Filomat 30:3 (2016), 699–709
DOI 10.2298/FIL1603699K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Collocation Finite Element Solution for Stefan Problems with Periodic Boundary Conditions

Hatice Karabenli[a], Yusuf Uçar[a], E. Nesligül Aksan[a]

[a]*Department of Mathematics, Faculty of Arts and Science, Inonu University, Turkey*

Abstract. In this study, we are going to obtain some numerical solutions of Stefan problems given together with time-dependent periodic boundary conditions. After using variable space grid method, we have presented a numerical finite element scheme based on collocation finite element method formed with cubic B-splines. The newly obtained numerical results are presented for temperature distribution, the position and the velocity of moving boundary. It is shown that the size of domain, oscillation amplitude and oscillation frequency which are situated at the boundary condition, strongly influence the temperature distribution and position of moving boundary. The numerical results are compared with other numerical solutions obtained by using finite difference method and they are found to be in good agreement with each other.

1. Introduction

Many problems in various areas of applied science can be modelled as partial differential equations posed in domains whose boundaries are determined as part of the problem. Such problems are usually referred to as moving boundary problems. We encounter these problems, known as Stefan problems, in various areas of industrial process, such as metal processing, melting of ice, solidification of moldings, evaporation of droplets, oxygen diffusion problems, several branches of metallurgical technology, etc. In these mentioned areas, the material has phase change with a moving boundary that has to be determined as part of the solution. For this reason the Stefan problems are non-linear problems, and thus have the limited analytical solutions. Due to the difficulty in obtaining analytical solutions, numerical methods have been used more commonly [1]-[5].

There are two main approaches to obtain the solution of the Stefan problems. The first one is the front-tracking method, in which the position of the phase boundary is continuously tracked. Variable space grid method [6], variable time step method [7] and the heat balance integral method improved by Goodman [8] are alternatives to track the moving boundary. The other approach is to use a fixed domain formulation. For example boundary immobilization method fixes the moving boundary by a suitable choice of new space coordinates, which has a lot of applications used by finite difference schemes. Furthermore, the isotherm migration method in which the dependent variable interchanges with space variable [4] and Entalpy method [10] in which an entalpy function is introduced, are examples of front-fixed methods.

2010 *Mathematics Subject Classification*. Primary 65D07; Secondary 65L60, 41A15, 35R37
Keywords. Stefan problems, Variable space grid methods; collocation finite element methods
Received: 06 August 2015; Revised: 13 December 2015; Accepted: 15 December 2015
Communicated by Ljubiša D.R. Kočinac and Ekrem Savaş
Email addresses: haticekarabenli@gmail.com (Hatice Karabenli), yusuf.ucar@inonu.edu.tr (Yusuf Uçar), nesligul.aksan.@inonu.edu.tr (E. Nesligül Aksan)

In this paper, we are going to deal with the one-dimensional Stefan problem with a time dependent periodic condition in which oscillation amplitude is situated at the fixed boundary. After applying variable space grid method, we obtain some numerical results by using collocation finite element schemes. And these results are compared with some numerical solutions obtained by using finite difference methods [5] in Section 6.

2. Governing Equation

In one-dimensional mathematical model of Stefan problem, the function $U(x,t)$ is temperature distribution and governed by the heat equation :

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{1}$$

subject to the boundary conditions

$$U(0,t) = 1 + \epsilon sin\omega t, \quad t > 0 \tag{2}$$

$$U(s(t),t) = 0, \quad t > 0, \tag{3}$$

where ϵ is the surface temperature oscillation amplitude and ω is the oscillation frequency. And the heat balance equation is known as Stefan condition

$$\frac{ds(t)}{dt} = -Ste\frac{\partial U}{\partial x}, \quad x = s(t), \quad t > 0, \tag{4}$$

where $s(t)$ is the position of moving boundary and Ste is the Stefan number given by $c\triangle T/L$. c is the specific heat capacity of liquid, $\triangle T$ is a reference temperature and L is the latent heat in equation $c\triangle T/L$ which is essential for melting of ice [13]. The initial condition for $s(t)$ is given by

$$s(0) = 0. \tag{5}$$

So, we have three important physical parameters (Ste, ϵ, ω) for this model problem. Savović and Caldwell [5] used finite difference method Rizwan-uddin used nodal integral method [11] to solve the Stefan problems defined by (1)-(5) equations. We compare their results and present results to see accuracy of finite element solution for parameters (Ste, ϵ, ω).

3. Variable Space Grid Method

Murray and Landis [6] kept constant the number of space intervals between $x = 0$ and $x = s(t)$, equal to N, for all time. As a result of this, the moving boundary is always on the N^{th} grid line. In that case, the grid size must be $x = s(t)/N$ which is changed with the time.

For the line x_i, partial differentiation with respect to time t,

$$\left.\frac{\partial U}{\partial t}\right|_i = \left.\frac{\partial U}{\partial x}\right|_t \left.\frac{dx}{dt}\right|_i + \left.\frac{\partial U}{\partial t}\right|_x \tag{6}$$

and at the i^{th} grid point variation of $\frac{dx}{dt}$

$$\frac{dx_i}{dt} = \frac{x_i}{s(t)}\frac{ds}{dt} \tag{7}$$

given by above equations. By substituting Eqn. (7) into (6) the one dimensional heat equation becomes

$$\left.\frac{\partial U}{\partial t}\right|_i = \frac{x_i}{s(t)}\frac{ds}{dt}\left.\frac{\partial U}{\partial x}\right|_t + \left.\frac{\partial U}{\partial t}\right|_x$$

So, dimensionless Stefan problem turns into following equation

$$\frac{\partial U}{\partial t} = \frac{x_i}{s(t)}\frac{ds}{dt}\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < s(t) \tag{8}$$

subject to (2)-(3) boundary conditions. $s(t)$ is updated at each time step by using a suitable finite difference form of the Stefan condition. For this aim, we are going to use the following three point backward difference at the moving boundary

$$\left.\frac{\partial U}{\partial x}\right|_{x=s(t)} = \frac{3U_N - 4U_{N-1} + U_{N-2}}{2\triangle x} + O(\triangle x^2). \tag{9}$$

Now, we can employ the collocation finite element method based on cubic B-splines to the equation given by (8).

4. Collocation Finite Element Method

Let us divide the interval $[a, b]$ into N uniform element consisting of the knots x_m such as $a = x_0 < x_1 < x_2 < ... < x_N = b$. Cubic B-splines ϕ_m, span $[a, b]$, are $\phi_{-1}, \phi_0, \phi_1, ..., \phi_{N+1}$. So, approximate solution for $U(x, t)$ can be written as

$$U_N(x, t) = \sum_{m=-1}^{N+1} \delta_m(t)\phi_m(x), \tag{10}$$

where ϕ_m are trial functions given by the following expressions and δ_m are time-dependent variables which are going to be determined by boundary and collocations conditions for Stefan problems. The cubic B-spline is defined by the relationship

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & [x_{m-2}, x_{m-1}] \\ h + 3h(x - x_{m-1}) + 3h(x - x_{m-1}) - 3(x - x_{m-1}), & [x_{m-1}, x_m] \\ h + 3h(x_{m+1} - x) + 3h(x_{m+1} - x) - 3(x_{m+1} - x), & [x_m, x_{m+1}] \\ (x_{m+2} - x), & [x_{m+1}, x_{m+2}] \\ 0, & otherwise, \end{cases}$$

where $\Delta x = h = x_m - x_{m-1}$ for all $m, m = -1, 0, 1, ..., N + 1$. The cubic spline ϕ_m and its principle derivatives ϕ'_m and ϕ''_m disappear outside the interval $[x_{m-2}, x_{m+2}]$. So, we can tabulate the ϕ_m, ϕ'_m and ϕ''_m values at the knots.

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
ϕ_m	0	1	4	1	0
ϕ'_m	0	$-\frac{3}{h}$	0	$\frac{3}{h}$	0
ϕ''_m	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

Table 1. Spline values at the knots

Now, we see that approximation of nodal values U_m, U'_m and U''_m at the knot x_m can be written as

$$\begin{aligned} U = U(x_m) &= \delta_{m-1} + 4\delta_m + \delta_{m+1} \\ U' = U'(x_m) &= \frac{3}{h}(\delta_{m+1} - \delta_{m-1}) \\ U'' = U''(x_m) &= \frac{6}{h^2}(\delta_{m+1} - 2\delta_m + \delta_{m-1}) \text{ [14].} \end{aligned}$$

By using space derivatives given in the above equations and applying Crank-Nicholson approach and forward difference approximation for two time levels n and $n+1$, such as

$$\delta_m = \frac{\delta_m^{n+1}+\delta_m^n}{2}$$
$$\frac{d\delta_m}{dt} = \frac{\delta_m^{n+1}-\delta_m^n}{\triangle t}$$

where δ_m^n are the parameters at the time $n\Delta t$. We obtain the finite element schemes as follows

$$\alpha_{m1}\delta_{m-1}^{n+1}+\alpha_{m2}\delta_m^{n+1}+\alpha_{m3}\delta_{m+1}^{n+1} = \alpha_{m4}\delta_{m-1}^{n}+\alpha_{m5}\delta_m^{n}+\alpha_{m6}\delta_{m+1}^{n} \quad (11)$$

$m = 0, 1, ..., N$ where

$$\alpha_{m1} = 1+\frac{3kx_m^n(\dot{s})^n}{2h^ns^n}-\frac{3k}{(h^n)^2}$$
$$\alpha_{m2} = 4+\frac{6k}{(h^n)^2}$$
$$\alpha_{m3} = 1-\frac{3kx_m^n(\dot{s})^n}{2h^ns^n}-\frac{3k}{(h^n)^2}$$
$$\alpha_{m4} = 1-\frac{3kx_m^n(\dot{s})^n}{2h^ns^n}+\frac{3k}{(h^n)^2}$$
$$\alpha_{m5} = 4-\frac{6k}{(h^n)^2}$$
$$\alpha_{m6} = 1+\frac{3kx_m^n(\dot{s})^n}{2h^ns^n}+\frac{3k}{(h^n)^2},$$

where s^n is position of moving boundary, $(\dot{s})^n$ is velocity of moving boundary, $\Delta t \equiv k$ is time step and $\Delta x^n \equiv h^n$ is size of grid which will be updated each time step. The tridiagonal matrix system (11) consists of $N+1$ linear equations and $N+3$ unknown parameters $\delta_m = (\delta_{-1}, \delta_0, ..., \delta_N, \delta_{N+1})^T$. To solve this system uniquely we must have two more equations. We can obtain these equations from the boundary conditions and use them in the system in order to eliminate δ_{-1} and δ_{N+1}. So the system becomes a matrix form

$$A\delta_m^{n+1} = B\delta_m^n + r$$

where A, B are $(N+1)\times(N+1)$ tridiagonal matrixes and r is a $(N+1)$ column vector. To start time evaluation of the approximate solution, δ_m^0 must be determined, firstly. To attain vector δ_m^0 , we require two conditions for

$$U_N(x,0) = \sum_{m=-1}^{N+1} \delta_m^0(t)\phi_m(x).$$

i) Initial condition $U(x, 0)$ and $U_N(x, 0)$ should be equal to each other for $N+1$ points.

ii) To be able to solve the $A\delta_m^0 = b$, we need further equations which can be obtained from the first and second derivatives of approximate initial conditions. After we find initial vector δ_m^0, we get vectors $\delta_m^1, \delta_m^2, ..., \delta_m^n$, respectively.

It is clearly seen that equation (11) includes the fictitious parameters δ_{-1} and δ_{N+1} for $m = 0$ and $m = N$. By using boundary conditions $U(x = 0, t) = 1 + \varepsilon sin\omega t$ and $U(s(t), t) = 0$, we obtain the following equations

$$\delta_{-1} = 1 + \varepsilon sin\omega t - 4\delta_0 - \delta_1$$

$$\delta_{N+1} = -4\delta_N - \delta_{N-1}.$$

By eliminating parameters δ_{-1} and δ_{N+1} for $m = 0$ and $m = N$, the model problems' finite element schemes for temperature distribution on $0 < x < s(t)$ become

for $m = 0, \quad n = 0, 1, 2, ...$

$$(\alpha_{02} - 4\alpha_{01})\delta_0^{n+1} + (\alpha_{03} - \alpha_{01})\delta_1^{n+1} = (\alpha_{04} - \alpha_{01})(1 + \varepsilon sin\omega t) + (\alpha_{05} - 4\alpha_{04})\delta_0^n + (\alpha_{06} - \alpha_{04})\delta_1^n$$

for $m = 1, 2, 3, ..., N - 1, \quad n = 0, 1, 2, ...$

$$\alpha_{m1}\delta_{m-1}^{n+1} + \alpha_{m2}\delta_m^{n+1} + \alpha_{m3}\delta_{m+1}^{n+1} = \alpha_{m4}\delta_{m-1}^n + \alpha_{m5}\delta_m^n + \alpha_{m6}\delta_{m+1}^n$$

for $m = N, \quad n = 0, 1, 2, ...$

$$(\alpha_{N1} - \alpha_{N3})\delta_{N-1}^{n+1} + (\alpha_{N2} - 4\alpha_{N3})\delta_N^{n+1} = (\alpha_{N4} - \alpha_{N6})\delta_{N-1}^n + (\alpha_{N5} - 4\alpha_{N6})\delta_N^n$$

and the heat balance at x=s(t)

$$s^{n+1} = s^n - Ste\frac{k}{2h^n}(3U_N^n - 4U_{N-1}^n + U_{N-2}^n) \quad n = 0, 1, 2, ...$$

and initial condition for $s(t)$, $s^0 = 0$. Through the numerical calculations $U_m^n = U(x_m^n, t^n)$ and $t^n = t_{in} + nk$ are used where t_{in} is initial time. Furthermore, location of interface s^{n+1} is calculated at with the above equation. And $h^{n+1} = \frac{s^{n+1}}{N}$ is used to update size of grids, for each time.

5. Stability Analysis

We implement the Von-Neumann theory in which the growth factor of typical Fourier mode defined as

$$\delta_m^n(t) = \xi^n e^{imkh},$$

where k is the mode number and h is element size. By substituting equation $\delta_m^n(t) = \xi^n e^{imkh}$ into equation given with (11), and by performing some simplification operations we obtain

$$\xi = \frac{a_1 - ib}{a_2 + ib},$$

where

$$a_1 = 6 - 4\sin^2\frac{kh}{2} - \frac{12k}{h^2}\sin^2\frac{kh}{2}$$

$$a_2 = 6 - 4\sin^2\frac{kh}{2} + \frac{12k}{h^2}\sin^2\frac{kh}{2}$$

$$b = \frac{3k}{h}\frac{x_m^n(\dot{s})^n}{s^n}.$$

The growth factor must satisfy $|\xi| \leq 1$, so $\left|\frac{a_1 - ib}{a_2 + ib}\right| \leq 1$ if only if $a_1^2 \leq a_2^2$. If we do essential operations we will see that the system (11) is unconditionally stable.

6. Numerical Results and Discussion

For $\epsilon \neq 0$, the exact solution of Stefan problem is not known. But, in case of $\epsilon = 0$, we have exact solution defined by the following expression

$$U(x,t) = 1 - \frac{erf(x/2\sqrt{t})}{erf(\lambda)}, \quad 0 \leq x \leq s(t), \quad t > 0 \tag{12}$$

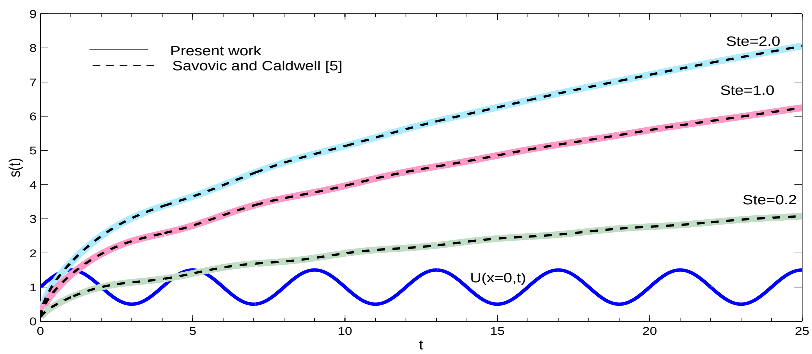
$$s(t) = 2\lambda\sqrt{t} \tag{13}$$

where λ is known melting/solidification number and root of non-algebraic equation

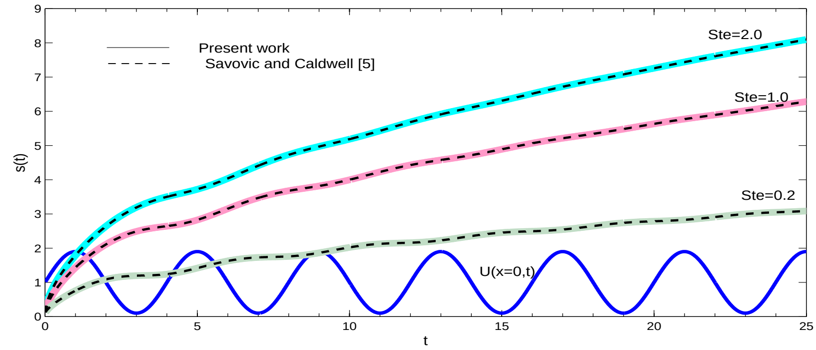
$$\sqrt{\pi}\lambda e^{\lambda^2} erf(\lambda) = Ste. \tag{14}$$

We will poignantly use this Ste number in our calculations obtained in the above equation. When finite element methods are applied, because of singularity at $t = 0$, all numerical calculations began at $t_{in} = 0.01$. Moreover, due to the lack of exact solution for $\epsilon \neq 0$ to start finite element procedure, we use equations (12-13) which are analytical solutions of temperature distribution and the position of moving boundary for $\epsilon = 0$. This opinion is early suggested by Rizwan-uddin [11, 12] for nodal integral method and used by Savović-Caldwell for finite difference method [5]. The same initialization procedure has been used in our study for the Stefan problem with time-dependent periodic boundary condition.

In this study, we compare numerical results found with collocation finite element method and finite difference method studied by Savović and Caldwell [5]. We obtain numerical results for two oscillation amplitudes $\varepsilon = 0.5$ and $\varepsilon = 0.9$, three Stefan numbers $Ste = 2.0$, $Ste = 1.0$ and $Ste = 0.2$. The initial time $t_{in} = 0.01$, the time step $\Delta t(\equiv k) = 0.00002$ and the number of element for defined domain $N = 10$ are chosen for all numerical calculations. For the Stefan numbers $Ste = 2.0$, $Ste = 1.0$ and $Ste = 0.2$, λ values can be obtained by using equation (14) as $\lambda = 0.30642$, $\lambda = 0.62006$ and $\lambda = 0.80060$, respectively [5].

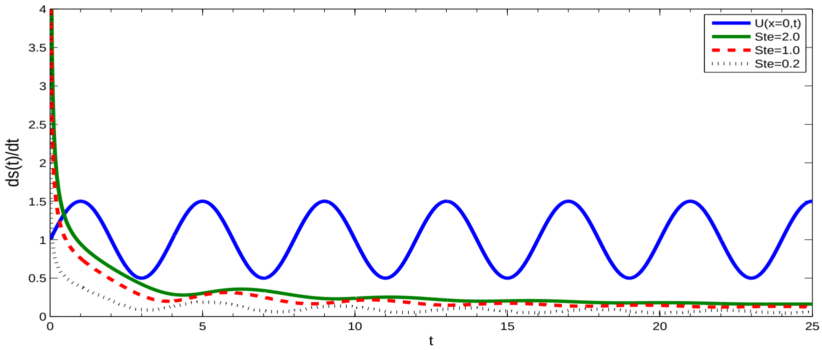


(a)

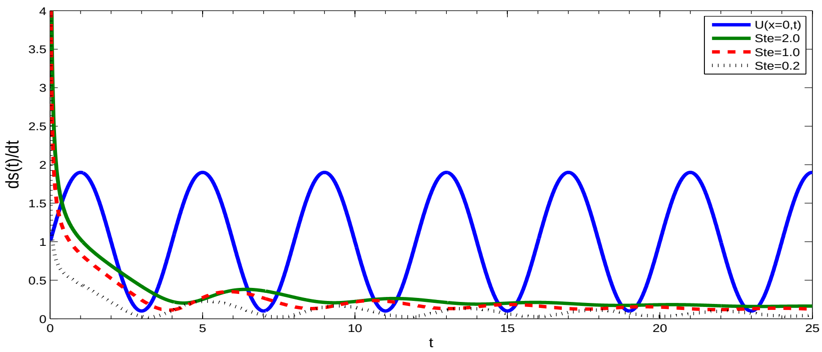


(b)

Figure 1. Position of the moving boundary for different Stefan numbers, for oscillation amplitudes (a) $\epsilon = 0.5$ (b) $\epsilon = 0.9$ and oscillation frequency $\omega = \pi/2$. Also the temperature oscillation at the $x = 0$ surface are plotted.



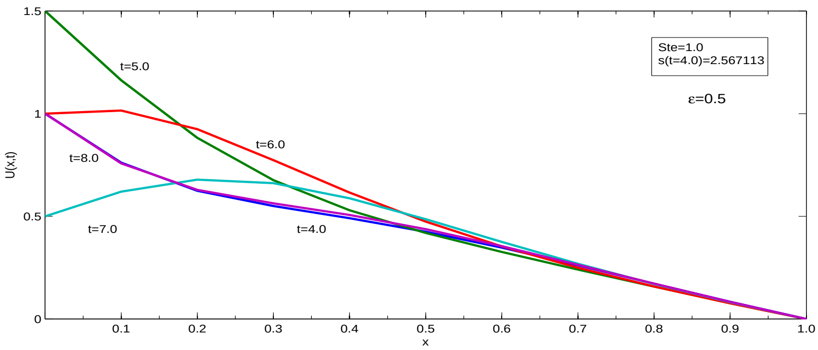
(a)



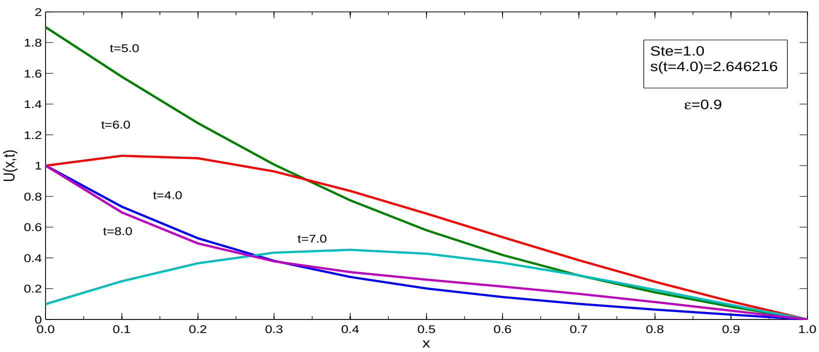
(b)

Figure 2. Velocity of the moving boundary for oscillation amplitudes (a) $\epsilon = 0.5$ and (b) $\epsilon = 0.9$, for oscillation frequency $\omega = \pi/2$ and for different Stefan numbers. Also the temperature oscillation at the $x = 0$ surface are plotted.

In order to verify the accuracy of finite element method, we compare the present results developed by using finite element method and numerical results obtained by using finite difference solution formed with boundary immobilisation method by Savović and Caldwell [5]. In Figures (1a) and (1b), the positions of moving boundary are plotted for oscillation amplitudes $\epsilon = 0.5$ and $\epsilon = 0.9$, different Stefan numbers $Ste = 2.0$, $Ste = 1.0$ and $Ste = 0.2$ and oscillation frequency $\omega = \pi/2$. Numerical results for the position of moving boundary obtained in present work and those obtained by finite difference method coincide with each other. For larger oscillation amplitudes $\epsilon = 0.9$ oscillation of moving boundary or superimposed humps increase as regards to $\epsilon = 0.5$ as shown reference [5] obtained by using finite difference method.



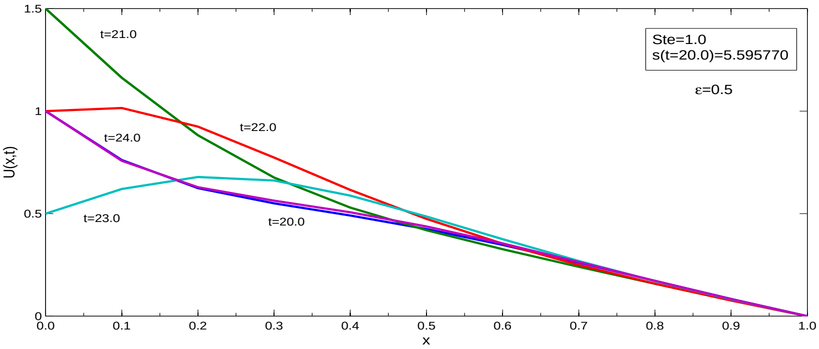
(a)



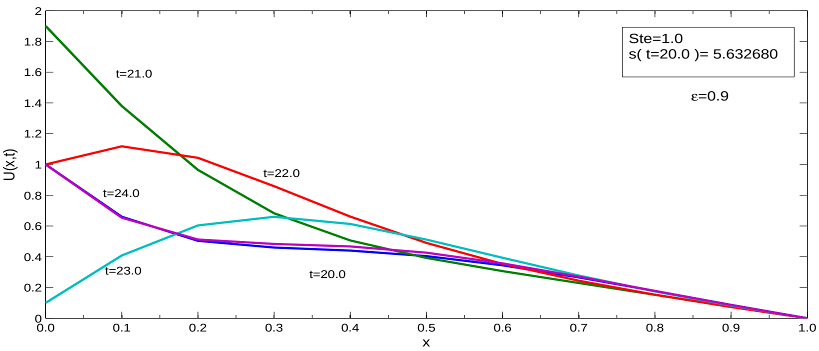
(b)

Figure 3. Temperature distribution for Ste $= 1.0$, *oscillation frequency* $\omega = \pi/2$ *and oscillation amplitudes (a)* $\epsilon = 0.5$, *(b)* $\epsilon = 0.9$ *when the domain size are (a)* $s(t = 4.0) = 2.567113$, *(b)* $s(t = 4.0) = 2.646216$ *at* $t = 4.0$.

In Figs (2a) and (2b), the velocity of moving boundary is plotted for different oscillation amplitudes $\epsilon = 0.5$ and $\epsilon = 0.9$, for different Stefan numbers $Ste = 2.0$, $Ste = 1.0$ and $Ste = 0.2$. With decreasing of Stefan numbers, the velocity of moving boundary oscillatory becomes to zero. For both oscillation amplitudes analyzed, the velocity of the moving boundary depends very strongly upon Ste numbers. And by increasing oscillation amplitudes, for the smaller Stefan numbers melting process periodically terminates. But for the larger Stefan numbers $Ste = 2.0$, $Ste = 1.0$ the melting process occurs without termination. So we can say both the oscillation amplitude and the Stefan numbers strongly influence the status and the velocity of moving boundary [5].

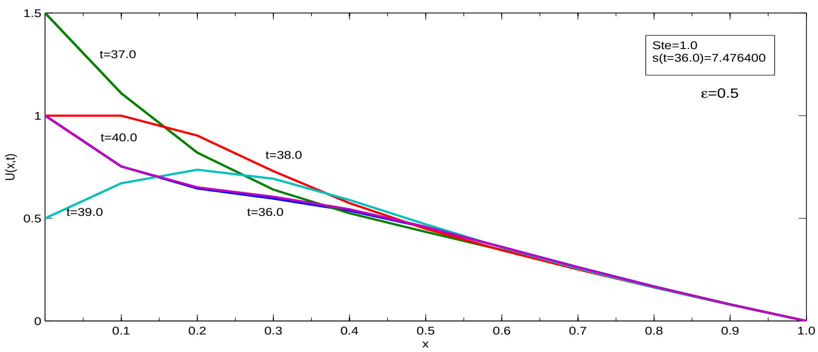


(a)

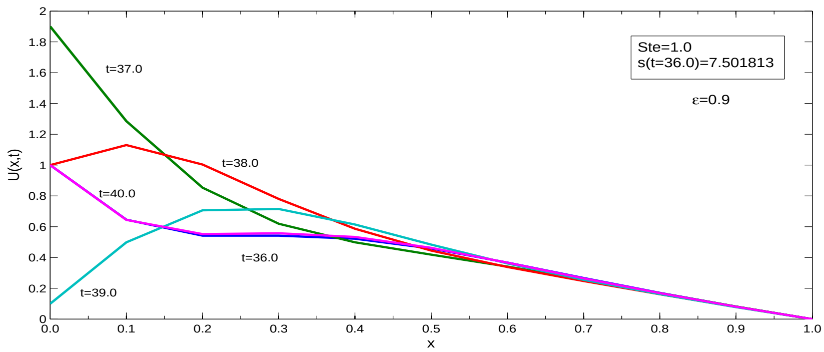


(b)

Figure 4. Temperature distribution for Ste = 1.0*, oscillation frequency* $\omega = \pi/2$ *and oscillation amplitudes (a)* $\epsilon = 0.5$*, (b)* $\epsilon = 0.9$ *when the domain size are (a)* $s(t = 20.0) = 5.595770$*, (b)* $s(t = 20.0) = 5.632680$ *at* $t = 20.0$*.*



(a)



(b)

Figure 5. Temperature distribution for Ste = 1.0*, oscillation frequency* $\omega = \pi/2$ *and oscillation amplitudes (a)* $\epsilon = 0.5$*, (b)* $\epsilon = 0.9$ *when the domain size are (a)* $s(t = 36.0) = 7.476400$*, (b)* $s(t = 36.0) = 7.501813$ *at* $t = 36.0$*.*

In Figures 3-5, in order to investigate impact of the size of time values for *Ste* = 1.0, temperature distribution is shown for different time values and the oscillation amplitudes $\epsilon = 0.5$ and $\epsilon = 0.9$. For the small size of time values the temperature distribution is changing in whole domain, but for the larger time values the temperature distribution is changing within the left half of the domain about $x = 0.5$. So, we can conclude that size of time values have very strong effect on time distribution. By expanding the size of time values, effective of boundary depending to time will be decrease. Consequently, as the size of time value sufficiently expand, temperature distribution near the moving boundary will be constant.

In Figure 6, we show the impact of oscillation frequency, which is taken constant $\omega = \pi/2$ whole of the other figures, for the position of moving boundary. As can be seen from Figures 6 and 7 for the smaller oscillation frequencies the oscillation of moving boundary will increase, for the larger oscillation frequencies the graphs for position of moving boundaries so close to each other, which is shown by Rizwan-uddin [11] for nodal integral method.

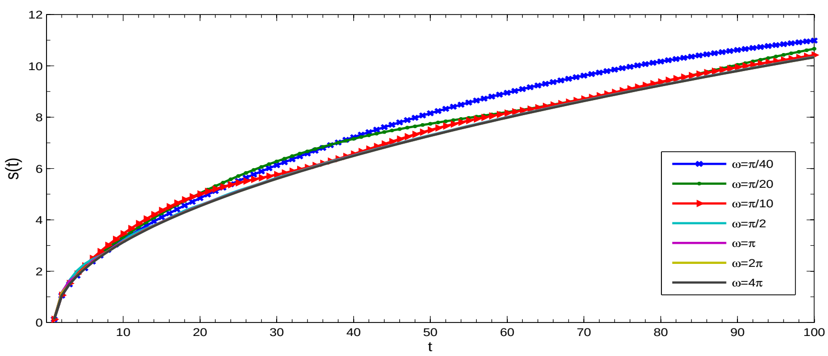


Figure 6. Position of the moving boundary for different Ste = 1.0*, oscillation amplitude* $\epsilon = 0.5$ *and for oscillation frequencies* $\omega = \pi/40$*,* $\omega = \pi/20$*,* $\omega = \pi/10$*,* $\omega = \pi/2$*,* $\omega = \pi$*,* $\omega = 2\pi$*,* $\omega = 4\pi$*.*

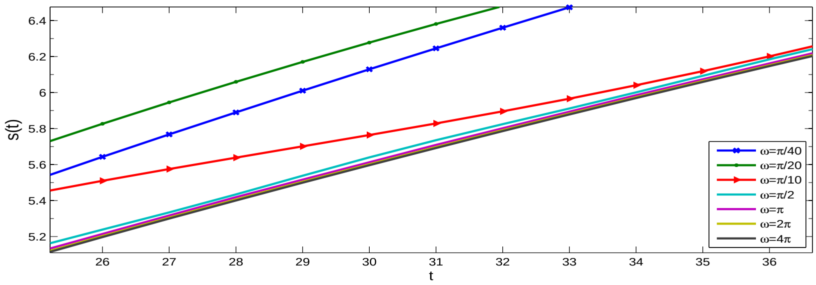


Figure 7. Position of the moving boundary values in figure 6 zoomed at time values between $t = 26.0$ and $t = 36.0$.

7. Conclusion

We have applied finite element method formed with cubic B-splines to solve the Stefan problems with time dependent periodic boundary condition. The position and velocity of moving boundary and temperature distribution are obtained for different oscillation amplitudes, different Stefan numbers and different oscillation frequency. It can be concluded that the size of domain, oscillation amplitude, oscillation frequency and Stefan numbers have strong effect on the movement of boundary, the velocity of moving boundary and time distribution. Furthermore, the effect of oscillation temperature distribution $x = 0$ is shown for different oscillation amplitudes. By increasing of oscillation amplitudes, for small Stefan numbers, the boundary movement terminates periodically. The present results show that finite element solutions are in good agreement with some other numerical solutions obtained by using finite difference method and nodal integral method [5,11,12].

Consequently, we can say that finite element method provide numerical solutions for the Stefan problems with time-dependent boundary condition. So it is important to achieve numerical solutions of Stefan problems with time-dependent boundary condition where the analytical solutions are not avaliable.

References

[1] J. Crank, Free and Moving Boundary Problems, (1st edition), Clarendon Press, Oxford, England, 1984.
[2] R.M. Furzeland, A comparative study of numerical methods for moving boundary problems, J. Inst. Maths. Appl. 26 (1980) 411–429.
[3] S. Kutluay, A.R. Bahadir, A. Ozdes, The numerical solution of one-phase classical Stefan problem, J. Comp. Appl. Math. 81 (1997) 35–44.
[4] A. Esen, S.Kutluay, An isotherm migration formulation for one-phase Stefan problem with a time dependent Neumann condition, Appl. Math. Comput. 150 (2004) 59–67.
[5] S. Savović, J. Caldwell, Finite difference solution of one-dimensional Stefan problem with periodic boundary conditions, Intern. J. Heat Mass Transfer 46 (2003) 2911–2916.
[6] W.D. Murray, F. Landis, Numerical and machine solutions of transient heat conduction problems involving melting or freezing, J.Heat Transfer 81 (1959) 106–112.
[7] N.S. Asaithambi, A variable time-step Galerkin method for a one-dimensional Stefan problem, Appl. Math. Comput. 81 (1997) 189–200.
[8] T.R. Goodman, The heat-balance integral method and its application to problems involving a change of phase, Trans. ASME 80 (1958) 335–342.
[9] N.S. Asaithambi, A Galerkin method for Stefan problems, Appl. Math. Comput. 52 (1992) 239–250.
[10] A. Esen, S. Kutluay, A numerical solution of the Stefan problem with a Neumann-type boundary condition by enthalpy method, Appl. Math. Comput. 148 (2004) 321–329.
[11] Rizwan-uddin, One-dimensional phase change with periodic boundary conditions, Numer. Heat Transfer A 35 (1999) 361–372.
[12] Rizwan-uddin, A nodal method for phase change moving boundary problems, Int. J. Comp. Fluid Dynam. 11 (1999) 211–221.
[13] L.S. Yao, J. Prusa, Melting and freezing, Adv. Heat Transfer 19 (1989) 1–95.
[14] A.H.A. Ali, G.A. Gardner, L.R.T. Gardner, A collocation solution for Burgers.equation using cubic B-spline finite elements, Comput. Meth. Appl. Mech. Eng. 100 (1992) 325–337.