19, 25, 29–32, 36]. One such very important generalization of this notion was introduced by Kostyrko et al. [26] by using an ideal $\mathcal{I}$ of subsets of the set of natural numbers, which they called $\mathcal{I}$-convergence. More recent applications of ideals can be seen in [8, 20, 21, 33–35, 37, 46, 48–50] where more references can be found. Different types of statistical convergence of sequences of real functions and related notions were first studied in [4], and some important results and references on statistical convergence and function sequences can be found in [6, 7, 10, 11]. Recently, in [23], Karakaya et al. studied the statistical convergence of sequences of functions with respect to the intuitionistic fuzzy normed spaces. Recently, in [24], Karakaya et al. introduced the concept of λ-statistical convergence of sequences of functions in the intuitionistic fuzzy normed spaces.

The notion of ideal convergence of sequences of functions has not been studied previously in the setting of random 2-normed spaces. Motivated by this fact, in this paper, as a variant of $\mathcal{I}$-convergence, the notion of ideal convergence of sequences of functions was introduced in a random 2-normed space, and some important results are established. Finally, the notions of $\mathcal{I}$-pointwise convergence and $\mathcal{I}$-uniform convergence in a random 2-normed space are introduced and studied.

First we recall some of the basic concepts, which will be used in this paper.

The notion of a statistically convergent sequence can be defined using the asymptotic density of subsets of the set of positive integers $\mathbb{N} = \{1, 2, ...\}$. For any $K \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we denote $K(n) := cardK \cap \{1, 2, ..., n\}$ and we define lower and upper asymptotic density of the set K by the formulas

$$\underline{\delta}(K) := \liminf_{n\to\infty} \frac{K(n)}{n};\ \overline{\delta}(K) := \limsup_{n\to\infty} \frac{K(n)}{n}.$$

If $\underline{\delta}(K) = \overline{\delta}(K) =: \delta(K)$, then the common value $\delta(K)$ is called the asymptotic density of the set K and

$$\delta(K) = \lim_{n\to\infty} \frac{K(n)}{n}.$$

Obviously all three densities $\underline{\delta}(K)$, $\overline{\delta}(K)$ and $\delta(K)$ (if they exist) lie in the unit interval $[0, 1]$.

$$\delta(K) = \lim_n \frac{1}{n} |K_n| = \lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),$$

if it exists, where χ_K is the characteristic function of the set K [13]. We say that a number sequence $x = (x_k)_{k\in\mathbb{N}}$ statistically converges to a point L if for each $\varepsilon > 0$ we have $\delta(K(\varepsilon)) = 0$, where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ and in such situation we will write $L = st\text{-}\lim x_k$.

The notion of statistical convergence was further generalized in the paper [26, 27] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a proper ideal. A proper ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N}\backslash A\}$, called the filter associated with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Recall the generalization of statistical convergence from [26, 27].

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and $x = (x_k)_{k\in\mathbb{N}}$ be a sequence of points in a metric space (X, ρ). We say that the sequence x is $\mathcal{I}$-convergent (or $\mathcal{I}$-converges) to a point $\xi \in X$, and we denote it by $\mathcal{I}\text{-}\lim x = \xi$, if for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, \xi) \geq \varepsilon\} \in \mathcal{I}.$$

This generalizes the notion of usual convergence, which can be obtained when we take for $\mathcal{I}$ the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. A sequence is statistically convergent if and only if it is $\mathcal{I}_\delta$-convergent, where $\mathcal{I}_\delta := \{K \subset \mathbb{N} : \delta(K) = 0\}$ is the admissible ideal of the sets of zero asymptotic density.

Definition 1.1. ([15]) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|(\cdot,\cdot)\| : X \times X \to \mathbb{R}$ which satisfies **(i)** $\left\|(x,y)\right\| = 0$ if and only if x and y are linearly dependent; **(ii)** $\left\|(x,y)\right\| = \left\|(y,x)\right\|$; **(iii)** $\left\|(\alpha x,y)\right\| = |\alpha| \left\|(x,y)\right\|$, $\alpha \in \mathbb{R}$; **(iv)** $\left\|(x,y+z)\right\| \leq \left\|(x,y)\right\| + \|(x,z)\|$. The pair $(X, \|(\cdot,\cdot)\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\left\|(x,y)\right\|$:= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\left\|(x,y)\right\| = \left|x_1 y_2 - x_2 y_1\right|, \quad x = (x_1,x_2), \quad y = (y_1,y_2).$$

Observe that in any 2-normed space $(X, \|(\cdot,\cdot)\|)$ we have $\left\|(x,y)\right\| \geq 0$ and $\left\|(x,y+\alpha x)\right\| = \left\|(x,y)\right\|$ for all $x,y \in X$ and $\alpha \in \mathbb{R}$. Also, if x,y and z are linearly independent, then $\left\|(x,y+z)\right\| = \left\|(x,y)\right\| + \|(x,z)\|$ or $\left\|(x,y-z)\right\| = \left\|(x,y)\right\| + \|(x,z)\|$. Given a 2-normed space $(X, \|(\cdot,\cdot)\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty} \left\|(x_n - x, y)\right\| = 0$ for every $y \in X$.

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [39].

Definition 1.2. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0,1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & \text{if } t > a, \\ 0 & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 1.3. A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S,*)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a,b,c,d \in S$. A triangle function τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 1.4. Let X be a linear space of dimension greater than one, τ be a triangle function, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:

(i) $F(x,y;t) = H_0(t)$ if x and y are linearly dependent, where $F(x,y;t)$ denotes the value of $F(x,y)$ at $t \in \mathbb{R}$,

(ii) $F(x,y;t) \neq H_0(t)$ if x and y are linearly independent,

(iii) $F(x,y;t) = F(y,x;t)$ for all $x,y \in X$,

(iv) $F(\alpha x,y;t) = F(x,y;\frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x,y \in X$,

(v) $F(x+y,z;t) \geq \tau(F(x,z;t), F(y,z;t))$ whenever $x,y,z \in X$, and $t > 0$.

If (v) is replaced by

(vi) $F(x+y,z;t_1+t_2) \geq F(x,z;t_1) * F(y,z;t_2)$ for all $x,y,z \in X$ and $t_1,t_2 \in \mathbb{R}_+$;

then $(X,F,*)$ is called a random 2-normed space (for short, RTN space).

As a standard example, we can give the following:

Example 1.5. Let $(X, \|(.,.)\|)$ be a 2-normed space, and let $a * b = ab$ for all $a,b \in S$. For all $x,y \in X$ and every $t > 0$, consider

$$F(x,y;t) = \frac{t}{t + \left\|(x,y)\right\|}.$$

Then observe that $(X,F,*)$ is a random 2-normed space.

Let $(X, F, *)$ be a RTN space. Since $*$ is a continuous t-norm, the system of (ε, η)-neighborhoods of θ (the null vector in X)

$$\left\{\mathcal{N}_{(\theta,z)}(\varepsilon, \eta) : \varepsilon > 0,\ \eta \in (0, 1),\ z \in X\right\},$$

where

$$\mathcal{N}_{(\theta,z)}(\varepsilon, \eta) = \left\{(x, z) \in X \times X : F_{(x,z)}(\varepsilon) > 1 - \eta\right\}.$$

determines a first countable Hausdorff topology on $X \times X$, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_{(\theta,z)}$ means $y \in \mathcal{N}_{(x,z)}$ and vice-versa.

A sequence $x = (x_k)$ in X is said to be F-convergence to $L \in X$ if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and for each nonzero $z \in X$ there exists a positive integer N such that

$$(x_k, z - L) \in \mathcal{N}_{(\theta,z)}(\varepsilon, \lambda) \text{ for each } k \geq N$$

or equivalently,

$$(x_k, z) \in \mathcal{N}_{(L,z)}(\varepsilon, \lambda) \text{ for each } k \geq N.$$

In this case we write $F\text{-}\lim (x_k, z) = L$.

We also recall that the concept of convergence and Cauchy sequence in a random 2-normed space is studied in [3].

Definition 1.6. Let $(X, F, *)$ be a RN space. Then, a sequence $x = \{x_k\}$ is said to be convergent to $L \in X$ with respect to the random norm F if, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $k_0 \in \mathbb{N}$ such that $F_{x_k - L}(\varepsilon) > 1 - \lambda$ whenever $k \geq k_0$. It is denoted by $F\text{-}\lim x = L$ or $x_k \to_F L$ as $k \to \infty$.

Definition 1.7. Let $(X, F, *)$ be a RN space. Then, a sequence $x = \{x_k\}$ is called a Cauchy sequence with respect to the random norm F if, for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $k_0 \in \mathbb{N}$ such that $F_{x_k - x_m}(\varepsilon) > 1 - \lambda$ for all $k, m \geq k_0$.

2. Kinds of $\mathcal{I}$-Convergence for Functions in RTNS

In this section we are concerned with convergence in $\mathcal{I}$-pointwise convergence and $\mathcal{I}$-uniform convergence of sequences of functions in a random 2-normed spaces. Particularly, we introduce the ideal analog of the Cauchy convergence criterion for pointwise and uniform ideal convergence in a random 2-normed space. Finally, we prove that pointwise and uniform ideal convergence preserves continuity.

2.1. *$\mathcal{I}$-pointwise convergence in RTNS*

Fix an admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ and a random 2-normed space $(Y, F', *)$. Assume that $(X, F, *)$ is a RTN space and that $\mathcal{N}'_{(\theta,z)}(\varepsilon, \eta) = \left\{(x, z) \in X \times X : F'_{(x,z)}(\varepsilon) > 1 - \eta\right\}$, called the F'-topology, is given.

Let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. A sequence of functions $(f_k)_{k \in \mathbb{N}}$ (on X) is said to be F-convergence to f (on X) if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and for each nonzero $z \in X$, there exists a positive integer $N = N(\varepsilon, \lambda, x)$ such that

$$(f_k(x) - f(x), z) \in \mathcal{N}'_{(\theta,z)}(\varepsilon, \eta) = \left\{x, z \in X \times X : F'_{((f_k(x) - f(x)),z)}(\varepsilon) > 1 - \eta\right\}$$

for each $k \geq N$ and for each $x \in X$ or equivalently,

$$(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon, \eta) \text{ for each } k \geq N \text{ and for each } x \in X.$$

In this case we write $f_k \to_{F^2} f$.

First we define $\mathcal{I}$-pointwise convergence in a random 2-normed space.

Definition 2.1. Let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. $(f_k)_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-pointwise convergent to a function f (on X) with respect to F-topology if for every $x \in X$, $\varepsilon > 0$, $\lambda \in (0,1)$ and each nonzero $z \in X$ the set

$$\left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda)\right\} \text{ belongs to } \mathcal{I}.$$

In this case we write $f_k \to_{\mathcal{I}(F^2)} f$.

Theorem 2.2. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal and let $(X, F, *)$, $(Y, F', *)$ be RTN spaces. Assume that $(f_k)_{k\in\mathbb{N}}$ is pointwise convergent (on X) with respect to F-topology where $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$. Then $f_k \to_{\mathcal{I}(F^2)} f$ (on X). But the converse of this is not true.*

Proof. Let $\varepsilon > 0$ and $\lambda \in (0,1)$. Suppose that $(f_k)_{k\in\mathbb{N}}$ is F-convergent on X. In this case the sequence $(f_k(x))$ is convergent with respect to F'-topology for each $x \in X$. Then, there exists a number $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda)$ for every $k \geq k_0$, every nonzero $z \in X$ and for each $x \in X$. This implies that the set

$$A(\varepsilon, \lambda) = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda)\right\} \subseteq \{1, 2, 3, ..., k_0 - 1\}.$$

Since the right hand side belongs to $\mathcal{I}$, we have $A(\varepsilon, \lambda) \in \mathcal{I}$. That is, $f_k \to_{\mathcal{I}(F^2)} f$ (on X). □

Example 2.3. Consider X as in Example 1.5, we have $(X, F, *)$ is a RTN space induced by the random 2-norm $F_{x,y}(\varepsilon) = \frac{\varepsilon}{\varepsilon + \|(x,y)\|}$. Define a sequence of functions $f_k : [0,1] \to \mathbb{R}$ via

$$f_k(x) = \begin{cases} x^{k^2} + 1 & \text{if } k = m^2 \ (m \in \mathbb{N}) \text{ and } x \in [0, \frac{1}{2}) \\ 0 & \text{if } k \neq m^2 \ (m \in \mathbb{N}) \text{ and } x \in [0, \frac{1}{2}) \\ 0 & \text{if } k = m^2 \ (m \in \mathbb{N}) \text{ and } x \in [\frac{1}{2}, 1) \\ x^k + \frac{1}{2} & \text{if } k \neq m^2 \ (m \in \mathbb{N}) \text{ and } x \in [\frac{1}{2}, 1) \\ 2 & \text{if } x = 1. \end{cases}$$

Then, for every $\varepsilon > 0$, $\lambda \in (0,1)$, $x \in [0, \frac{1}{2})$ and each nonzero $z \in X$, let $A_n(\varepsilon, \lambda) = \left\{k \leq n : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda)\right\}$. We observe that

$$\begin{aligned} (f_k(x), z) \notin \mathcal{N}'_{(\theta,z)}(\varepsilon, \lambda) &\Rightarrow F'_{(f_k(x),z)}(\varepsilon) \leq 1 - \lambda \\ &\Rightarrow \frac{\varepsilon}{\varepsilon + \|(f_k(x), z)\|} \leq 1 - \lambda \\ &\Rightarrow \|(f_k(x), z)\| \leq \frac{\varepsilon\lambda}{1 - \varepsilon} > 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} A_n(\varepsilon, \lambda) &= \left\{k \leq n : \|(f_k(x), z)\| > 0\right\} \\ &= \left\{k \leq n : f_k(x) = x^{k^2} + 1\right\} \\ &= \left\{k \leq n : k = m^2 \text{ and } m \in \mathbb{N}\right\} \end{aligned}$$

which yields $A_n(\varepsilon, \lambda) \in \mathcal{I}$. Therefore, for each $x \in [0, \frac{1}{2})$, $(f_k)_{k\in\mathbb{N}}$ is $\mathcal{I}$-convergence to 0 with respect to F-topology. Similarly, if we take $x \in [\frac{1}{2}, 1)$ and $x = 1$, it can be seen easily that $(f_k)_{k\in\mathbb{N}}$ is $\mathcal{I}$-convergence to $\frac{1}{2}$ and 2 with respect to F-topology, respectively. Hence $(f_k)_{k\in\mathbb{N}}$ is pointwise convergent with respect to F-topology (on X).

Theorem 2.4. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k:(X,F,*)\rightarrow(Y,F',*)$, $k\in\mathbb{N}$, be a sequence of functions. Then the following statements are equivalent:*

(i) $f_k \rightarrow_{\mathcal{I}(F^2)} f$.

(ii) $\left\{k\le n:(f_k(x),z)\notin \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)\right\}\in\mathcal{I}$ *for every* $\varepsilon>0$, $\lambda\in(0,1)$, *for each* $x\in X$ *and each nonzero* $z\in X$.

(iii) $\left\{k\le n:(f_k(x),z)\in \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)\right\}\in\mathcal{F}(\mathcal{I})$ *for every* $\varepsilon>0$, $\lambda\in(0,1)$, *for each* $x\in X$ *and each nonzero* $z\in X$.

(iv) $\mathcal{I}$-$\lim F'_{(f_k(x)-f(x),z)}(\varepsilon)=1$ *for every* $x\in X$ *and each nonzero* $z\in X$.

Proof is standard.

Theorem 2.5. *Let $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be two sequences of functions from $(X,F,*)$ to $(Y,F',*)$ with $a*a>a$ for every $a\in(0,1)$. If $f_k\rightarrow_{\mathcal{I}(F^2)} f$ and $g_k\rightarrow_{\mathcal{I}(F^2)} g$, then $(\alpha f_k+\beta g_k)\rightarrow_{\mathcal{I}(F^2)}(\alpha f+\beta g)$ where $\alpha,\beta\in\mathbb{R}$ (or $\mathbb{C}$).*

Proof. Let $\varepsilon>0$ and $\lambda\in(0,1)$. Since $f_k\rightarrow_{\mathcal{I}(F^2)} f$ and $g_k\rightarrow_{\mathcal{I}(F^2)} g$ for each $x\in X$, we have

$$A=\left\{k\in\mathbb{N}:(f_k(x),z)\notin\mathcal{N}'_{(f(x),z)}(\tfrac{\varepsilon}{2},\lambda)\right\} \text{ and } B=\left\{k\in\mathbb{N}:(g_k(x),z)\notin\mathcal{N}'_{(g(x),z)}(\tfrac{\varepsilon}{2},\lambda)\right\}$$

belong to $\mathcal{I}$. This implies that $A^c=\left\{k\in\mathbb{N}:(f_k(x),z)\in\mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2},\lambda)\right\}$ and $B^c=\left\{k\in\mathbb{N}:(g_k(x),z)\in\mathcal{N}'_{(g(x),z)}(\frac{\varepsilon}{2},\lambda)\right\}$ belong to $\mathcal{F}(\mathcal{I})$. Let

$$C=\left\{k\in\mathbb{N}:((\alpha f_k(x)+\beta g_k(x)),z)\notin\mathcal{N}'_{((\alpha f(x)+\beta g(x)),z)}(\varepsilon,\lambda)\right\}.$$

Since $\mathcal{I}$ is an ideal it is sufficient to show that $C\subset A\cup B$. This is equivalent to show that $C^c\supset A^c\cap B^c$. Let $k\in A^c\cap B^c$. For the case $\alpha,\beta=0$, we have

$$F'_{(0\cdot f_k(x)-0\cdot g_k(x),z)}(\varepsilon)=F'_0(\varepsilon)=1>1-\lambda$$

and for the case $\alpha,\beta\neq0$, we have

$$\begin{aligned}F'_{((\alpha f_k(x)+\beta g_k(x)-\alpha f(x)+\beta g(x)),z)}(\varepsilon)&\ge F'_{((\alpha f_k(x)-\alpha f(x)),z)}\left(\frac{\varepsilon}{2}\right)*F'_{((\beta g_k(x)-\beta g(x)),z)}\left(\frac{\varepsilon}{2}\right)\\&=F'_{((f_k(x)-f(x)),z)}\left(\frac{\varepsilon}{2\alpha}\right)*F'_{((g_k(x)-g(x)),z)}\left(\frac{\varepsilon}{2\beta}\right)\\&>(1-\lambda)*(1-\lambda)\\&>1-\lambda.\end{aligned}$$

Hence, $k\in C^c\supset A^c\cap B^c\in\mathcal{F}(\mathcal{I})$ which implies $C\subset A\cup B\in\mathcal{I}$ and the result follows. □

Definition 2.6. Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k:(X,F,*)\rightarrow(Y,F',*)$, $k\in\mathbb{N}$, be a sequence of functions. Then a sequence $(f_k)_{k\in\mathbb{N}}$ is called $\mathcal{I}$-pointwise Cauchy sequence in RTN space if for every $\varepsilon>0$, $\lambda\in(0,1)$ and each nonzero $z\in X$ there exists $M=M(\varepsilon,\lambda,x)\in\mathbb{N}$ such that

$$\left\{k\in\mathbb{N}:(f_k(x)-f_M(x),z)\notin\mathcal{N}'_\theta(\varepsilon,\lambda)\right\}\in\mathcal{I}.$$

Theorem 2.7. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces such that $a*a>a$ for every $a\in(0,1)$ and let $f_k:(X,F,*)\rightarrow(Y,F',*)$, $k\in\mathbb{N}$, be a sequence of functions. If $(f_k)_{k\in\mathbb{N}}$ is an $\mathcal{I}$-pointwise convergent sequence with respect to F-topology, then $(f_k)_{k\in\mathbb{N}}$ is an $\mathcal{I}$-pointwise Cauchy sequence with respect to F-topology.*

Proof. Suppose that $(f_k)_{k\in\mathbb{N}}$ is an $\mathcal{I}$-pointwise convergent to f with respect to F-topology. Let $\varepsilon > 0$ and $\lambda \in (0,1)$ be given. We have

$$A = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \lambda)\right\} \in \mathcal{I}.$$

This implies that $A^c \in \mathcal{F}(\mathcal{I})$. Now, for every $k, m \in A^c$,

$$\begin{aligned} F'_{(f_k(x)-f_m(x),z)}(\varepsilon) &\geq F'_{(f_k(x)-f(x),z)}\left(\frac{\varepsilon}{2}\right) * F'_{(f_m(x)-f(x),z)}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\lambda) * (1-\lambda) \\ &> 1-\lambda. \end{aligned}$$

So, $\left\{k \in \mathbb{N} : (f_k(x) - f_m(x), z) \in \mathcal{N}'_{(\theta,z)}(\varepsilon, \lambda)\right\} \in \mathcal{F}(\mathcal{I})$. Therefore

$$\left\{k \in \mathbb{N} : (f_k(x) - f_m(x), z) \notin \mathcal{N}'_{(\theta,z)}(\varepsilon, \lambda)\right\} \in \mathcal{I},$$

i.e., $(f_k)_{k\in\mathbb{N}}$ is an $\mathcal{I}$-pointwise Cauchy sequence with respect to F-topology. □

The next result is a modification of a well-known result.

Theorem 2.8. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces such that $a * a > a$ for every $a \in (0,1)$. Assume that $f_k \to_{\mathcal{I}(F^2)} f$ (on X) where functions $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, are equi-continuous (on X) and $f : (X, F, *) \to (Y, F', *)$. Then f is continuous (on X) with respect to F-topology.*

Proof. We will prove that f is continuous with respect to F-topology. Let $x_0 \in X$ and $(x - x_0, z) \in \mathcal{N}_{(\theta,z)}(\varepsilon, \lambda)$ be fixed. By the equi-continuity of f_k's, for every $\varepsilon > 0$ and each nonzero $z \in X$, there exists a $\gamma \in (0,1)$ with $\gamma < \lambda$ such that $(f_k(x) - f_k(x_0), z) \in \mathcal{N}'_{(\theta,z)}(\frac{\varepsilon}{3}, \gamma)$ for every $k \in \mathbb{N}$. Since $f_k \to_{\mathcal{I}(F^2)} f$, the set

$$\left\{k \in \mathbb{N} : (f_k(x_0), z) \notin \mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3}, \gamma)\right\} \cup \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3}, \gamma)\right\}$$

is in $\mathcal{I}$ and different from $\mathbb{N}$. So, there exists $k \in \mathcal{F}(\mathcal{I})$ such that $(f_k(x_0), z) \in \mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3}, \gamma)$ and $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3}, \gamma)$. We have

$$\begin{aligned} F'_{(f(x_0)-f(x),z)}(\varepsilon) &\geq F'_{(f(x_0)-f_k(x_0),z)}\left(\frac{\varepsilon}{3}\right) * \left[F'_{(f_k(x_0)-f_k(x),z)}\left(\frac{\varepsilon}{2}\right) * F'_{(f_k(x)-f(x),z)}\left(\frac{\varepsilon}{3}\right)\right] \\ &> (1-\gamma) * [(1-\gamma) * (1-\gamma)] \\ &> (1-\gamma) * (1-\gamma) \\ &> 1-\gamma \\ &> 1-\lambda \end{aligned}$$

and the contiunity of f with respect to F-topology is proved. □

2.2. $\mathcal{I}$-uniform convergence in RTNS

Now we define $\mathcal{I}$-uniform convergence in a random 2-normed space.

Definition 2.9. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal and let $(X, F, *)$, $(Y, F', *)$ be RTN spaces. We say that a sequence of functions $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, is $\mathcal{I}$-uniform convergence to a function f (on X) with respect to F-topology if and only if $\forall \varepsilon > 0$, $\exists M \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ and $\exists k_0 = k_0(\varepsilon, \lambda, x) \in M \ni \forall k > k_0$ and $k \in M$, $\forall z \in X$ and $\forall x \in X$, $\lambda \in (0,1)$ $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda)$.

In this case we write $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$.

Theorem 2.10. *Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k : (X,F,*) \to (Y,F',*)$, $k \in \mathbb{N}$, be a sequence of functions. Then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, the following statements are equivalent:*

(i) $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$.

(ii) $\left\{k \leq n : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)\right\} \in \mathcal{I}$ *for every $x \in X$ and each nonzero $z \in X$.*

(iii) $\left\{k \leq n : (f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)\right\} \in \mathcal{F}(\mathcal{I})$ *for every $x \in X$ and each nonzero $z \in X$.*

(iv) $\mathcal{I}\text{-}\lim F'_{(f_k(x)-f(x),z)}(\varepsilon) = 1$ *for every $x \in X$ and each nonzero $z \in X$.*

Proof is standard, so omitted.

Definition 2.11. Let $(X,F,*)$ be a RTN space. A subset Y of X is said to be bunded on RTN spaces if for every $\lambda \in (0,1)$ there exists $\varepsilon > 0$ such that $(x,z) \in \mathcal{N}_{(\theta,z)}(\varepsilon,\lambda)$ for all $x \in Y$ and every nonzero $z \in X$.

Definition 2.12. Let $(X,F,*)$, $(Y,F',*)$ be RTN spaces and let $f_k : (X,F,*) \to (Y,F',*)$, $k \in \mathbb{N}$, and $f : (X,F,*) \to (Y,F',*)$ be bounded functions. Then $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$ if and only if $\mathcal{I}\text{-}\lim\left(\inf_{x\in X} F'_{(f_k(x)-f(x),z)}(\varepsilon)\right) = 1$.

Example 2.13. Let $(X,F,*)$ be as in Example 1.5. Define a sequence of functions $f_k : [0,1) \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} x^k + 1 & \text{if } k \neq m^2 \ (m \in \mathbb{N}) \\ 2 & \text{otherwise.} \end{cases}$$

Then, for every $\varepsilon > 0$, $\lambda \in (0,1)$ and each nonzero $z \in X$, let $A_n(\varepsilon,\lambda) = \left\{k \leq n : (f_k(x), z) \notin \mathcal{N}'_{(1,z)}(\varepsilon,\lambda)\right\}$. For all $x \in X$, we have $A_n(\varepsilon,\lambda) \in \mathcal{I}$. Since $f_k \to_{\mathcal{I}(F^2)} 1$ for all $x \in X$, $f_k \rightrightarrows_{\mathcal{I}(F^2)} 1$ (on $[0,1)$).

Remark 2.14. If $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$ then $f_k \to_{\mathcal{I}(F^2)} f$. But the converse of this is not true.

We prove this with the following example.

Example 2.15. Let's define the sequence of functions

$$f_k(x) = \begin{cases} 0 & \text{if } k = n^2 \\ \frac{k^2 x}{1+k^3x^2} & \text{otherwise} \end{cases}$$

on $[0,1]$. Since $f_k\left(\frac{1}{k}\right) \to_{\mathcal{I}(F^2)} 1$ and $f_k(0) \to_{\mathcal{I}(F^2)} 0$, this sequence of functions is $\mathcal{I}$-pointwise convergence to 0 with respect to F-topology. But by Definition 2.12, it is not $\mathcal{I}$-uniformly by convergent with respect to F-topology.

Theorem 2.16. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal and let $(X,F,*)$, $(Y,F',*)$ be RTN spaces. Assume that $(f_k)_{k\in\mathbb{N}}$ is uniformly convergent (on X) with respect to F-topology where $f_k : (X,F,*) \to (Y,F',*)$, $k \in \mathbb{N}$. Then $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$ (on X).*

Proof. Assume that $(f_k)_{k\in\mathbb{N}}$ is uniformly convergent to f on X with respect to F-topology. In this case, for every $\varepsilon > 0$, $\lambda \in (0,1)$ and every nonzero $z \in X$, there exists a positive integer $k_0 = k_0(\varepsilon,\lambda)$ such that $\forall x \in X$ and $\forall k > k_0$, $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)$. That is, for $k \leq k_0$

$$A(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)\right\} \subseteq \{1,2,3,...,k_0\} \in \mathcal{I}$$

and $A^c = A^c(\varepsilon,\lambda)$ belongs to $\mathcal{F}(\mathcal{I})$. Hence for every $\varepsilon > 0$ and every nonzero $z \in X$, there exists $A^c \subset \mathbb{N}$, $A^c \in \mathcal{F}(\mathcal{I})$ and $\exists k_0 = k_0(\varepsilon,\lambda) \in A^c$ such that $\forall k > k_0$ and $k \in A^c$ and $\forall x \in X$, $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon,\lambda)$. This implies that $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$ (on X). $\square$

Definition 2.17. Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces and let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. Then a sequence $(f_k)_{k\in\mathbb{N}}$ is called $\mathcal{I}$-uniform Cauchy sequence in RTN space if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and each nonzero $z \in X$ there exists $N = N(\varepsilon, \lambda) \in \mathbb{N}$ such that

$$\left\{k \in \mathbb{N} : (f_k(x) - f_N(x), z) \notin \mathcal{N}'_{(\theta,z)}(\varepsilon, \lambda)\right\} \in \mathcal{I}.$$

Theorem 2.18. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces such that $a * a > a$ for every $a \in (0,1)$ and let $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be a sequence of functions. If $(f_k)_{k\in\mathbb{N}}$ is an $\mathcal{I}$-uniform convergence sequence with respect to F-topology, then $(f_k)_{k\in\mathbb{N}}$ is an $\mathcal{I}$-uniform Cauchy sequence with respect to F-topology.*

Proof. Suppose that $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$. Let $A = \left\{k \in \mathbb{N} : (f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\varepsilon, \lambda)\right\}$. By Definition 2.9, for every $\varepsilon > 0$, $\lambda \in (0,1)$ and each nonzero $z \in X$, there exists $A \subset \mathbb{N}$, $A \in \mathcal{F}(\mathcal{I})$ and $\exists k_0 = k_0(\varepsilon, \lambda) \in A$ such that for all $k > k_0$, $k \in A$ and for all $x \in X$, $(f_k(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \lambda)$. Choose $N = N(\varepsilon, \lambda) \in A$, $N > k_0$. So, $(f_N(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{2}, \lambda)$. For every $k \in A$, we have

$$\begin{aligned} F'_{(f_k(x)-f_N(x),z)}(\varepsilon) &\geq F'_{(f_k(x)-f(x),z)}\left(\frac{\varepsilon}{2}\right) * F'_{(f(x)-f_N(x),z)}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\lambda) * (1-\lambda) \\ &> 1-\lambda. \end{aligned}$$

Hence, $\left\{k \in \mathbb{N} : (f_k(x) - f_N(x), z) \in \mathcal{N}'_{(\theta,z)}(\varepsilon, \lambda)\right\} \in \mathcal{F}(\mathcal{I})$. Therefore

$$\left\{k \in \mathbb{N} : (f_k(x) - f_N(x), z) \notin \mathcal{N}'_{(\theta,z)}(\varepsilon, \lambda)\right\} \in \mathcal{I},$$

i.e., (f_k) is an $\mathcal{I}$-uniformly Cauchy sequence in RTN space. □

The next result is a modification of a well-known result.

Theorem 2.19. *Let $(X, F, *)$, $(Y, F', *)$ be RTN spaces such that $a*a > a$ for every $a \in (0,1)$ and the map $f_k : (X, F, *) \to (Y, F', *)$, $k \in \mathbb{N}$, be continuous (on X) with respect to F-topology. If $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$ (on X) then $f : (X, F, *) \to (Y, F', *)$ is continuous (on X) with respect to F-topology.*

Proof. Let $x_0 \in X$ and $(x_0 - x, z) \in \mathcal{N}_{(\theta,z)}(\varepsilon, \lambda)$ be fixed. By F-continuity of f_k's, for every $\varepsilon > 0$ and each nonzero $z \in X$, there exists a $\gamma \in (0,1)$ with $\gamma < \lambda$ such that $(f_k(x_0) - f_k(x), z) \in \mathcal{N}'_{(\theta,z)}(\frac{\varepsilon}{3}, \gamma)$ for every $k \in \mathbb{N}$. Since $f_k \rightrightarrows_{\mathcal{I}(F^2)} f$, for all $x \in X$, the set

$$\left\{k \in \mathbb{N} : (f_k(x), z) \notin \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3}, \gamma)\right\} \cup \left\{k \in \mathbb{N} : (f_k(x_0), z) \notin \mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3}, \gamma)\right\}$$

is in $\mathcal{I}$ and different from $\mathbb{N}$. So, there exists $m \in \mathcal{F}(\mathcal{I})$ such that $(f_m(x), z) \in \mathcal{N}'_{(f(x),z)}(\frac{\varepsilon}{3}, \gamma)$ and $(f_m(x_0), z) \in \mathcal{N}'_{(f(x_0),z)}(\frac{\varepsilon}{3}, \gamma)$. It follows that

$$\begin{aligned} F'_{(f(x)-f(x_0),z)}(\varepsilon) &\geq F'_{(f(x)-f_m(x),z)}\left(\frac{\varepsilon}{3}\right) * \left[F'_{(f_m(x_0)-f_m(x_0),z)}\left(\frac{\varepsilon}{2}\right) * F'_{(f_m(x_0)-f(x_0),z)}\left(\frac{\varepsilon}{3}\right)\right] \\ &> (1-\gamma) * [(1-\gamma) * (1-\gamma)] \\ &> (1-\gamma) * (1-\gamma) \\ &> 1-\gamma \\ &> 1-\lambda. \end{aligned}$$

This implies that f is continuous (on X) with respect to F-topology. □

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