Some Approximation Results on Bleimann-Butzer-Hahn Operators Defined by (p, q)-Integers

Mohammad Mursaleen[a], Md Nasiruzzaman[a], Asif Khan[a], Khursheed Jamal Ansari[a]

[a]*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*

Abstract. In this paper we introduce a generalization of the Bleimann-Butzer-Hahn operators based on (p, q)-integers and obtain Korovkin's type approximation theorem for these operators. Furthermore, we compute convergence of these operators by using the modulus of continuity.

1. Introduction

Bleimann, Butzer and Hahn (BBH) introduced the following operators in [2] as follows;

$$L_n(f; x) = \frac{1}{(1 + x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n - k + 1}\right) \left[\begin{array}{c} n \\ k \end{array} \right] x^k, x \geq 0. \tag{1}$$

In approximation theory, q-type generalization of Bernstein polynomials was introduced by Lupaş [7]. In 1997, Phillips [11] introduced another modification of Bernstein polynomials. Also he obtained the rate of convergence and the Voronovskaja's type asymptotic expansion for these polynomials.

The BBH-type operators based on q-integers are defined as follows

$$L_n^q(f; x) = \frac{1}{\ell_n(x)} \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n - k + 1]_q q^k}\right) q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k, \tag{2}$$

where $\ell_n(x) = \prod_{s=0}^{n-1} (1 + q^s x)$.

Recently, Mursaleen et al [9] applied (p, q)-calculus in approximation theory and introduced first (p, q)-analogue of Bernstein operators. They also introduced and studied approximation properties of (p, q)-analogue of Bernstein-Stancu operators in [8], and of Bernstein-Shurer operators in [10].

Let us recall certain notations on (p, q)-calculus.

The (p, q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \qquad n = 0, 1, 2, \cdots, \ 0 < q < p \leq 1.$$

Whereas q-integers are given by

$$[n]_q = \frac{1 - q^n}{1 - q}, \qquad n = 0, 1, 2, \cdots, \ 0 < q < 1.$$

It is very clear that q-integers and (p, q)-integers are different, that is we cannot obtain (p, q) integers just by replacing q by $\frac{q}{p}$ in the definition of q-integers but if we put $p = 1$ in definition of (p, q) integers then q-integers becomes a particular case of (p, q) integers. Thus we can say that (p, q)-calculus can be taken as a generalization of q-calculus.

Now by some simple calculation and induction on n, we have (p, q)-binomial expansion as follows

$$(ax + by)_{p,q}^n := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x + y)_{p,q}^n = (x + y)(px + qy)(p^2 x + q^2 y) \cdots (p^{n-1} x + q^{n-1} y),$$

$$(1 - x)_{p,q}^n = (1 - x)(p - qx)(p^2 - q^2 x) \cdots (p^{n-1} - q^{n-1} x)$$

and the (p, q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n - k]_{p,q}!}.$$

Again it can be easily verified that (p, q)-binomial expansion is different from q-binomial expansion and is not a replacement of q by $\frac{q}{p}$.

By some simple calculation, we have the following relation

$$q^k[n - k + 1]_{p,q} = [n + 1]_{p,q} - p^{n-k+1}[k]_{p,q}.$$

For details on q-calculus and (p, q)-calculus, one can refer [15], [5, 12, 13], respectively.

Now based on (p, q)-integers, we construct (p, q)-analogue of BBH-operators, and we call it as (p, q)-Bleimann-Butzer-Hahn-Operators and investigate its Korovokin's-type approximation properties, by using the test functions $\left(\frac{t}{1+t}\right)^\nu$ for $\nu = 0, 1, 2$. Also for a space of generalized Lipschitz-type maximal functions we give a pointwise estimation.

Let $C_B(\mathbb{R}_+)$ be the set of all bounded and continuous functions on $\mathbb{R}_+$, then $C_B(\mathbb{R}_+)$ is linear normed space with

$$\| f \|_{C_B} = \sup_{x \geq 0} | f(x) | .$$

Let ω denotes modulus of continuity satisfying the following condition:

1. ω is a non-negative increasing function on $\mathbb{R}_+$
2. $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$
3. $\lim_{\delta \to 0} \omega(\delta) = 0$.

Let H_ω be the space of all real-valued functions f defined on the semiaxis $\mathbb{R}_+$ satisfying the condition

$$| f(x) - f(y) | \leq \omega \left(\left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| \right),$$

for any $x, y \in \mathbb{R}_+$.

Theorem 1.1. ([4]) *Let $\{A_n\}$ be the sequence of positive linear operators from H_ω into $C_B(\mathbb{R}_+)$, satisfying the conditions*

$$\lim_{n\to\infty} \| A_n\left(\left(\frac{t}{1+t}\right)^\nu ; x\right) - \left(\frac{x}{1+x}\right)^\nu \|_{C_B},$$

for $\nu = 0, 1, 2$. Then for any function $f \in H_\omega$

$$\lim_{n\to\infty} \| A_n(f) - f \|_{C_B} = 0.$$

We define (p, q)-Bleimann-Butzer and Hahn-type operators based on (p, q)-integers as follows:

$$L_n^{p,q}(f; x) = \frac{1}{\ell_n^{p,q}(x)} \sum_{k=0}^n f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \tag{3}$$

where, $x \geq 0$, $0 < q < p \leq 1$

$$\ell_n^{p,q}(x) = \prod_{s=0}^{n-1} (p^s + q^s x)$$

and f is defined on semiaxis $\mathbb{R}_+$.

And also by induction, we construct the Euler identity based on (p, q)-analogue defined as follows:

$$\prod_{s=0}^{n-1} (p^s + q^s x) = \sum_{k=0}^n p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k. \tag{4}$$

If we put $p = 1$, then we obtain q-BBH-operators. If we take $f\left(\frac{[k]_{p,q}}{[n-k+1]_{p,q}}\right)$ instead of $f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)$ in (3), then we obtain usual generalization of Bleimann, Butzer and Hahn operators based on (p, q)-integers, then in this case it is impossible to obtain explicit expressions for the monomials t^ν and $\left(\frac{t}{1+t}\right)^\nu$ for $\nu = 1, 2$. But if we define the Bleimann, Butzer and Hahn operators as in (3), then we can obtain explicit formulas for the monomials $\left(\frac{t}{1+t}\right)^\nu$ for $\nu = 0, 1, 2$. We emphasize that these operators are more flexible than the classical BBH-operators and q-analogue of BBH-operators. That is depending on the selection of (p, q)-integers, the rate of convergence of (p, q)-BBH-operators is at least as good as the classical one.

2. Preliminary Results

Lemma 2.1. *Let $L_n^{p,q}(f; x)$ be given by (3), then for any $x \geq 0$ and $0 < q < p \leq 1$ we have the following identities*

1. $L_n^{p,q}(1; x) = 1$,
2. $L_n^{p,q}(\frac{t}{1+t}; x) = \frac{p[n]_{p,q}}{[n+1]_{p,q}}\left(\frac{x}{1+x}\right)$,
3. $L_n^{p,q}\left(\left(\frac{t}{1+t}\right)^2; x\right) = \frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \frac{x^2}{(1+x)(p+qx)} + \frac{p^{n+1}[n]_{p,q}}{[n+1]_{p,q}^2}\left(\frac{x}{1+x}\right)$.

Proof. 1. $L_n^{p,q}(1; x) = \frac{1}{\ell_n^{p,q}(x)} \sum_{k=1}^n p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k$.

For $0 < q < p \leq 1$, we have

$$\sum_{k=0}^n p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k = \prod_{s=0}^{n-1} (p^s + q^s x) = \ell_n^{p,q}(x).$$

2. Let $t = \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}$, then $\frac{t}{t+1} = \frac{[k]_{p,q}p^{n+1-k}}{[n+1]_{p,q}}$

$$
\begin{aligned}
L_n^{p,q}\left(\frac{t}{1+t};x\right) &= \frac{1}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \frac{[k]_{p,q}p^{n-k+1}}{[n+1]_{p,q}} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k \\
&= \frac{1}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \frac{[n]_{p,q}p^{n-k+1}}{[n+1]_{p,q}} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} x^k \\
&= x\left(\frac{1}{\ell_n^{p,q}(x)} \cdot \frac{[n]_{p,q}}{[n+1]_{p,q}}p\right) \sum_{k=0}^{n-1} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (qx)^k \\
&= p\frac{[n]_{p,q}}{[n+1]_{p,q}} \frac{x}{1+x}.
\end{aligned}
$$

3. We have

$L_n^{p,q}\left(\frac{t^2}{(1+t)^2};x\right) = \frac{1}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \frac{[k]_{p,q}^2 p^{2(n-k+1)}}{[n+1]_{p,q}^2} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$

By some simple calculation, we have

$$[k]_{p,q} = p^{k-1} + q[k-1]_{p,q}, \quad \text{and} \quad [k]_{p,q}^2 = q[k]_{p,q}[k-1]_{p,q} + p^{k-1}[k]_{p,q}.$$

Then by using it we get,

$$
L_n^{p,q}\left(\frac{t^2}{(1+t)^2};x\right) = \frac{1}{\ell_n^{p,q}(x)} \sum_{k=2}^{n} \frac{q[k]_{p,q}[k-1]_{p,q}p^{2n-2k+2}}{[n+1]_{p,q}^2} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k
$$

$$
+\frac{1}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} p^{k-1} \frac{[k]_{p,q}p^{2n-2k+2}}{[n+1]_{p,q}^2} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k
$$

$$
= \frac{1}{\ell_n^{p,q}(x)} \frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=2}^{n} p^{\left((2n-2k+2)+\frac{(n-k)(n-k-1)}{2}\right)} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{p,q} x^k
$$

$$
+\frac{1}{\ell_n^{p,q}(x)} \frac{[n]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=1}^{n} p^{\left((k-1)+(2n-2k+2)+\frac{(n-k)(n-k-1)}{2}\right)} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} x^k
$$

$$
= x^2 \frac{1}{\ell_n^{p,q}(x)} \frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=0}^{n-2} p^{\left((2n-2k-2)+\frac{(n-k-2)(n-k-3)}{2}\right)} q^{\frac{(k+1)(k+2)}{2}} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} x^k
$$

$$
+x \frac{1}{\ell_n^{p,q}(x)} \frac{[n]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=0}^{n-1} p^{\left(k+(2n-2k)+\frac{(n-k-1)(n-k-2)}{2}\right)} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} x^k
$$

$$
= x^2 \frac{1}{\ell_n^{p,q}(x)} \frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=0}^{n-2} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} (q^2 x)^k
$$

$$
+x \frac{1}{\ell_n^{p,q}(x)} \frac{p^{n+1}[n]_{p,q}}{[n+1]_{p,q}^2} \sum_{k=0}^{n-1} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} (qx)^k
$$

$$
= \frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \frac{x^2}{(1+x)(p+qx)} + \frac{p^{n+1}[n]_{p,q}}{[n+1]_{p,q}^2} \left(\frac{x}{1+x}\right).
$$

This completes the proof of Lemma 2.1. $\quad\square$

3. Main Results

3.1. Korovkin type approximation properties

In this section we obtain the Korovkin's type approximation properties for our operators defined by (3), with the help of Theorem 1.1.

In order to obtain the convergence results for the operators $L_n^{p,q}$, we take $q = q_n$, $p = p_n$ where $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ satisfying,

$$\lim_n p_n = 1, \qquad \lim_n q_n = 1. \tag{5}$$

Theorem 3.1. *Let $p = p_n$, $q = q_n$ satisfying (5), for $0 < q_n < p_n \le 1$ and if $L_n^{p_n, q_n}$ is defined by (3), then for any function $f \in H_\omega$,*

$$\lim_n \| L_n^{p_n, q_n}(f; x) - f \|_{C_B} = 0.$$

Proof. Using Theorem 1.1, we see that it is sufficient to verify that following three conditions:

$$\lim_{n \to \infty} \| L_n^{p_n, q_n} \left(\left(\frac{t}{1+t} \right)^\nu ; x \right) - \left(\frac{x}{1+x} \right)^\nu \|_{C_B} = 0, \quad \nu = 0, 1, 2. \tag{6}$$

From Lemma 2.1, the first condition of (6) is fulfilled for $\nu = 0$. Now it is easy to see that from (2) of Lemma 2.1

$$\| L_n^{p_n, q_n} \left(\left(\frac{t}{1+t} \right)^\nu ; x \right) - \left(\frac{x}{1+x} \right)^\nu \|_{C_B} \le \left| \frac{p_n [n]_{p_n, q_n}}{[n+1]_{p_n, q_n}} - 1 \right|$$

$$\le \left| \left(\frac{p_n}{q_n} \right) \left(1 - p_n^n \frac{1}{[n+1]_{p_n, q_n}} \right) - 1 \right|.$$

Since $q_n [n]_{p_n, q_n} = [n+1]_{p_n, q_n} - p_n^n$, $[n+1]_{p_n, q_n} \to \infty$ as $n \to \infty$, the condition (6) holds for $\nu = 1$. To verify this condition for $\nu = 2$, consider (3) of Lemma 2.1. Then, we see that

$$\| L_n^{p_n, q_n} \left(\left(\frac{t}{1+t} \right)^2 ; x \right) - \left(\frac{x}{1+x} \right)^2 \|_{C_B}$$

$$= \sup_{x \ge 0} \left\{ \frac{x^2}{(1+x)^2} \left(\frac{p_n q_n^2 [n]_{p_n, q_n} [n-1]_{p_n, q_n}}{[n+1]_{p_n, q_n}^2} \cdot \frac{1+x}{p_n + q_n x} - 1 \right) + \frac{p_n^{n+1} [n]_{p_n, q_n}}{[n+1]_{p_n, q_n}^2} \cdot \frac{x}{1+x} \right\}.$$

A small calculation leads to

$$\frac{[n]_{p_n, q_n} [n-1]_{p_n, q_n}}{[n+1]_{p_n, q_n}^2} = \frac{1}{q_n^3} \left\{ 1 - p_n^n \left(2 + \frac{q_n}{p_n} \right) \frac{1}{[n+1]_{p_n, q_n}} + (p_n^n)^2 \left(1 + \frac{q_n}{p_n} \right) \frac{1}{[n+1]_{p_n, q_n}^2} \right\},$$

and

$$\frac{[n]_{p_n, q_n}}{[n+1]_{p_n, q_n}^2} = \frac{1}{q_n} \left(\frac{1}{[n+1]_{p_n, q_n}} - p_n^n \frac{1}{[n+1]_{p_n, q_n}^2} \right).$$

Thus, we have

$$\| L_n^{p_n, q_n} \left(\left(\frac{t}{1+t} \right)^2 ; x \right) - \left(\frac{x}{1+x} \right)^2 \|_{C_B}$$

$$\le \frac{p_n}{q_n} \left\{ 1 - p_n^n \left(2 + \frac{q_n}{p_n} \right) \frac{1}{[n+1]_{p_n, q_n}} + (p_n^n)^2 \left(1 + \frac{q_n}{p_n} \right) \frac{1}{[n+1]_{p_n, q_n}^2} - 1 \right\}$$

$$+ p_n^n \frac{p_n}{q_n} \left(\frac{1}{[n+1]_{p_n, q_n}} - p_n^n \frac{1}{[n+1]_{p_n, q_n}^2} \right).$$

This implies that the condition (6) holds for $\nu = 2$ and the proof is completed by Theorem 1.1. $\square$

3.2. Rate of convergence

In this section we calculate the rate of convergence of operators (3) by means of modulus of continuity and Lipschitz type maximal functions.

The modulus of continuity for $f \in H_\omega$ is defined by

$$\widetilde{\omega}(f;\delta) = \sum_{\substack{|\frac{t}{1+t} - \frac{x}{1+x}| \leq \delta, \\ x,t \geq 0}} | f(t) - f(x) |,$$

where $\widetilde{\omega}(f;\delta)$ satisfies the following conditions. For all $f \in H_\omega(\mathbb{R}_+)$

1. $\lim_{\delta \to 0} \widetilde{\omega}(f;\delta) = 0$
2. $| f(t) - f(x) | \leq \widetilde{\omega}(f;\delta) \left(\frac{|\frac{t}{1+t} - \frac{x}{1+x}|}{\delta} + 1 \right).$

Theorem 3.2. *Let $p = p_n$, $q = q_n$ satisfy (5), for $0 < q_n < p_n \leq 1$ and if $L_n^{p_n,q_n}$ is defined by (3). Then for each $x \geq 0$ and for any function $f \in H_\omega$, we have*

$$| L_n^{p_n,q_n}(f;x) - f | \leq 2\widetilde{\omega}(f; \sqrt{\delta_n(x)}),$$

where

$$\delta_n(x) = \frac{x^2}{(1+x)^2} \left(\frac{p_n q_n^2 [n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{1+x}{p_n + q_n x} - 2\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1 \right) + \frac{p_n^{n+1}[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{x}{1+x}.$$

Proof.

$$
\begin{aligned}
| L_n^{p_n,q_n}(f;x) - f | &\leq L_n^{p_n,q_n}(| f(t) - f(x) |;x) \\
&\leq \widetilde{\omega}(f;\delta) \left\{ 1 + \frac{1}{\delta} L_n^{p_n,q_n}\left(\left| \frac{t}{1+t} - \frac{x}{1+x} \right|;x \right) \right\}.
\end{aligned}
$$

Now by using the Cauchy-Schwarz inequality, we have

$$| L_n^{p_n,q_n}(f;x) - f | \leq \widetilde{\omega}(f;\delta_n) \left\{ 1 + \frac{1}{\delta_n} \left[\left(L_n^{p_n,q_n}\left(\frac{t}{1+t} - \frac{x}{1+x} \right)^2;x \right) \right]^{\frac{1}{2}} \left(L_n^{p_n,q_n}(1;x) \right)^{\frac{1}{2}} \right\}$$

$$\leq \widetilde{\omega}(f;\delta_n) \left\{ 1 + \frac{1}{\delta_n} \left[\frac{x^2}{(1+x)^2} \left(\frac{p_n q_n^2 [n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{1+x}{p_n+q_n x} - 2\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1 \right) + \frac{p_n^{n+1}[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}} \frac{x}{1+x} \right]^{\frac{1}{2}} \right\}.$$

This completes the proof of Theorem 3.2. $\square$

Now we will give an estimate concerning the rate of convergence by means of Lipschitz type maximal functions. In [1], the Lipschitz type maximal function space on $E \subset \mathbb{R}_+$ is defined as

$$\widetilde{W}_{\alpha,E} = \{f : \sup(1+x)^\alpha \widetilde{f_\alpha}(x) \leq M \frac{1}{(1+y)^\alpha} : x \leq 0, \text{ and } y \in E\}, \tag{7}$$

where f is bounded and continuous function on $\mathbb{R}_+$, M is a positive constant, $0 < \alpha \leq 1$.

In [6], B.Lenze introduced a Lipschitz type maximal function f_α as follows:

$$f_\alpha(x,t) = \sum_{\substack{t>0 \\ t\neq x}} \frac{| f(t) - f(x) |}{| x - t |^\alpha}. \tag{8}$$

We denote by $d(x,E)$ the distance between x and E, that is

$$d(x,E) = \inf\{| x - y |; y \in E\}.$$

Theorem 3.3. *For all $f \in \widetilde{W}_{\alpha,E}$, we have*

$$| L_n^{p_n,q_n}(f;x) - f(x) | \leq M\left(\delta_n^{\frac{\alpha}{2}}(x) + 2\left(d(x,E)\right)^\alpha\right), \tag{9}$$

where $\delta_n(x)$ is defined as in Theorem 3.2.

Proof. Let $\overline{E}$ denote the closure of the set E. Then there exits a $x_0 \in \overline{E}$ such that $| x - x_0 |= d(x,E)$, where $x \in \mathbb{R}_+$. Thus we can write

$$| f - f(x) | \leq | f - f(x_0) | + | f(x_0) - f(x) | .$$

Since $L_n^{p_n,q_n}$ is a positive linear operator, $f \in \widetilde{W}_{\alpha,E}$ and by using the previous inequality, we have

$$| L_n^{p_n,q_n}(f;x) - f(x) | \leq | L_n^{p_n,q_n}(| f - f(x_0) |;x) + | f(x_0) - f(x) | L_n^{p_n,q_n}(1;x)$$

$$\leq M\left(L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^\alpha;x\right) + \frac{| x - x_0 |^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x)\right).$$

Since $(a+b)^\alpha \leq a^\alpha + b^\alpha$, which consequently imply

$$L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^\alpha;x\right) \leq L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^\alpha;x\right) + L_n^{p_n,q_n}\left(\left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^\alpha;x\right)$$

$$L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^\alpha;x\right) \leq L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^\alpha;x\right) + \frac{| x - x_0 |^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x).$$

By using the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^\alpha;x\right) \leq L_n^{p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right)^{\frac{\alpha}{2}} (L_n^{p_n,q_n}(1;x))^{\frac{2-\alpha}{2}}$$

$$+ \frac{| x - x_0 |^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x)$$

$$= \delta_n^{\frac{\alpha}{2}}(x) + \frac{| x - x_0 |^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}.$$

This completes the proof of Theorem 3.3. $\square$

Corollary 3.4. *If we take $E = \mathbb{R}_+$ as a particular case of Theorem 3.3, then for all $f \in \widetilde{W}_{\alpha,\mathbb{R}_+}$, we have*

$$| L_n^{p_n,q_n}(f;x) - f(x) | \leq M\delta_n^{\frac{\alpha}{2}}(x),$$

where $\delta_n(x)$ is defined in Theorem 3.2.

Theorem 3.5. *If $x \in (0,\infty) \setminus \left\{p^{n-k+1}\frac{[k]_{p,q}}{[n-k+1]_{p,q}q^k}\Big| k = 0,1,2,\cdots,n\right\}$, then*

$$L_n^{p,q}(f;x) - f\left(\frac{px}{q}\right) = -\frac{x^{n+1}}{\ell_n^{p,q}(x)}\left[\frac{px}{q}; \frac{p[n]_{p,q}}{q^n}; f\right]pq^{\frac{n(n-1)}{2}-1}$$

$$+ \frac{x}{\ell_n^{p,q}(x)}\sum_{k=0}^{n-1}\left[\frac{px}{q}; p^{n-k+1}\frac{[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f\right]\frac{1}{[n-k]_{p,q}}p^{\frac{(n-k)(n-k+1)}{2}+1}q^{\frac{k(k-3)}{2}-2}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}x^k. \tag{10}$$

Proof. By using (3), we have

$$L_n^{p,q}(f;x) - f\left(\frac{px}{q}\right) = \frac{1}{\ell_n^{p,q}(x)} \sum_{k=0}^{n} \left[f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right) - f\left(\frac{px}{q}\right) \right] p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k$$

$$= -\frac{1}{\ell_n^{p,q}(x)} \sum_{k=0}^{n} \left(\frac{px}{q} - \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k} \right) \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right] p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

By using $\dfrac{[k]_{p,q}}{[n-k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}$, we have

$$L_n^{p,q}(f;x) - f\left(\frac{px}{q}\right) = -\frac{x}{\ell_n^{p,q}(x)} \sum_{k=0}^{n} \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right] p^{\frac{(n-k)(n-k-1)}{2}+1} q^{\frac{k(k-1)}{2}-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

$$+ \frac{1}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right] p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)} q^{\frac{k(k-1)}{2}-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} x^k$$

$$= -\frac{x}{\ell_n^{p,q}(x)} \sum_{k=0}^{n} \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right] p^{\frac{(n-k)(n-k-1)}{2}+1} q^{\frac{k(k-1)}{2}-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

$$+ \frac{x}{\ell_n^{p,q}(x)} \sum_{k=0}^{n-1} \left[\frac{px}{q}; \frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}; f \right] p^{\frac{(n-k-1)(n-k-2)}{2}-(k-n)} q^{\frac{k(k+1)}{2}-(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k$$

$$= -\frac{x^{n+1}}{\ell_n^{p,q}(x)} \left[\frac{px}{q}; \frac{p[n]_{p,q}}{q^n}; f \right] p q^{\frac{n(n-1)}{2}-1}$$

$$+ \frac{x}{\ell_n^{p,q}(x)} \sum_{k=0}^{n-1} \left\{ \left[\frac{px}{q}; \frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}; f \right] - \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right] \right\} p^{\frac{(n-k)(n-k-1)}{2}+1} q^{\frac{k(k-1)}{2}-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

Now by using the result

$$\left[\frac{px}{q}; \frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}; f \right] - \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right]$$

$$= \left(\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}} - \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k} \right) f \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; \frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}; f \right]$$

and

$$\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}} - \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k} = [n+1]_{p,q},$$

we have

$$L_n^{p,q}(f;x) - f\left(\frac{px}{q}\right) = -\frac{x^{n+1}}{\ell_n^{p,q}(x)} \left[\frac{px}{q}; \frac{p[n]_{p,q}}{q^n}; f \right] p q^{\frac{n(n-1)}{2}-1}$$

$$+ \frac{x}{\ell_n^{p,q}(x)} \sum_{k=0}^{n-1} \left\{ \left[\frac{px}{q}; \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}; f \right] \frac{p^{n-k}[n+1]_{p,q}}{[n-k]_{p,q}[n-k+1]_{p,q}q^{k+1}} \right\} p^{\frac{(n-k)(n-k-1)}{2}+1} q^{\frac{k(k-1)}{2}-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

This completes the proof of Theorem 3.5. □

4. Generalization of $L_n^{p,q}$

In this section we present some generalization of the operators $L_n^{p,q}$ based on (p,q)-integers similar to work done in [1, 3].

We consider a sequence of linear positive operators based on (p,q)-integers as follows:

$$L_n^{(p,q),\gamma}(f;x) = \frac{1}{\ell_n^{p,q}(x)} \sum_{k=0}^{n} f\left(\frac{p^{n-k+1}[k]_{p,q} + \gamma}{b_{n,k}}\right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k, \qquad (\gamma \in \mathbb{R}), \tag{11}$$

where $b_{n,k}$ satisfies the following conditions:

$$p^{n-k+1}[k]_{p,q} + b_{n,k} = c_n \quad \text{and} \quad \frac{[n]_{p,q}}{c_n} \to 1, \quad \text{for} \quad n \to \infty.$$

It is easy to check that if $b_{n,k} = q^k[n-k+1]_{p,q} + \beta$ for any n,k and $0 < q < p \le 1$, then $c_n = [n+1]_{p,q} + \beta$. If we choose $p = 1$, then operators reduce to generalization of q-BBH opeartors defined in [1], and which turn out to be D.D. Stancu-type generalization of Bleimann, Butzer, and Hahn operators based on q-integers [14]. If we choose $\gamma = 0$, $q = 1$ as in [1] for $p = 1$, then the operators become the special case of the Balázs-type generalization of the q-BBH operators [1] given in [3].

Theorem 4.1. *Let $p = p_n$, $q = q_n$ satisfying (5), for $0 < q_n < p_n \le 1$ and if $L_n^{(p_n,q_n),\gamma}$ is defined by (11), then for any function $f \in \widetilde{W}_{\alpha,[0,\infty)}$, we have*

$$\lim_n \| L_n^{(p_n,q_n),\gamma}(f;x) - f(x) \|_{C_B} \le 3M$$

$$\times \max\left\{ \left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha}, \left|1 - \frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha}\left(\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^{\alpha}, 1 - 2\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + \frac{q_n[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2}\right\}.$$

Proof. Using (3) and (11), we have

$$| L_n^{(p,q),\gamma}(f;x) - f(x) |$$

$$\le \frac{1}{\ell_n^{p_n,q_n}(x)} \sum_{k=0}^{n} \left| f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{b_{n,k}}\right) - f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_{n,k}}\right) \right| p_n^{\frac{(n-k)(n-k-1)}{2}} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_n,q_n} x^k$$

$$+ \frac{1}{\ell_n^{p_n,q_n}(x)} \sum_{k=0}^{n} \left| f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_{n,k}}\right) - f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n-k+1]_{p_n,q_n}q_n^k}\right) \right| p_n^{\frac{(n-k)(n-k-1)}{2}} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_n,q_n} x^k$$

$$+ | L_n^{p_n,q_n}(f;x) - f(x) | .$$

Since $f \in \widetilde{W}_{\alpha,[0,\infty)}$ and by using Corollary 3.4, we can write

$$| L_n^{(p,q),\gamma}(f;x) - f(x) |$$

$$\le \frac{M}{\ell_n^{p_n,q_n}(x)} \sum_{k=0}^{n} \left| \frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma+b_{n,k}} - \frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+p_n^{n-k+1}[k]_{p_n,q_n}+b_{n,k}} \right|^{\alpha} p_n^{\frac{(n-k)(n-k-1)}{2}} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_n,q_n} x^k$$

$$+ \frac{M}{\ell_n^{p_n,q_n}(x)} \sum_{k=0}^{n} \left| \frac{p_n^{n-k+1}[k]_{p_n,q_n}}{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma+b_{n,k}} - \frac{p_n^{n-k+1}[k]_{p_n,q_n}}{p_n^{n-k+1}[k]_{p_n,q_n}+[n-k+1]_{p_n,q_n}q_n^k} \right|$$

$$\times p_n^{\frac{(n-k)(n-k-1)}{2}} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_n,q_n} x^k + M\delta_n^{\frac{\alpha}{2}}(x).$$

This implies that

$$| L_n^{(p,q),\gamma}(f;x) - f(x) | \le M\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha}$$

$$+ \frac{M}{\ell_n^{p_n,q_n}(x)}\left|1 - \frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha} \sum_{k=0}^{n}\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^{\alpha} p_n^{\frac{(n-k)(n-k-1)}{2}} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_n,q_n} x^k + M\delta_n^{\frac{\alpha}{2}}(x)$$

$$= M\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha} + M\left|1 - \frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha} L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^{\alpha};x\right) + M\delta_n^{\frac{\alpha}{2}}(x).$$

Using the Hölder inequality for $p = \frac{1}{\alpha}$, $\quad q = \frac{1}{1-\alpha}$, we get

$$| L_n^{(p,q),\gamma}(f;x) - f(x) |$$

$$\leq M \left(\frac{[n]_{p_n,q_n}}{c_n+\gamma} \right)^{\alpha} \left(\frac{\gamma}{[n]_{p_n,q_n}} \right)^{\alpha} + M \left| 1 - \frac{[n+1]_{p_n,q_n}}{c_n+\gamma} \right|^{\alpha} L_n^{p_n,q_n} \left(\frac{t}{1+t};x \right)^{\alpha} \left(L_n^{p_n,q_n}(1;x) \right)^{1-\alpha} + M\delta_n^{\frac{\alpha}{2}}(x)$$

$$\leq M \left(\frac{[n]_{p_n,q_n}}{c_n+\gamma} \right)^{\alpha} \left(\frac{\gamma}{[n]_{p_n,q_n}} \right)^{\alpha} + M \left| 1 - \frac{[n+1]_{p_n,q_n}}{c_n+\gamma} \right|^{\alpha} \left(\frac{p_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} \frac{x}{1+x} \right)^{\alpha} + M\delta_n^{\frac{\alpha}{2}}(x).$$

This completes the proof of Theorem 4.1. $\quad\square$

References

[1] A. Aral, O. Doğru, Bleimann Butzer and Hahn operators based on *q*-integers, J. Inequal. Appl. (2007) 1–12, Art. ID 79410.
[2] G. Bleimann, P.L. Butzer, L. Hahn, A Bernstein-type operator approximating continuous functions on the semi-axis, Indag. Math. 42 (1980) 255–262.
[3] O. Doğru, On Bleimann, Butzer and Hahn type generalization of Balázs operators, Studia Univ. Babeş-Bolyai. Math. 47:4 (2002) 37–45.
[4] A.D. Gadjiev, Ö. Çakar, On uniform approximation by Bleimann, Butzer and Hahn operators on all positive semi-axis, Trans. Acad. Sci. Azerb.Ser. Phys. Tech. Math. Sci. 19 (1999) 21–26.
[5] M.N. Hounkonnou, J. Désiré, B. Kyemba, $\mathcal{R}(p,q)$-calculus: differentiation and integration, SUT Jour. Math. 49:2 (2013) 145–167.
[6] B. Lenze, Bernstein-Baskakov-Kantorovich operators and Lipschitz-type maximal functions, In: Colloq. Math. Soc. Janos Bolyai 58, Approx. Theory (1990) 469–496.
[7] A. Lupaş, A *q*-analogue of the Bernstein operator, University of Cluj- Napoca, Sem. Numer. Stat. Calc. 9 (1987) 85–92.
[8] M. Mursaleen, K.J. Ansari, A. Khan, Some approximation results by (p,q)-analogue of Bernstein-Stancu operators, Appl. Math. Comput. 264 (2015) 392–402.
[9] M. Mursaleen, K.J. Ansari, A. Khan, On (p,q)-analogue of Bernstein operators, Appl. Math. Comput. 266 (2015) 874–882.
[10] M. Mursaleen, Md Nasiruzzaman, Ashirbayev Nurgali, Some approximation results on Bernstein-Schurer operators defined by (p,q)-integers, Journal of Inequalities and Applications 2015:249, DOI10.1186/s13660-015-0767-4.
[11] G.M. Phillips, Bernstein polynomials based on the *q*-integers, The heritage of P.L.Chebyshev: Ann. Numer. Math. 4 (1997) 511–518.
[12] P.N. Sadjang, On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas, arXiv:1309.3934 [math.QA].
[13] V. Sahai, S. Yadav, Representations of two parameter quantum algebras and *p,q*-special functions, J. Math. Anal. Appl. 335 (2007) 268–279.
[14] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roumaine Math. Pures Appl. 13 (1968) 1173–1194.
[15] K. Victor, C. Pokman, Quantum Calculus, Springer-Verlag, New York-Berlin-Heidelberg, 2002.