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Characterizations of H-J Matrices

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Abstract. In this paper we provide two characterizations of H-J generalized Hausdorff matrices. The first is a recursion relationship among each nonzero triangular array consisting of three terms, and the second is a generalization of the classical Hurwitz-Silverman theorem to H-J Hausdorff matrices.

1. Introduction

A Hausdorff matrix is a lower triangular matrix with nonzero entries of the form

$$h_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k,$$

where $\{\mu_n\}$ is a real or complex sequence and Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\Delta_{n+1}\mu_k = \Delta(\Delta^n)\mu_k$ for $n > 1$.

The Cesáro matrix of order one, $(C, 1)$, is the Hausdorff matrix generated by the sequence

$$\mu_n = \frac{1}{n+1}.$$

Hurwitz and Silverman [3] showed that a lower triangular matrix commutes with $(C, 1)$ if and only if it is a Hausdorff matrix although they did not use that term. They also showed that each such matrix has the decomposition

$$H = \delta\mu\delta,$$

where

$$\delta_{nk} = (-1)^k\binom{n}{k},$$

Δ is its own inverse, and μ is the diagonal matrix with diagonal entries μ_k.

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In 1921 Hausdorff [4] established a number of properties of these matrices, which now bear his name, including necessary and sufficient conditions for regularity. A matrix is called regular if and only if it maps every convergent sequence into a convergent sequence with the same limit.

There are at least two well-known generalizations of Hausdorff matrices. The first of these was defined by Hausdorff [5]. An H-J matrix is defined as follows. Let $\{\lambda_n\}$ be a positive sequence satisfying

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \tag{1}$$

with $\lim_n \lambda_n = \infty$ and

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

The nonzero entries of an H-J matrix are defined by

$$h_{n,k}(\lambda;\mu) = \lambda_{k+1} \ldots \lambda_n [\mu_k, \mu_{k+1}, \ldots, \mu_n], \tag{2}$$

where $[\cdot]$ is the symmetric difference operator defined by

$$[\mu_k, \mu_{k+1}] = \frac{\mu_k - \mu_{k+1}}{\lambda_{k+1} - \lambda_k},$$

and, for $n > 1$,

$$[\mu_k, \ldots, \mu_{n+1}] = \frac{[\mu_k, \ldots, \mu_n] - [\mu_{k+1}, \ldots, \mu_{n+1}]}{\lambda_{n+1} - \lambda_k}.$$

Hausdorff considered those methods for which $\lambda_0 = 0$, and Jakimovski [6] investigated such matrices for $\lambda_0 > 0$.

The other generalization we shall consider is the class of E-J matrices, which were defined independently by Jakimovski [6] and Endl [2].

The nonzero entries of an E-J matrix are

$$h_{n,k}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k.$$

Thus, the H-J matrices reduce to the E-J matrices by setting $\lambda_n = n + \alpha$, and the choice $\lambda_n = n$ yields the ordinary Hausdorff matrices.

The purpose of this paper is to provide two characterizations of H-J matrices. The first is a new characterization that has not been considered before, even for ordinary Hausdorff matrices, and the second is a slight generalization of the characterization established by Hurwitz and Silverman for ordinary Hausdorff matrices.

For simplicity of notation we shall also denote the entries of an H-J matrix by h_{nk}.

2. Results

Theorem 2.1. *A lower triangular matrix A is an H-J matrix if and only if there exists a sequence λ_n satisfying* (1) *such that*

$$a_{n+1,k} = \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_k} a_{nk} - \frac{\lambda_{k+1}}{\lambda_{n+1} - \lambda_k} a_{n+1,k+1} \tag{3}$$

for each $0 \le k \le n$.

Proof. Suppose that A is an H-J matrix. Then, from the definition of an H-J matrix,

$$\begin{aligned} h_{n+1,k} &= \lambda_{k+1}\cdots\lambda_{n+1}[\mu_k,\dots,\mu_{n+1}] \\ &= \frac{\lambda_{k+1}\cdots\lambda_{n+1}}{\lambda_{n+1}-\lambda_k}([\mu_k,\dots,\mu_n]-[\mu_{k+1},\dots,\mu_{n+1}]) \\ &= \frac{\lambda_{n+1}}{\lambda_{n+1}-\lambda_k}h_{nk}-\frac{\lambda_{k+1}}{\lambda_{n+1}-\lambda_k}h_{n+1,k+1}, \end{aligned}$$

and (3) is satisfied.

With $n = k$, from (3), one has

$$a_{k+1,k} = \frac{\lambda_{k+1}}{\lambda_{k+1}-\lambda_k}a_{kk}-\frac{\lambda_{k+1}}{\lambda_{k+1}-\lambda_k}a_{k+1,k+1}. \tag{4}$$

Since the diagonal entries of A only depend on k we can write $a_{kk} = \mu_k$, for some sequence μ_k. Therefore (4) becomes

$$a_{k+1.k} = \frac{\lambda_{k+1}}{\lambda_{k+1}-\lambda_k}(\mu_k-\mu_{k+1}) = \lambda_{k+1}[\mu_k,\mu_{k+1}],$$

and the first diagonal below the main diagonal has the entries of an H-J matrix.

Assume the induction hypotheses; i.e., assume that, for any j satisfying $0 \le j \le k$ that (3) is satisfied with $n = k + j$. Then, with $n = k + j + 1$, (3) becomes

$$\begin{aligned} a_{k+j+2,k} &= \frac{\lambda_{k+j+2}}{\lambda_{k+j+2}-\lambda_k}a_{k+j+1,k}-\frac{\lambda_{k+1}}{\lambda_{k+j+2}-\lambda_k}a_{k+j+2,k+1} \\ &= \frac{1}{\lambda_{k+j+2}-\lambda_k}\Big(\lambda_{k+j+2}\lambda_{k+1}\cdots\lambda_{k+j+1}[\mu_k,\dots,\mu_{k+j+1}] \\ &- \lambda_{k+1}\lambda_{k+2}\cdots\lambda_{k+j+2}[\mu_{k+1},\dots,\mu_{k+j+2}]\Big) \\ &= \lambda_{k+1},\dots,\lambda_{k+j+2}[\mu_k,\dots,\mu_{k+j+2}], \end{aligned}$$

and the $(n + k + 1)$st diagonal elements are the corresponding terms of an H-J matrix. □

To prove the analogue of the Hurwitz-Silverman theorem, which states that a lower triangular matrix A commutes with $(C, 1)$ if and only if A is a Hausdorff matrix, it will first be necessary to compute the entries of the H-J analogue of $(C, 1)$. For a regular H-J matrix it is known that

$$\mu_n = \int_0^1 x^{\lambda_n} d\chi(x),$$

where χ is a function of bounded variation over $[0, 1]$. The function χ is normally called the mass function for the moment generating sequence $\{\mu_n\}$. Since the mass function for $(C, 1)$ is $\chi(t) = t$, we shall use the same mass function to compute the entries of the H-J analogue.

$$\mu_n = \int_0^1 x^{\lambda_n} dx = \frac{1}{\lambda_n+1}.$$

A routine computation verifies that

$$c_{nk} = \frac{\lambda_{k+1}\cdots\lambda_n}{\prod_{i=k}^{n}(\lambda_i+1)} = \frac{\lambda_1\cdots\lambda_n}{\prod_{i=0}^{n}(\lambda_i+1)}\cdot\frac{\prod_{i=0}^{k-1}(\lambda_i+1)}{\lambda_1\cdots\lambda_k}. \tag{5}$$

We shall also use the notation C to denote the H-J analogue of $(C, 1)$.

A lower triangular matrix A is called a triangle if $a_{nn} \neq 0$ for all $n \geq 0$. A matrix A is called factorable if it is lower triangular with entries $a_{nk} = c_n b_k, 0 \leq k \leq n$, where b_k depends only on k and c_n depends only on n. A simple example of a factorable matrix is C, the Hausdorff Cesáro matrix of order one. Then C is a factorable matrix with each $b_k = 1$ and each $c_n = 1/(n+1)$

The following lemma allows us to omit the assumption that a matrix A which commutes with C must be lower triangular.

Lemma 2.2. *Let B be a factorable triangle with distinct diagonal entries. Then, if A is any row finite infinite matrix that commutes with B, A must be lower triangular.*

Proof. A matrix A commute with B if and only if A commutes with B^{-1}. From Lemma 2.1 of [1], a triangle is factorable if and only if its inverse is bidiagonal. Therefore B^{-1} is bidiagonal. Moreover, if $b_{nk} = c_n d_k$, then $b^{-1}_{nn} = 1/(c_n d_n)$ and $b^{-1}_{n,n-1} = 1/(c_{n-1} b_n)$, which are defined and nonzero for each n, since B is a triangle.

The proof is by repeated induction.

$$(AB^{-1})_{00} = \sum_j a_{oj} b^{-1}_{j0} = a_{00} b^{-1}_{00} + a_{01} b^{-1}_{10},$$

and

$$(B^{-1}A)_{00} = \sum_j b^{-1}_{0j} a_{j0} = b^{-1}_{00} a_{00},$$

which implies that $a_{01} = 0$. Assume that $a_{0k} = 0$ for $0 < k \leq n$. Then

$$(AB^{-1})_{0n} = \sum_j a_{0j} b^{-1}_{jn} = a_{0n} b^{-1}_{nn} + a_{0,n+1} b^{-1}_{n+1,n},$$

and

$$(B^{-1}A)_{0n} = \sum_j b^{-1}_{0j} a_{jn} = b^{-1}_{00} a_{0n},$$

which implies that $a_{0,n+1} = 0$.

Now assume that $a_{km} = 0$ for $0 < k < n$ and $m > k$. Then

$$(AB^{-1})_{nn} = \sum_j a_{nj} b^{-1}_{jn} = a_{nn} b^{-1}_{nn} + a_{n,n+1} b^{-1}_{n+1,n},$$

and

$$(B^{-1}A)_{nn} = \sum_j b^{-1}_{nj} a_{jn} = \sum_j b^{-1}_{n,n-1} a_{n-1,n} + b^{-1}_{nn} a_{nn},$$

and it follows that $a_{n,n+1} = 0$.

Assume now that $a_{nm} = 0$ for $n < m \leq n+k$. Then

$$(AB^{-1})_{n,n+k} = \sum_j a_{nj} b^{-1}_{j,n+k} = a_{n,n+k} b^{-1}_{n+k,n+k} + a_{n,n+k+1} b^{-1}_{n+k+1,n+k},$$

and

$$(B^{-1}A)_{n,n+k} = \sum_j b^{-1}_{nj} a_{j,n+k} = b^{-1}_{n,n-1} a_{n-1,n+k} + b^{-1}_{nn} a_{n,n+k},$$

which implies that $a_{n,n+k+1} = 0$. □

Theorem 2.3. *A row finite infinite matrix A commutes with C if and only if A is an H-J matrix.*

Proof. From (5) it is clear that C is a factorable triangle. Therefore by Lemma 1, A must be a lower triangular matrix. Since the principal diagonal entries of A depend only on k, one can regard $a_{kk} = \mu_k$, for some sequence $\{\mu_k\}$. Using (5),

$$(AC)_{k+1,k} = \sum_j a_{k+1,j}c_{jk} = a_{k+1,k}c_{kk} + a_{k+1,k+1}c_{k+1,k}$$
$$= \frac{a_{k+1,k}}{\lambda_k + 1} + \frac{\lambda_{k+1}\mu_{k+1}}{(\lambda_k + 1)(\lambda_{k+1} + 1)},$$

and

$$(CA)_{k+1,k} = \sum_j c_{k+1,j}a_{j,k} = c_{k+1,k}a_{k,k} + c_{k+1,k+1}a_{k+1,k}$$
$$= \frac{\mu_k}{(\lambda_k + 1)(\lambda_{k+1} + 1)} + \frac{a_{k+1,k}}{(\lambda_{k+1} + 1)}.$$

Equating $(AC)_{k+1,k}$ and $(CA)_{k+1,k}$ gives

$$\frac{a_{k+1,k}}{\lambda_k + 1} + \frac{\lambda_{k+1}\mu_{k+1}}{(\lambda_k + 1)(\lambda_{k+1} + 1)} = \frac{\lambda_{k+1}\mu_k}{(\lambda_k + 1)(\lambda_{k+1} + 1)} + \frac{a_{k+1,k}}{(\lambda_{k+1} + 1)},$$

or

$$\Big(\frac{1}{\lambda_k + 1} - \frac{1}{\lambda_{k+1} + 1}\Big)a_{k+1,k} = \frac{\lambda_{k+1}\mu_k}{(\lambda_k + 1)(\lambda_{k+1} + 1)} - \frac{\lambda_{k+1}\mu_{k+1}}{(\lambda_k + 1)(\lambda_{k+1} + 1)}.$$

Hence

$$(\lambda_{k+1} - \lambda_k)a_{k+1,k} = \lambda_{k+1}(\mu_k - \mu_{k+1}),$$

or $a_{k+1,k} = \lambda_{k+1}[\mu_k, \mu_{k+1}]$.

Now assume the induction hypothesis. Then Theorem 1 applies and A is an H-J matrix. □

Remark 2.4. Setting $\lambda_n = n$ in Theorem 2 gives a generalization of the Hurwitz-Silverman theorem, and setting $\lambda_n = n + \alpha$ yields the corresponding result for the E-J matrices.

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